(I) $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (from example 2)

(J) $\lim_{x \to \infty} x^n \exp(-x) = 0$ for all $n \in \mathbb{N}_0$

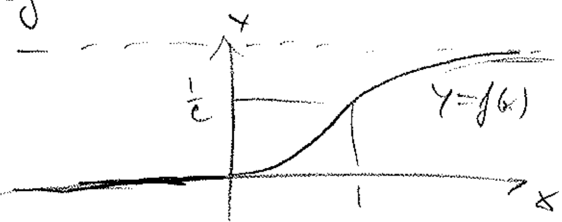
<u>Proof</u> from (I) $\exp(x) > \frac{x^{n+1}}{(n+1)!}$ for $x > 0$.

Thus $0 < x^n \exp(-x) < \frac{(n+1)!}{x}$, and taking the limit of $x \to \infty$

$$0 \leq \lim_{x \to \infty} x^n \exp(-x) \leq \lim_{x \to \infty} \frac{(n+1)!}{x} = 0 \qquad \square$$

<u>Theorem 2.3</u> Let $f: \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$



Then $f^{(k)}(x) = \begin{cases} P_{2k}\left(\frac{1}{x}\right) e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$

where P_{2k} is a polynomial of degree $\leq 2k$.

<u>Corollary</u> The n-th degree Taylor polynomial of f at zero is $T_{n,0}(x) = 0$

<u>Remark</u> While the Taylor polynomial can be a good approximation to a function, it need not be. As $f(x) = T_{n,0}(x) + R_n$, so $R_n = f(x)$.

|| Continue on Bottom of page 30! ||

Proof (of Thm 23): by mathematical induction over k.

$k=0$: $P_0 = 1$ ✓

$k \to k+1$: $x<0$: ✓

$$x>0: \quad f^{(k+1)}(x) = P_{2k}'\left(\tfrac{1}{x}\right)\left(-\tfrac{1}{x^2}\right)e^{-\frac{1}{x}} + P_{2k}\left(\tfrac{1}{x}\right)e^{-\frac{1}{x}}\left(+\tfrac{1}{x^2}\right)$$

$$= \underbrace{\tfrac{1}{x^2}\left(P_{2k}\left(\tfrac{1}{x}\right) - P_{2k}'\left(\tfrac{1}{x}\right)\right)}_{P_{2k+2}\left(\frac{1}{x}\right) \text{ pol of degree } \le 2k+2} e^{-\frac{1}{x}}$$

$$x=0: \quad f^{(k+1)}(0) = \lim_{x\to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x-0}$$

now $$\lim_{x\to 0^+} \frac{f^{(k)}(x)-f^{(k)}(0)}{x-0} = \lim_{x\to 0^+} \frac{1}{x} P_{2k}\left(\tfrac{1}{x}\right) e^{-\frac{1}{x}}$$

$$= \lim_{t\to\infty} t\, P_{2k}(t)\, e^{-t} = 0 \quad \text{by } (J)$$

and $$\lim_{x\to 0^-} \frac{f^{(k)}(x)-f^{(k)}(0)}{x-0} = \lim_{x\to 0^-} \frac{0-0}{x} = 0$$ □

Close to zero, derivatives of f become arbitrarily large: $\forall n\ \exists c_n \in (0,x)$

s.t. $$e^{-\frac{1}{x}} = R_{n-1} = \frac{f^{(n)}(c_n)}{n!} x^n, \text{ i.e.}$$

$$f^{(n)}(c_n) = \frac{n!}{x^n} e^{-\frac{1}{x}} \to \infty \quad \text{as } n\to\infty \quad (x \text{ fixed})$$

No matter how small x is, $\exists$ sequence (c_n) s.t. $f^{(n)}(c_n) \to \infty$.
$(|c_n| < x)$

Theorem 24 (L'Hospital's rule)

Let $f, g : D \to \mathbb{R}$ be differentiable for $|x-a| < \varepsilon$ and

let $g'(x) \neq 0$ for $0 < |x-a| < \varepsilon$. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

and if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Proof (a) need to show that $g(x) \neq 0$ for $0 < |x-a| < \varepsilon$

$g(a) = 0$, and if $g(b) = 0$ for some b, then there exists

a c between a and b such that $g'(c) = 0$ by Rolle,

but this is a contradiction to $g'(x) \neq 0$ for $0 < |x-a| < \varepsilon$

(b) by the second MVT, there exists a c between a and x such

that $$g'(c)\,(f(x) - \underbrace{\cancel{f(a)}}_{=0}) = f'(c)\,(g(x) - \underbrace{\cancel{g(a)}}_{=0})$$

Thus $$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

And $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{c\to a} \frac{f'(c)}{g'(c)}$$

as $c \to a$ if $x \to a$. $\square$

Examples

1) $$\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1+x}}{x} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1+2x}} - \frac{1}{2\sqrt{1+x}}}{1} = 1-\frac{1}{2} = \frac{1}{2}$$

2) $$\lim_{x\to 0} \frac{e^x-1-x}{x^2} = \lim_{x\to 0} \frac{e^x-1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$ (twice)

The rule also holds if $f(x), g(x) \to \infty$:

3) $$\lim_{x\to 0} x\log|x| = \lim_{x\to 0} \frac{\log|x|}{x^{-1}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$$

4) $$\lim_{x\to 0} |x|^x = \lim_{x\to 0} \exp\left(x\log|x|\right) = \exp\left(\lim_{x\to 0} x\log(|x|)\right) = \exp(0) = 1$$

Questionnaires