<u>Theorem 54</u> If a sequence of continuous functions converges uniformly, then the limiting function is continuous.

<u>Remark</u> If the limiting function is discontinuous, the convergence cannot be uniform.

<u>Examples</u> (1) $f_n : [0,1] \to \mathbb{R}$ $f(x) = x^n$ are continuous

the limiting function $f : [0,1] \to \mathbb{R}$ $f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$ is not

the convergence cannot be uniform. (2) the convergence is uniform, the limiting function is continuous.

(3) the limiting function is continuous, but this does not imply uniform convergence.

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<u>Theorem 55</u> Let $f_n : [a,b] \to \mathbb{R}$ be differentiable. If (f_n) converges pointwise to $f : [a,b] \to \mathbb{R}$ and (f_n') converges uniformly to $g : [a,b] \to \mathbb{R}$, then f is differentiable and $f' = g$.

<u>Proof</u> Fix $x_0 \in [a,b]$. To show that f is differentiable at x_0 with $f'(x_0) = g(x_0)$, define auxiliary functions

$$h_n : [a,b] \to \mathbb{R}$$

$$h_n(x) = \begin{cases} \dfrac{f_n(x) - f_n(x_0)}{x - x_0} & x \neq x_0 \\ f_n'(x_0) & x = x_0 \end{cases}$$

so that $$h(x) = \lim_{n\to\infty} h_n(x) = \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0} & x \neq x_0 \\ g(x_0) & x = x_0 \end{cases}$$

If we can show that (1) h_n are continuous (2) h_n converge uniformly to h

then by Theorem 54 h is continuous and therefore $\lim_{x \to x_0} h(x) = h(x_0)$

But $h(x_0) = g(x_0)$ and $\lim_{x \to x_0} h(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$, so

this implies $g(x_0) = f'(x_0)$ as needed.

To prove (1): h_n is continuous for $x \neq x_0$ by construction,

and $\lim_{x \to x_0} h_n(x) = \lim_{x \to x_0} \frac{f_n(x) - f_n(x_0)}{x - x_0} = f_n'(x_0) = h_n(x_0)$ implies continuity at x_0.

(2) $$h_m(x) - h_n(x) = \frac{f_m(x) - f_m(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} \qquad (x \neq x_0)$$

$$= \frac{(f_m(x) - f_n(x)) - (f_m(x_0) - f_n(x_0))}{x - x_0}$$

apply MVT to $f_m - f_n$ to get $\exists\, c \in (a,b)$ such that

$$h_m(x) - h_n(x) = f_m'(c) - f_n'(c) \qquad \text{for } x \neq x_0$$

$$h_m(x_0) - h_n(x_0) = f_m'(x_0) - f_n'(x_0) \qquad \text{for } x = x_0$$

Now f_n' converge uniformly to g, so that

$\forall \varepsilon > 0\ \exists n_0\ \forall n \geq n_0 : |f_m'(x) - f_n'(x)| < \varepsilon$ indep. of x,

and thus $|h_m(x) - h_n(x)| < \varepsilon$ indep of x, i.e. $h_n \to h$ uniformly $\square$

<u>Remark</u> This implies that under the assumption of <u>uniform convergence of the derivative f_n'</u>

we have $\left(\lim_{n \to \infty} f_n\right)' = \lim_{n \to \infty} \left(f_n'\right)$

Remarks (1) We need only convergence of f_n to f at one point x_0.

Moreover, it follows that f_n converges to f uniformly:

Proof: $(f_n - f)(x) = (f_n - f)(x_0) + (x - x_0)(f_n' - f')(c_n) \qquad c_n \in (a,b)$ by MVT

so that
$$|f_n(x) - f(x)| \leq \underbrace{|f_n(x_0) - f(x_0)|}_{< \frac{\varepsilon}{2} \text{ as } f_n(x_0) \to f(x_0)} + (b-a)\underbrace{|f_n'(c_n) - f'(c_n)|}_{< \frac{\varepsilon}{2} \text{ as } f_n' \to f' \text{ uniformly.}} < \varepsilon$$

which implies uniform convergence of f_n to f.

(2) Even if f_n are differentiable and $f_n \to f$ uniformly, the limiting function need not be differentiable:

Jout!

Lemma There exists a sequence of polynomials $p_n(t)$ which converge uniformly to $p(t) = \sqrt{t}$ on $[0,1]$.

Proof Define $p_0(t) = 0$, $p_{n+1}(t) = p_n(t) + \frac{1}{2}(t - p_n^2(t))$

$\left[\text{i.e. } p_1(t) = \frac{t}{2},\ p_2(t) = \frac{t}{2} + \frac{1}{2}\left(t - \frac{t^2}{4}\right) = t - \frac{t^2}{8} \text{ etc}\right]$

We have

(a)
$$\sqrt{t} - p_{n+1}(t) = \sqrt{t} - p_n(t) - \frac{1}{2}\left(\sqrt{t} - p_n(t)\right)\left(\sqrt{t} + p_n(t)\right)$$
$$= \left(\sqrt{t} - p_n(t)\right)\left(1 - \frac{1}{2}\left(\sqrt{t} + p_n(t)\right)\right)$$

Next we show

(b) $0 \leq \sqrt{t} - p_n(t) \leq \dfrac{2\sqrt{t}}{2 + n\sqrt{t}}$ for $0 \leq t \leq 1$

Proof of (2) by induction: $n=0$: $0 \le \sqrt{t} \le \frac{2\sqrt{t}}{2}$ ✓

$n \to n+1$: $$1-\frac{1}{2}\left(\sqrt{t}+p_n(t)\right) \geq 1-\frac{1}{2}\left(\underbrace{\sqrt{t}+\sqrt{t}}_{p_n \le \sqrt{t} \text{ by assumption}}\right) = 1-\sqrt{t} \geq 0$$

$$1-\frac{1}{2}\left(\sqrt{t}+p_n(t)\right) \le 1-\frac{1}{2}\sqrt{t} \le 1-\frac{\sqrt{t}}{2+(n+1)\sqrt{t}} = \frac{2+n\sqrt{t}}{2+(n+1)\sqrt{t}}$$

so that $$0 \le 1-\frac{1}{2}\left(\sqrt{t}+p_n(t)\right) \le \frac{2+n\sqrt{t}}{2+(n+1)\sqrt{t}}$$

Therefore by (a) $$0 \le \sqrt{t}-p_{n+1}(t) \le \left(\sqrt{t}-p_n\right)\frac{2+n\sqrt{t}}{2+(n+1)\sqrt{t}}$$

$$\le \frac{2\sqrt{t}}{\cancel{2+n\sqrt{t}}}\,\frac{\cancel{2+n\sqrt{t}}}{2+(n+1)\sqrt{t}} = \frac{2\sqrt{t}}{2+(n+1)\sqrt{t}}$$ ✓

As $\frac{2\sqrt{t}}{2+n\sqrt{t}} \le \frac{2}{n}$, (b) implies the weaker, but simpler

(c) $$0 \le \sqrt{t}-p_n(t) \le \frac{2}{n} \quad \text{for all } t \in [0,1]$$

but this implies uniform convergence of $p_n(t)$ to $p(t)=\sqrt{t}$ on $[0,1]$ □

Applying the lemma to $g_n(x)=p_n(x^2)$, we find

$$0 \le |x|-g_n(x) \le \frac{2}{n} \quad \text{for all } x \in [-1,1]$$

i.e. $g_n \to g$ uniformly on $[-1,1]$, where $g(x)=|x|$ is not differentiable

In fact $g_n'(0)=0$ for all n, but g is not differentiable at 0.

<u>Theorem 56</u> Let $f_n : [a,b] \to \mathbb{R}$ be Riemann-integrable.

If (f_n) converge uniformly to $f : [a,b] \to \mathbb{R}$ then f is Riemann-integrable and $$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$$

<u>Remark</u> This implies that under the assumption of uniform convergence of R-integrable f_n we have $$\int_a^b \lim_{n\to\infty} f_n(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$$

<u>Proof</u> Let $\varepsilon > 0$. We want to show that there exists a partition $P \in \mathcal{P}[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

(a) f_n converges uniformly to f : $\exists n$ s.t. $|f(x) - f_n(x)| < \frac{\varepsilon}{3(b-a)}$ $\forall x \in [a,b]$

once n is chosen,

(b) f_n is R-integrable : $\exists P \in \mathcal{P}$ s.t. $U(f_n, P) - L(f_n, P) < \frac{\varepsilon}{3}$

f_n is bounded, and (a) implies $f - f_n$ is bounded, so that we can construct upper and lower sums $U(f,P)$, $L(f,P)$:

$$M_i = \sup_{x\in I_i} f(x) \leq \sup_{x\in I_i} f_n(x) + \sup_{x\in I_i}\left(f(x) - f_n(x)\right) \leq M_i^{(n)} + \frac{\varepsilon}{3(b-a)}$$

$$m_i = \inf_{x\in I} f(x) \geq \inf_{x\in I_i} f_n(x) + \inf_{x\in I_i}\left(f(x) - f_n(x)\right) \geq m_i^{(n)} - \frac{\varepsilon}{3(b-a)}$$

Therefore

$$U(f,P) - U(f_n,P) \leq \sum_{i=1}^{n} (M_i - M_i^{(n)}) \Delta x_i \leq \frac{\varepsilon}{3(b-a)} \sum_{i=1}^{n} \Delta x_i = \frac{\varepsilon}{3}$$

$$L(f,P) - L(f_n,P) \geq \sum_{i=1}^{n} (m_i - m_i^{(n)}) \Delta x_i \geq -\frac{\varepsilon}{3(b-a)} \sum_{i=1}^{n} \Delta x_i = -\frac{\varepsilon}{3}$$

and thus

$$\begin{aligned} U(f,P) - L(f,P) &= U(f,P) - U(f_n,P) \\ &\quad + U(f_n,P) - L(f_n,P) \\ &\quad + L(f_n,P) - L(f,P) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

Thus f is R-integrable.

Moreover

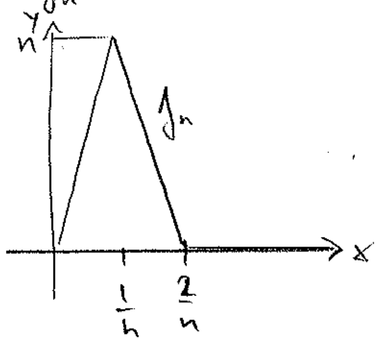
$$\left| \int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx \right| = \left| \int_a^b f(x) - f_n(x)\,dx \right|$$

$$\leq \int_a^b |f(x) - f_n(x)|\,dx \leq (b-a) \sup_{x \in [a,b]} |f(x) - f_n(x)| \to 0$$

as $n \to \infty$ $\square$

Example (4)

$f_n : [0,2] \to \mathbb{R}$



$$f_n(x) = \begin{cases} n^2 x & 0 \le x \le \frac{1}{n} \\ 2n - n^2 x & \frac{1}{n} < x \le \frac{2}{n} \\ 0 & \frac{2}{n} < x \le 2 \end{cases}$$

as in (3), $f_n(x) \to f(x) = 0$ pointwise, but not uniformly.

$$\int_0^2 f_n(x)\,dx = \int_0^{1/n} n^2 x\,dx + \int_{1/n}^{2/n} (2n - n^2 x)\,dx = 1, \quad \text{but} \quad \int_0^2 f(x)\,\cancel{dx} = 0$$

Definition 57

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(a) $\sum_{n=1}^{\infty}{}' f_n(x)$ is pointwise convergent if

$$S_k(x) = \sum_{n=1}^{k}{}' f_n(x) \quad \text{converges pointwise as } k \to \infty$$

(b) $\sum_{n=1}^{\infty}{}' f_n(x)$ is uniformly convergent if

$$S_k(x) = \sum_{n=1}^{k}{}' f_n(x) \quad \text{converges uniformly as } k \to \infty.$$

Example $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2+x^2)^n}$

$$S_k(x) = \sum_{n=1}^{k} \frac{1}{(2+x^2)^n} = \frac{1}{(2+x^2)} \frac{1 - \frac{1}{(2+x^2)^k}}{1 - \frac{1}{(2+x^2)}} = \frac{1}{1+x^2}\left(1 - \frac{1}{(2+x^2)^k}\right)$$

As $\frac{1}{2+x^2} \le \frac{1}{2}$ for all $x \in \mathbb{R}$, we have $S_k(x) \to \frac{1}{1+x^2}$

The convergence is uniform, as

$$\left| \frac{1}{1+x^2} - S_k(x) \right| = \frac{1}{1+x^2} \frac{1}{(2+x^2)^k} \le \frac{1}{2^k} \to 0 \quad \text{indep of } x.$$

Theorem 58 (Weierstraß M-test)

Let $\sum_{n=1}^{\infty} a_n$ be convergent. If $|f_n(x)| \le a_n$ for all $x \in D$

then $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on D.

Proof $\left|\sum_{n=1}^{\infty} f_n(x) - \sum_{n=1}^{k} f_n(x)\right| = \left|\sum_{n=k+1}^{\infty} f_n(x)\right| \le \sum_{n=k+1}^{\infty} a_n \to 0$ as $n \to \infty$

indep. of x □

Return to example $f_n(x) = \frac{1}{(2+x^2)^n}$ $\quad |f_n(x)| \le \frac{1}{2^n} = a_n$

and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

Theorem 59

(a) Let $\sum_{n=1}^{\infty} f_n$ be uniformly convergent with continuous f_n

Then $f = \sum_{n=1}^{\infty} f_n$ is continuous

(b) Let $\sum_{n=1}^{\infty} f_n$ be convergent and $\sum_{n=1}^{\infty} f_n'$ be uniformly convergent.

Then $f = \sum_{n=1}^{\infty} f_n$ is differentiable and $f' = \sum_{n=1}^{\infty} f_n'$

(c) Let $\sum_{n=1}^{\infty} f_n$ be uniformly convergent with R-integrable f_n on $[a,b]$

Then $f = \sum_{n=1}^{\infty} f_n$ is R-integrable and $\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$

10. Power series

Definition 60: $\sum_{n=0}^{\infty} a_n x^n$ with $a_n \in \mathbb{R}$ is called a *power series*

$r = \sup \left\{ |x| : \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\}$ is its *radius of convergence*

(r may not exist if $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in \mathbb{R}$).

Theorem 61 (a) If $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = c$, then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all $x \in \mathbb{R}$ with $|x| < |c|$

(b) If $\sum_{n=0}^{\infty} a_n x^n$ diverges for $x = c$, then $\sum_{n=0}^{\infty} a_n x^n$ diverges for all $x \in \mathbb{R}$ with $|x| > |c|$

Proof (a) Convergence of $\sum_{n=0}^{\infty} a_n c^n$ implies that $\lim_{n\to\infty} a_n c^n = 0$

Thus $$|a_n x^n| = \underbrace{|a_n c^n|}_{\leq 1 \text{ for } n \geq n_0} \left|\frac{x}{c}\right|^n \leq \left|\frac{x}{c}\right|^n \text{ for } n \geq n_0$$

Therefore $\sum_{n=0}^{\infty} |a_n x^n|$ is majorized by $\sum_{n=0}^{\infty} \left|\frac{x}{c}\right|^n$ which converges absolutely, and thus $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.

(b) if $\sum_{n=0}^{\infty} a_n x^n$ converged for some x with $|x| > |c|$ then by (a) $\sum_{n=0}^{\infty} a_n x^n$ would converge for all $|y| < |x|$; in particular for $y = c$, a contradiction. $\square$