Theorem 13 For all $x \in \mathbb{R}$, $\exp(x) > x$

Proof $x < 0$: $\exp(x) > 0 > x$

$x = 0$: $\exp(0) = 1 > 0$

$x > 0$: There exists a $c \in (0, x)$ such that $\dfrac{\exp(x) - \exp(0)}{x - 0} = \exp(c)$ (MVT). Therefore $\exp(x) - 1 = x \exp c > x$, as

$\exp c > \exp 0 = 1$ for $c > 0$ by (F). So $\exp(x) > x + 1 > x$ □

(G) $\exp(\mathbb{R}) = \mathbb{R}^+ \quad (= \{x \in \mathbb{R} : x > 0\})$

Proof (E) implies $\exp(\mathbb{R}) \subseteq \mathbb{R}^+$. We need to show that

$$\forall c > 0 \; \exists x \in \mathbb{R} : \exp(x) = c$$

case 1: $c \geq 1$. Then $\exp(0) = 1 \leq c < \exp(c)$ so by the IVT there exists an $x \in (0, c)$ such that $\exp(x) = c$

case 2: $0 < c < 1$: $c^{-1} > 1$ and there exists $x \in (0, c^{-1})$ s.t. $\exp(x) = c^{-1}$

As $\exp(x)\exp(-x) = 1$, we have $\exp(-x) = c$ □

(H) $\exp(1) = e$, where $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

> Bernoulli inequality
> $(1+x)^n \geq 1+nx \quad \forall x \geq -1$
> $\forall n \in \mathbb{N}_0$

Proof: 1) show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists:

(a) $a_n = \left(1+\frac{1}{n}\right)^n$ is increasing:

As $\left(1-\frac{1}{n^2}\right)\left(1+\frac{1}{n-1}\right) = \frac{n^2-1}{n}\;\frac{n}{n-1} = \frac{n+1}{n} = 1+\frac{1}{n}$, we have

$$a_n = \left(1+\frac{1}{n}\right)^n = \underbrace{\left(1-\frac{1}{n^2}\right)^n}\left(1+\frac{1}{n-1}\right)^n$$

$$\geq \left(1-\frac{1}{n}\right)\left(1+\frac{1}{n-1}\right)\left(1+\frac{1}{n-1}\right)^{n-1} = \left(1+\frac{1}{n-1}\right)^{n-1} = a_{n-1}$$

(b) $b_n = \left(1+\frac{1}{n}\right)^{n+1}$ is decreasing:

As $\left(1+\frac{1}{n^2-1}\right)^n \geq 1+\frac{n}{n^2-1} \geq 1+\frac{1}{n}$, we have

$$b_n = \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) \leq \left(\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n^2-1}\right)\right)^n = \left(1+\frac{1}{n-1}\right)^n = b_{n-1}$$

(c) each b_m is an upper bound for (a_n) and vice versa.
Therefore $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(a_n\left(1+\frac{1}{n}\right)\right) = \lim_{n\to\infty} a_n.$$

2) show that $\exp(1) = e$: We shall show

that $a_n = \left(1+\frac{1}{n}\right)^n \leq \exp(1) \leq \left(1+\frac{1}{n}\right)^{n+1} = b_n$:

MVT on $\left[0,\frac{1}{n}\right]$: $\exists c \in \left(0,\frac{1}{n}\right)$ s.t. $\dfrac{\exp\left(\frac{1}{n}\right)-\exp(0)}{\frac{1}{n}-0} = \exp(c)$,

so that $\exp\left(\frac{1}{n}\right) = 1+\frac{1}{n}\exp(c)$. As $1 \leq \exp c \leq \exp\left(\frac{1}{n}\right)$,

$$1+\frac{1}{n} \leq \exp\left(\frac{1}{n}\right) \leq 1+\frac{1}{n}\exp\left(\frac{1}{n}\right)$$

This implies firstly that

$$\left(1+\frac{1}{n}\right)^n \leq \left(\exp\left(\frac{1}{n}\right)\right)^n = \exp(1)$$

Secondly $\left(1-\frac{1}{n}\right)\exp\left(\frac{1}{n}\right) \leq 1$, so that $\exp\left(\frac{1}{n}\right) \leq \frac{n}{n-1}$ $(n \geq 2)$

Shifting by 1, $\exp\left(\frac{1}{n+1}\right) \leq \frac{n+1}{n} = 1+\frac{1}{n}$, and

$$\left(1+\frac{1}{n}\right)^{n+1} \geq \left(\exp\left(\frac{1}{n+1}\right)\right)^{n+1} = \exp(1)$$ □

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<u>Corollary</u> $\exp(n) = e^n$ for $n \in \mathbb{Z}$

<u>Proof</u>: $n \in \mathbb{N}$: $\exp(n) = (\exp(1))^n = e^n$

$n = 0$: $\exp(0) = 1 = e^0$

$-n \in \mathbb{N}$: $\exp(-n) = (\exp(n))^{-1} = e^{-n}$ □

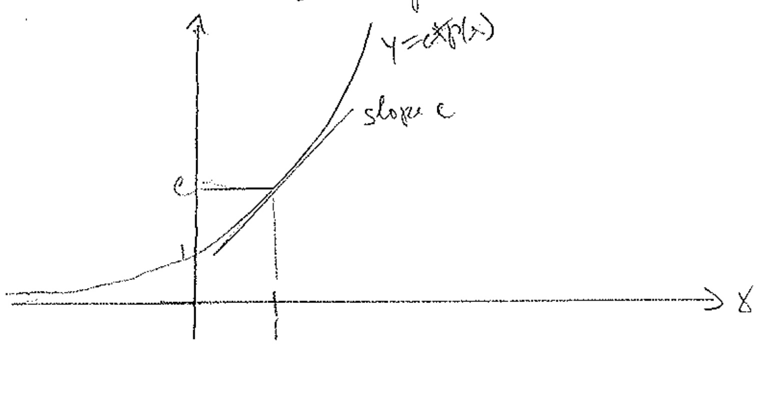
Also $\left(\exp\left(\frac{n}{m}\right)\right)^m = \exp(n) = e^n$, so that $\exp\left(\frac{n}{m}\right) = e^{\frac{n}{m}}$

Summarising, we have

<u>Theorem 14</u> (1) exp is strictly increasing

(2) $\exp(\mathbb{R}) = \mathbb{R}^+$

(3) $\exp(x) = e^x \quad \forall x \in \mathbb{Q}$



4. Inverse functions

<u>Definition 15</u> Let $f : D \to \mathbb{R}$, $E = f(D)$ the image of f.

Then f is <u>invertible</u> if there exist $g : E \to \mathbb{R}$ such that

$$g \circ f(x) = x \quad \text{for all } x \in D \quad \text{and} \quad f \circ g(x) = x \text{ for all } x \in E.$$

g is an <u>inverse</u> of f

<u>Properties</u> 1) The inverse is uniquely defined

<u>Proof</u> Let $E = f(D)$ and $g_1, g_2 : E \to \mathbb{R}$ be inverses of f.

Let $y \in E$. There exists an $x \in D$ with $y = f(x)$, and

$$g_1(y) = g_1 \circ f(x) = x = g_2 \circ f(x) = g_2(y), \text{ so } g_1 = g_2 \qquad \square$$

As the inverse is uniquely defined, we can write $g = f^{-1}$.

2) If f is invertible, then f^{-1} is invertible as well, and $(f^{-1})^{-1} = f$.

3) The graphs of f and f^{-1} are mirror images with respect to the straight line $y = x$.

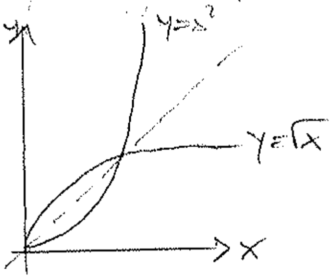
<u>Proof</u>

$$E = f(D)$$

$$\text{Graph}(f) = \{ (x, f(x)) : x \in D \}$$
$$\text{Graph}(f^{-1}) = \{ (y, f^{-1}(y)) : y \in E \} = \{ (f(x), f^{-1} \circ f(x)) : x \in D \}$$
$$= \{ (f(x), x) : x \in D \} \quad \text{is its mirror image} \qquad \square$$

<u>Example:</u> $f : \mathbb{R}_0^+ \to \mathbb{R}$, $f(x) = x^2$; $f(\mathbb{R}_0^+) = \mathbb{R}_0^+$

$f^{-1} : \mathbb{R}_0^+ \to \mathbb{R}$ $f^{-1}(x) = \sqrt{x}$



Theorem 16 $f : D \to \mathbb{R}$ is invertible if and only if it is injective (one-to-one)

Proof "$\Rightarrow$": Let f be invertible and $f(x_1) = f(x_2)$. Then

$$x_1 = f^{-1} \circ f(x_1) = f^{-1} \circ f(x_2) = x_2$$

"$\Leftarrow$" Let f be injective and $E = f(D)$. Then for each $y \in E$ there is a unique $x \in D$ such that $y = f(x)$. Define

$$g : E \to \mathbb{R} \text{ via } g(y) = x. \text{ Then}$$

$$g \circ f(x) = g(y) = x \quad \forall x \in D$$

$$\text{and } f \circ g(y) = f(x) = y \quad \forall y \in E$$

□

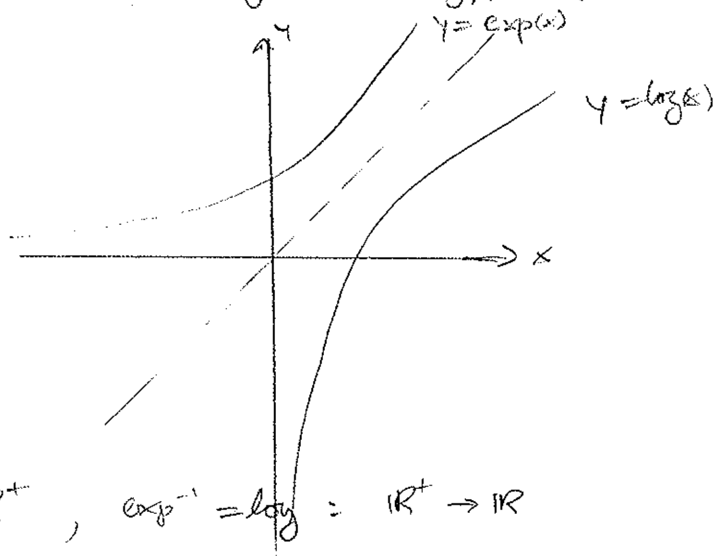
Corollary If $f : D \to \mathbb{R}$ is strictly increasing (or decreasing) then f is invertible.

Proof:

$$\left.\begin{array}{l} x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \\ x_1 > x_2 \Rightarrow f(x_1) > f(x_2) \end{array}\right\} \text{ implies } x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

□

Example $\exp : \mathbb{R} \to \mathbb{R}$ is strictly increasing, therefore invertible.



$$\exp(\mathbb{R}) = \mathbb{R}^+ , \quad \exp^{-1} = \log : \mathbb{R}^+ \to \mathbb{R}$$

Let I be an interval ($a, b \in I$, $a \leq c \leq b \Rightarrow c \in I$).

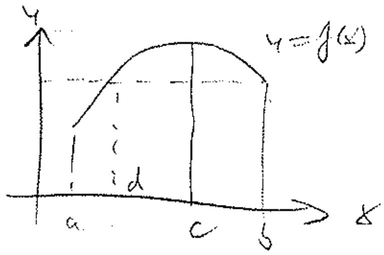
If $f: I \to \mathbb{R}$ is continuous, then $f(I)$ is an interval (by IVT).

<u>Theorem 17</u> Let $f: [a,b] \to \mathbb{R}$ be continuous and injective. Then f attains its minimum and maximum ~~at~~ a or b.

<u>Proof</u> Let $f(a) \leq f(b)$ [if $f(b) \leq f(a)$, the proof is analogous].

f is continuous, therefore f attains its maximum in $c \in [a,b]$.

If $c < b$, then $f(a) \leq f(b) \leq f(c)$



and by the IVT on $[a,c]$, there exists a $d \in [a,c]$ such that $f(d) = f(b)$.

Now $d \leq c < b$ implies $d \neq b$, a contradiction to injectivity.

The proof for the minimum is similar. $\square$

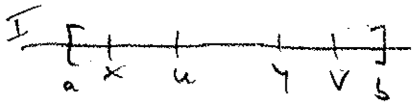
<u>Theorem 18.</u> Let I be an interval and $f: I \to \mathbb{R}$ be continuous and injective. Then f is strictly increasing or decreasing.

<u>Proof</u> (1) consider $I = [a,b]$ and assume $f(a) < f(b)$. Let $x, y \in I$, $x < y$.

Then, by Theorem 17, f attains its maximum in b and therefore $f(x) \leq f(b)$.

Considering the interval $[x, b]$, the minimum of f is attained on x, and thus $f(x) \leq f(y)$. [In fact, injectivity implies $f(x) < f(y)$ strictly]

(2) Let I be an arbitrary interval. Pick $u, v \in I$ with $u < v$,



Pick $x, y \in I$ with $x < y$,

and choose closed interval $[a,b]$ containing x, y, u, v.

Part (1) shows that f is strictly increasing or decreasing on $[a,b]$,

so if $f(u) < f(v)$ then $f(x) < f(y)$ $\forall x,y \in I$ $\square$

<u>Examples</u>: 1) $f : (0,2) \to \mathbb{R}$ $\quad f(x) = \begin{cases} x & x \in (0,1] \\ 3-x & x \in (1,2) \end{cases}$

injective, but not strictly increasing / decreasing (not continuous)

2) $f : (0,1) \cup (1,2) \to \mathbb{R}$ $\quad f(x) = \begin{cases} x & x \in (0,1) \\ 3-x & x \in (1,2) \end{cases}$

injective, continuous, but not strictly increasing / decreasing

(($(0,1) \cup (1,2)$ not interval!))