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<u>Example</u> $\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$

- using quotient rule with constant '1' in numerator:

$$\left(\frac{1}{f}\right)' = \frac{0 \cdot f - 1 \cdot f'}{f^2} = -\frac{f'}{f^2}$$

- using product rule with $g = \frac{1}{f}$, or $fg = 1$:

$$0 = (fg)' = f'g + fg', \text{ so } g' = -\frac{f'g}{f} = -\frac{f'}{f^2}$$

<u>Remark</u> All the derivatives from Calculus we assume known. This is not cheating, as we can prove every single one in principle.

Theorem 5 (Chain rule)

Let $f: D \to \mathbb{R}$ be differentiable at a, $g: f(D) \to \mathbb{R}$ be differentiable at $b = f(a)$. Then $g \circ f : D \to \mathbb{R}$ is differentiable at a and $(g \circ f)'(a) = g'(f(a))\, f'(a)$

Idea for formula: $\dfrac{g\circ f(x) - g\circ f(a)}{x-a} = \dfrac{g\circ f(x) - g \circ f(a)}{f(x) - f(a)} \cdot \dfrac{f(x)-f(a)}{x-a}$

and it looks like we can easily take limits on the right-hand side. The problem is that $f(x) - f(a)$ might be zero for $x \neq a$. We need a different proof:

Proof By Lemma 2 we have

(1) $f(x) = f(a) + f'(a)(x-a) + r(x)(x-a)$

(2) $g(y) = g(b) + g'(b)(y-b) + s(y)(y-b)$

with $\lim\limits_{x\to a} r(x) = 0$ and $\lim\limits_{y \to b} s(y) = 0$ and we define $s(b) = 0$.

Let $y = f(x)$ in (2) to get

$$\begin{aligned} g\circ f(x) &= g(b) + (g'(b) + s(f(x)))\,(f(x) - b) \\ &= g(b) + (g'(b) + s(f(x)))\,(f'(a) + r(x))\,(x-a) \\ &= g(b) + g'(b) f'(a)(x-a) + t(x)(x-a) \end{aligned}$$

where $t(x) = s(\cancel{f(x)}^{\,b}) f'(a) + g'(b)\, r(x) + s(\cancel{f(x)}^{\,b})\, r(x)$

Now $\lim\limits_{x \to a} t(x) = 0$ and thus $g \circ f$ is differentiable at a with $(g\circ f)'(a) = g'(b) f'(a) = g'(f(a))\, f'(a)$ $\square$

2. The Mean Value Theorem

<u>Theorem 6</u> If a function $f : [a,b] \to \mathbb{R}$ has a maximum (or minimum) at $c \in (a,b)$ and is differentiable at c, then $f'(c) = 0$

<u>Proof</u> (for maximum only): Let $d = f'(c) = \lim\limits_{x \to a} \dfrac{f(x) - f(c)}{x - c}$.

Then
$$d = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

and
$$d = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0$$

as
$$f(x) - f(c) \leq 0 \quad \text{for all } x \in D. \qquad \text{Therefore } d = 0 \; \square$$

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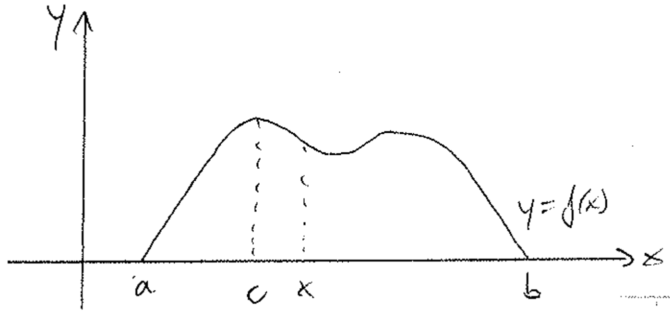
<u>Theorem 7</u> (Rolle) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b) = 0$ then there exists $c \in (a,b)$ such that $f'(c) = 0$



<u>Proof</u> We consider three cases:

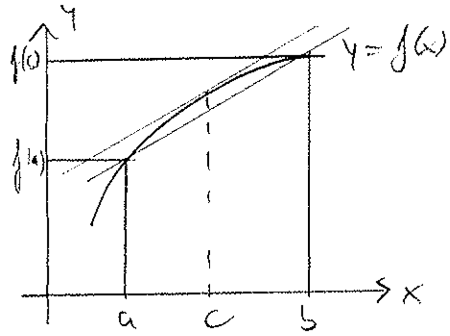
(1) $f(x) = 0$ for all $x \in (a,b)$. Then $f'(x) = 0$ for all $x \in (a,b)$

(2) $f(x) > 0$ for some $x \in (a,b)$. Then f is maximal at some $c \in [a,b]$ and $f(c) \geq f(x) > 0 = f(a) = f(b)$. Therefore $c \in (a,b)$ and, by Theorem 6, $f'(c) = 0$.

(3) $f(x) < 0$ for some $x \in (a,b)$. Continue as in (2). $\square$

Theorem 8 (Mean Value Theorem) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b).
Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Proof Consider the auxiliary function

$$h(x) = (x-a)\big(f(b) - f(a)\big) - (b-a)\big(f(x) - f(a)\big)$$

h is continuous on $[a,b]$ and differentiable on (a,b),

and $h(a) = 0 = h(b)$. By Rolle's theorem

there exists a $c \in (a,b)$ such that $h'(c) = 0$. As

$$h'(x) = f(b) - f(a) - (b-a) f'(x),$$ we have

$$0 = h'(c) = f(b) - f(a) - (b-a) f'(c),$$ so that

$$f'(c) = \frac{f(b) - f(a)}{b-a} \qquad \square$$

Geometric interpretation: There exists a tangent to the graph of f which is parallel to the secant through $(a, f(a))$ and $(b, f(b))$.

This theorem has many important consequences. For now, we give a simple application.

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b).

<u>Theorem 9</u> (a) if $f'(x) > 0$ for all $x \in (a,b)$, then

f is strictly increasing on $[a,b]$: $x_1 < x_2$ implies $f(x_1) < f(x_2)$

(b) if $f'(x) < 0$ for all $x \in (a,b)$ then f is strictly decreasing on $[a,b]$:

$x_1 > x_2$ implies $f(x_1) > f(x_2)$.

<u>Proof</u> (a) Let $x_1, x_2 \in [a,b]$ with $x_1 < x_2$. Applying the Mean Value Theorem

to f on $[x_1, x_2]$, we have that there exists a $c \in (x_1, x_2)$ with

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0$$

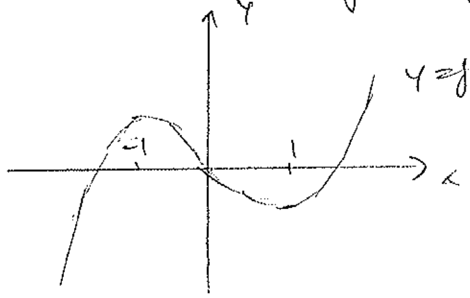
Therefore $f(x_2) - f(x_1) > 0$. (b) similarly. $\square$

<u>Example</u> $f : \mathbb{R} \to \mathbb{R}$; $f(x) = \frac{x^3}{3} - x$: $f'(x) = x^2 - 1$

$f'(x) < 0$ on $(-1,1)$, $f'(x) > 0$ on $(-\infty, -1) \cup (1, \infty)$

Therefore f is strictly decreasing on $(-1,1)$ and strictly increasing on $\{x : |x| > 1\}$

$f(0) = 0$, $f(\pm 1) = \mp \frac{2}{3}$



Theorem 10 Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f'(x)=0$ for all $x\in(a,b)$, then f is constant on $[a,b]$, i.e.

$f(x)=f(a)$ for all $x\in[a,b]$.

Proof Let $x\in(a,b]$ and apply the Mean Value Theorem to f on $[a,x]$: There exists a $c\in(a,x)$ such that $\frac{f(x)-f(a)}{x-a}=f'(c)=0$.

Therefore $f(x)=f(a)$ □

3. The Exponential Function

Definition 11 A differentiable function $f:\mathbb{R}\to\mathbb{R}$ with

(a) $f'(x)=f(x)$ for all $x\in\mathbb{R}$ (b) $f(0)=1$

is called exponential function

Remark We will show later that $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ satisfies this definition. For now, we shall assume existence of such a function.

Properties of an exponential function

(A) $f(x)\,f(-x)=1$

Proof: Differentiate $h(x)=f(x)\,f(-x)$: $h'(x)=f'(x)f(-x)+f(x)f'(-x)(-1)=0$.

By Theorem 10, h is constant, and $h(0)=f(0)f(0)=1$, so $h(x)=1$ □

(B) $f(x) \neq 0$ for all $x \in \mathbb{R}$

Proof: If $f(x) = 0$ for some $x \in \mathbb{R}$ then $0 = f(x)\, f(-x) = 1$, a contradiction. □

(C) Let $g: \mathbb{R} \to \mathbb{R}$ be differentiable with $g' = g$. Then there exists a $c \in \mathbb{R}$ such that $g = c f$

Proof Consider $h(x) = \dfrac{g(x)}{f(x)}$. By (B), h is defined on $\mathbb{R}$ and differentiable.

$$h'(x) = \frac{g'(x)\, f(x) - g(x)\, f'(x)}{f^2(x)} = 0$$, therefore h is constant, $h(x) = c$

Thus $g(x) = c\, f(x)$ □

(D) Definition 11 determines f uniquely

Proof Assume g satisfies Definition 11. Then (C) implies $g = cf$

As $g(0) = 1 = f(0)$ we have $c = 1$, so $g = f$. □

We will write $f(x) = \exp(x)$ for f defined by Definition 11.

Theorem 12 For all $a, b \in \mathbb{R}$, $\exp(a+b) = \exp(a)\exp(b)$

Proof Consider $g(x) = \exp(a+x)$. Then $g'(x) = \exp(a+x) = g(x)$,

so $\exp(a+x) = c \exp(x)$ by (C).

For $x = 0$, $\exp(a) = c$, so that $\exp(a+b) = c\exp(b) = \exp(a)\exp(b)$

□

Corollary For $a \in \mathbb{R}$ and $n \in \mathbb{N}$, $\exp(na) = (\exp(a))^n$

Proof: Math induction: $n=1$: $\exp(a) = (\exp(a))^1$ $n \to n+1$: $\exp((n+1)a) = \exp(na)\exp(a)$ etc. □

(E) $\exp(x) > 0$ for all $x \in \mathbb{R}$

Proof exp is differentiable, therefore continuous. $\exp(x) \neq 0$ for all $x \in \mathbb{R}$. $\exp(0) = 1$, and if there was an $x \in \mathbb{R}$ with $\exp(x) < 0$, then the Intermediate Value Theorem (IVT) would imply that there was a $c \in \mathbb{R}$ such that $\exp(c) = 0$. □

(F) $\exp(x)$ is strictly increasing

Proof $\exp'(x) = \exp(x) > 0$ and Theorem 9 □