Theorem 19 Let I be an interval and $f: I \to \mathbb{R}$ be continuous and injective. Then $f^{-1}: f(I) \to \mathbb{R}$ is continuous. (26 Jan 9)

Proof By Theorem 18, f is strictly increasing or decreasing. Assume increasing.

Let $a \in I$, $b = f(a) \in f(I)$ and show f^{-1} is continuous in b:

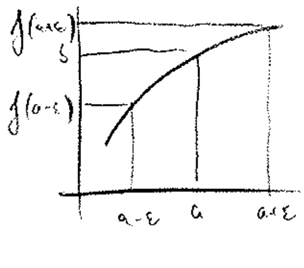
Fix $\varepsilon > 0$. If $y = f(x) \in f(I)$ satisfies

$$f(a-\varepsilon) < y < f(a+\varepsilon),$$ it follows that $a - \varepsilon < x < a + \varepsilon$

Let $\delta := \min\left(f(a+\varepsilon) - b,\ b - f(a-\varepsilon)\right)$

Then, $|y-b| < \delta$ implies $|x-a| < \varepsilon$,

so f^{-1} is continuous in b. $\square$



<u>Theorem 20</u>: Let I be an interval and $f: I \to \mathbb{R}$ be continuous and injective. Let f be differentiable at a and $b = f(a)$.

(a) If $f'(a) = 0$ then f^{-1} is not differentiable at b

(b) If $f'(a) \neq 0$ then f^{-1} is differentiable at b and

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

<u>Proof</u> (a) Let $f'(a) = 0$ and assume f^{-1} is differentiable at b.

Then differentiating $x = f^{-1}(f(x))$ gives

$$1 = (f^{-1})'(f(a))\, \underbrace{f'(a)}_{=0} = 0 \quad , \text{ a contradiction.}$$

(b) Let $f'(a) \neq 0$. Define, for $y \neq b$, $A(y) = \dfrac{f^{-1}(y) - f^{-1}(b)}{y-b}$

and
$$B(x) = \begin{cases} \dfrac{f(x)-f(a)}{x-a} & x \neq a \\ f'(a) & x = a \end{cases}$$

$\lim_{x \to a} B(x) = f'(a) = B(a)$, so B is continuous at a, and therefore continuous on all of I

f^{-1} is continuous on $f(I)$, so $B \circ f^{-1}$ is continuous on $f(I)$.

$$B \circ f^{-1}(y) = \begin{cases} \dfrac{y-b}{f^{-1}(y) - f^{-1}(b)} & y \neq b \\ f'(a) & y = b \end{cases} , \text{ so } B \circ f^{-1}(y) = \frac{1}{A(y)} \quad y \neq b$$

Thus $\lim_{y \to b} \dfrac{1}{A(y)} = B \circ f^{-1}(y) = f'(a)$, i.e.

$(f^{-1})'(b)$ exists and equals $\dfrac{1}{f'(a)}$. $\square$

Examples

1) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ is continuous and injective, $f(\mathbb{R}) = \mathbb{R}$

Therefore $f^{-1} : \mathbb{R} \to \mathbb{R}$ exists and is given by $x \mapsto x^{1/3}$

$\Rightarrow f^{-1}$ is continuous by Theorem 19

f is differentiable, $f'(x) = 3x^2$

$\Rightarrow f^{-1}$ is not differentiable at $x=0$ and differentiable elsewhere by Theorem 20

$$(f^{-1})'(x) = \frac{1}{3(x^{1/3})^2} = \frac{1}{3x^{2/3}} .$$

2) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \exp(x)$ is differentiable, $f(\mathbb{R}) = \mathbb{R}^+$, $f'(x) = \exp(x) > 0$

$f^{-1} : \mathbb{R}^+ \to \mathbb{R}$, $x \mapsto \log(x)$ is differentiable, and

$$(f^{-1})'(x) = \frac{1}{\exp(\log(x))} = \frac{1}{x} .$$

General powers

$a \in \mathbb{R}$, $x \in \mathbb{R}^+$ $\qquad x^a := \exp(a \log(x))$

$$(x^a)' = \exp(a \log(x)) \frac{a}{x} = a x^{a-1}$$

General exponential function

$a \in \mathbb{R}^+$, $x \in \mathbb{R}$ $\qquad a^x := \exp(x \log(a))$

$$(a^x)' = \exp(x \log(a)) \log(a) = \log(a)\, a^x$$

General logarithmic function for $a > 0$, $a \neq 1$

$$\log_a : \mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \frac{\log x}{\log a}$$

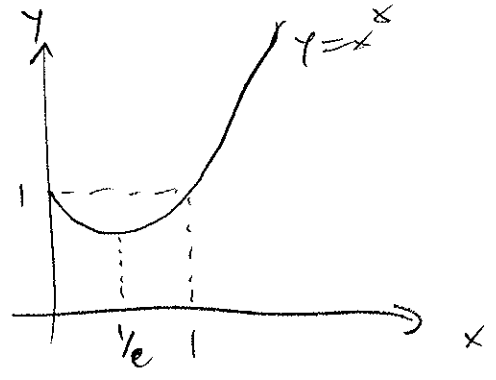
$$a^{\log_a(x)} = \exp\left(\log(a) \frac{\log(x)}{\log(a)}\right) = \exp(\log(x)) = x \quad \text{for } x > 0$$

$$\log_a(a^x) = \frac{1}{\log(a)} \log(\exp(\log(a)\, x)) = x \quad \text{for } x \in \mathbb{R}$$

Example

$f : \mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto x^x$ is differentiable,

$$f'(x) = \left(e^{x \log(x)}\right)' = e^{x \log x}\left(\log(x) + \frac{x}{x}\right) = x^x(\log(x) + 1)$$



5. Higher order derivatives

Theorem 21 (second Mean Value Theorem)

Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a $c \in (a,b)$ such that

$$g'(c)\,(f(b)-f(a)) = f'(c)\,(g(b)-g(a))$$

Proof: Consider the auxiliary function

$$h(x) = f(x)\,(g(b)-g(a)) - g(x)\,(f(b)-f(a))$$

h is continuous on $[a,b]$ and differentiable on (a,b). Thus, by the MVT, exists a $c \in (a,b)$ such that

$$h'(c) = \frac{h(b)-h(a)}{b-a},$$

i.e.

$$f'(c)\,(g(b)-g(a)) - g'(c)\,(f(b)-f(a))$$

$$= \frac{1}{b-a}\Big(f(b)(g(\cancel{b})-g(a)) - g(b)(\cancel{f(b)}-f(a)) - f(a)(g(b)-\cancel{g(a)}) + g(a)(f(b)-\cancel{f(a)})\Big) = 0$$

Remark: for $g(x)=x$ this reduces to the MVT. $\square$

Let $f : D \to \mathbb{R}$ be differentiable. If the derivative $f' : D \to \mathbb{R}$ is also differentiable, we can consider $(f')' = f''$, and so forth. In general we define the n-th derivative of an n-times differentiable function as

$$f^{(n)} = \left(f^{(n-1)}\right)' , \quad f^{(0)} = f .$$

<u>Example</u> $f : \mathbb{R} \to \mathbb{R}$, $x \to |x|\, x^n$ is differentiable for $n \in \mathbb{N}$

and $f'(x) = (n+1)\,|x|\, x^{n-1}$.

<u>Proof</u>

$x > 0$: $f(x) = x^{n+1}$, $f'(x) = (n+1)x^n$

$x < 0$: $f(x) = -x^{n+1}$, $f'(x) = -(n+1)x^n$

$x = 0$:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} |x|\, x^{n-1} = 0 \quad (\text{if } n > 0)$$

$$= \lim_{x \to 0} f'(x)$$

successive differentiation gives

$$f''(x) = (n+1)\, n\, |x|\, x^{n-2}$$

$$\vdots$$

$$f^{(n)}(x) = (n+1)!\, |x|$$ which is not differentiable at $x = 0$.

So f is precisely n times differentiable.

(Taylor's Theorem)

Theorem 22 Let $f : [a,x] \to \mathbb{R}$ be n times continuously differentiable (i.e. $f^{(n)}$ exists and is continuous) on $[a,x]$ and $(n+1)$ times differentiable on (a,x). Then there exists a $c \in (a,x)$ such that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$+ \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Remark A similar statement holds for $x < a$ (replace $[a,x]$ by $[x,a]$ etc)

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Proof Let $F(t) = f(t) + \frac{f'(t)}{1!}(x-t) + \dots + \frac{f^{(n)}(t)}{n!}(x-t)^n$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$$

Then $F(t)$ is continuous on $[a,x]$ and differentiable on (a,x), and

$$F'(t) = \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}$$

$$= \frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

Apply Theorem 21 to $F(t)$ and $g(t) = (x-t)^{n+1}$ on $[a,x]$:

There exists a $c \in (a,x)$ such that $F'(c)(g(x)-g(a)) = g'(c)(F(x)-F(a))$.

As $F(x) = f(x)$ and $g(x) = 0$, we find that

$$\frac{f^{(n+1)}(c)}{n!}\cancel{(x-c)^n}\left(0-(x-a)^{n+1}\right) = -(n+1)\cancel{(x-c)^n}\left(f(x)-F(a)\right)$$

so that $f(x) = F(a) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ $\square$

<u>Remark</u> We call

$$T_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

the <u>n-th ~~degree~~ Taylor polynomial of f at a</u> and

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

the <u>Lagrange form of the remainder term.</u>

$f(x) = T_{n,a}(x) + R_n$ is also called <u>Taylor's formula</u>.

$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is called the <u>Taylor series of f at a</u>

<u>Examples</u>

1) Estimate $e = \exp(1)$ using Taylor's formula:

$f(x) = \exp(x)$, $f^{(k)}(x) = \exp(x)$

$$T_{n,0}(x) = \sum_{k=0}^{n} \frac{\exp(0)}{k!}(x-0)^k = \sum_{k=0}^{n} \frac{x^k}{k!}$$

$$R_n = \frac{\exp(c)}{(n+1)!} x^{n+1}$$

Taylor's theorem for $x=1$ says that there exists a $c \in (0,1)$

such that $$e = \exp(1) = \sum_{k=0}^{n} \frac{1}{k!} + \frac{\exp(c)}{(n+1)!}$$

Thus $$\sum_{k=0}^{n+1} \frac{1}{k!} < e < \sum_{k=0}^{n+1} \frac{1}{k!} + \frac{3}{(n+1)!}$$ $e < \left(1+\frac{1}{1}\right)^2 = 4$

$n = 10$ gives $2.718281826 < e < 2.718281901$

Moreover, taking $n \to \infty$ gives $e = \sum_{k=0}^{\infty} \frac{1}{k!}$.

2) show $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$:

$$T_{n,0}(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \quad , \quad R_n = \frac{\exp(c)}{(n+1)!} x^{n+1}$$

Taylor's theorem says that there exists a $|c| < |x|$ such that

$$\left|\exp(x) - T_{n,0}(x)\right| = |R_n| = \left|\frac{\exp(c)}{(n+1)!} x^{n+1}\right| \leq C \left|\frac{x^{n+1}}{(n+1)!}\right|$$

Now $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, so $R_n \to 0$ as $n \to \infty$.

3) show $\log(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (x-1)^k$ for $1 < x \leq 2$

$$f(x) = \log(x), \; f'(x) = \frac{1}{x}, \; f''(x) = -\frac{1}{x^2}, \ldots \; f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{x^k}, \; k \geq 1$$

$$T_{n,1}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!} (x-1)^k = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} (x-1)^k$$

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!} (x-1)^{n+1} = \frac{(-1)^n}{n+1} \left(\frac{x-1}{c}\right)^{n+1}$$

So $\left|\log(x) - T_{n,1}(x)\right| = |R_n| \leq \frac{1}{n+1} \left|\frac{x-1}{c}\right|^{n+1}$

Now $0 < x-1 \leq 1$ and $1 < c < x \leq 2$, so that $\left|\frac{x-1}{c}\right| < 1$

Thus $R_n \to 0$ as $n \to \infty$ (also holds for $0 < x < 1$, so $|x-1| < 1$)