<u>Corollary</u> $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all $x \in \mathbb{R}$ s.t. $|x| < r$ and diverges for all $x \in \mathbb{R}$ s.t. $|x| > r$, where r is the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$

<u>Remark</u> Convergence for $x = \pm r$ needs to be considered separately.

<u>Theorem 62</u> Let $r > 0$ be the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ and $0 < \rho < r$. Then $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $D = \{x : |x| \leq \rho\}$

<u>Proof</u> As $\rho < r$, $\sum_{n=0}^{\infty} a_n \rho^n$ converges absolutely. As $|a_n x^n| \leq |a_n \rho^n|$ on D the Weierstraß M-Test implies uniform convergence of $\sum_{n=0}^{\infty} a_n x^n$ on D.

<u>Theorem 63</u> Let $r > 0$ be the radius of convergence of $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then for all x with $|x| < r$

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$$

<u>Proof</u> Choose $0 < \rho < r$. Then $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $D = \{x : |x| \leq \rho\}$.

As $f_n(x) = a_n x^n$ is Riemann-integrable, Theorem 59 (c) implies that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is R-integrable on D and

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

$\square$

<u>Theorem 64</u> Let $r > 0$ be the radius of convergence of

$f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then for all x with $|x| < r$

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

<u>Proof</u> Choose $0 < \rho < r$. Then $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $D = \{x : |x| \leq \rho\}$

To apply Theorem 59 (b), we need to show that $\sum_{n=0}^{\infty} n a_n x^n$ also converges uniformly on D. Once this is established, it follows that f is differentiable on D and $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$.

Now pick $\rho < \rho' < r$, then $\sum_{n=0}^{\infty} a_n \rho'^n$ converges absolutely,

and $|n a_n x^n| \leq |n a_n \rho^n| = |a_n \rho'^n| \, \underbrace{\left| n \left(\frac{\rho}{\rho'}\right)^n \right|}_{\leq 1 \text{ for } n \geq n_0} \leq |a_n \rho'^n|$

implies by the Weierstraß M-Test uniform convergence of $\sum_{n=0}^{\infty} n a_n x^n$ for $|x| \leq \rho$, as needed. $\square$

<u>Corollary</u> $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is for $|x| < r$ infinitely often differentiable

and $$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, a_n x^{n-k}$$

<u>Remark</u> We find $f^{(k)}(0) = k!\, a_k$, so that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$, the Taylor-series of f about zero.

<u>Examples:</u> 1) for $|x|<1$ we have

$$\frac{1}{1+x} = 1-x+x^2-x^3+\dots$$

and integration gives by Theorem 63

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}$$

for all x such that $|x|<1$ (we had only proved this for $0\le x<1$)

[Note that for $x=1$ the first series diverges $(1-1+1-1+\dots)$ but the second one converges $(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots)$, whereas for $x=-1$ both series diverge]

2) for $|x|<1$ we have $\displaystyle\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$

As $\displaystyle\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)$, we have

$$\frac{1}{2}\log\frac{1+x}{1-x} = \int_0^x \frac{dx}{1-x^2} = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1} \qquad \text{for } |x|<1$$

Thus, for example, $x=\frac{1}{2}$ gives

$$\log 3 = 2\left(\frac{1}{2}+\frac{1}{3\cdot 2^3}+\frac{1}{5\cdot 2^5}+\dots\right)$$

3) $$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \quad \text{for all } t \in \mathbb{R}, \text{ so that}$$

$$\int_0^x e^{-t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(2n+1)} = x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \cdots$$

4) $$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{for } |x|<1 \text{ gives}$$

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \text{for } |x|<1$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Return to Taylor series : $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ converges for $|x-a|<r$

and we identify $f^{(k)}(a) = k!\, a_k$, so that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k \quad \text{, the Taylor series of } f \text{ about } a$$

<u>Theorem 65</u> (Taylor's Theorem with Integral Form of Remainder)

Let $f : [a,x] \to \mathbb{R}$ be n times continuously differentiable on $[a,x]$

and $(n+1)$ times differentiable on (a,x). Then

$$f(x) = T_{n,a}(x) + \int_a^x \frac{f^{(n+1)}(t)}{n!} (x-t)^n \, dt$$

Proof   As in the proof of Theorem 22 we write

$$F(t) = T_{n,t}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k \quad \text{and compute}$$

$$F'(t) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n . \qquad \text{Therefore by FTC}$$

$$F(x) - F(a) = \int_a^x F'(t)\,dt = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt$$

and with $F(x) = f(x)$ and $F(a) = T_{n,a}(x)$ we have

$$f(x) - T_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt$$ $\square$

Remark An analogous result holds if $[a,x]$ is replaced by $[x,a]$.

Theorem 66   For $\alpha \in \mathbb{R}$ we have

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \quad \text{for} \quad |x|<1$$

where $$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}$$

Proof $f(x) = (1+x)^\alpha$ gives $f^{(k)}(x) = \alpha(\alpha-1)\dots(\alpha-k+1)(1+x)^{\alpha-k}$

so that $f^{(k)}(0) = \alpha(\alpha-1)\dots(\alpha-k+1)$. Thus

$$(1+x)^\alpha = \sum_{k=0}^{n} \binom{\alpha}{k} x^k + \int_0^x \frac{\alpha(\alpha-1)\dots(\alpha-n)}{n!}(1+t)^{\alpha-n-1}(x-t)^n\,dt$$

Now $\left| \int_0^x \underbrace{\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}}_{\text{bounded}} (1+t)^{\alpha+1} \left(\frac{x-t}{1+t}\right)^n dt \right| \to 0$

$\left| \frac{x-t}{1+t} \right| < |x|^n \to 0$ $\square$

<u>Examples</u>

$$\frac{1}{\sqrt{1+x}} = \sum_{k=0}^{\infty} \binom{-1/2}{k} x^k \qquad |x|<1$$

$$\frac{1}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} \binom{-1/2}{k} (-1)^k x^{2k} \qquad |x|<1$$

$$\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \sum_{k=0}^{\infty} \binom{-1/2}{k} \frac{(-1)^k}{2k+1} x^{2k+1} \qquad |x|<1$$

$$= x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots$$