<u>Example</u> $f:[-1,1]\to\mathbb{R}$ $f(x)=\begin{cases}0 & x\in[-1,0]\\ 1 & x\in(0,1]\end{cases}$

$$F(t)=\int_{-1}^{t} f(x)\,dx=\begin{cases}0 & t\in[-1,0]\\ t & t\in(0,1]\end{cases}$$

$F(t)$ is continuous on $[-1,1]$, differentiable on $[-1,0)\cup(0,1]$ but not differentiable at $t=0$.

<u>Corollary</u> Every continuous function $f:[a,b]\to\mathbb{R}$ has an antiderivative

<u>Proof</u> By Theorem 4.7, $F(t)=\int_a^t f(t)\,dt$ is an antiderivative of f □

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<u>Definition 4.8</u> If F is an antiderivative of f, we define

$$\int f(x)\,dx = F(x)+c,$$ the indefinite integral of f

(in contrast to the definite Riemann integral $\int_a^b f(x)\,dx$)

<u>Theorem 4.9</u> If f and g have antiderivatives, then so do $f+g$ and cf for $c\in\mathbb{R}$, and

$$\int (f(x)+g(x))\,dx=\int f(x)\,dx+\int g(x)\,dx \qquad \int cf(x)\,dx = c\int f(x)\,dx$$

<u>Proof</u> (a) $F'=f$ $G'=g$ imply $(F+G)'=F'+G'=f+g$

Thus $\int f(x)+g(x)\,dx = F(x)+G(x)=\int f(x)\,dx+\int g(x)\,dx$ □

(b) cf analogous

Theorem 50 Let $f,g : I \to \mathbb{R}$ be differentiable.

If fg' has an antiderivative, then so does $f'g$

and
$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx \quad \text{holds.}$$

Proof Let $h = fg'$ have an antiderivative H, i.e. $H' = h = fg'$

Then $(fg)' = f'g + fg'$ implies that

$$f'g = (fg)' - fg' = (fg)' - H' = (fg - H)'$$

Thus $f'g$ has $fg - H$ as antiderivative, and

$$\int f'(x)g(x)\,dx = f(x)g(x) - H(x) = f(x)g(x) - \int f(x)g'(x)\,dx \qquad \square$$

Theorem 51 Let $g : I \to \mathbb{R}$ be differentiable and let $f : g(I) \to \mathbb{R}$ have an antiderivative F. Then $(f\circ g)g'$ has an antiderivative $F\circ g$

i.e. $$F(g(x)) = \int f(g(x))g'(x)\,dx$$

Proof $$(F\circ g)'(x) = F'(g(x))\,g'(x) = f(g(x))\,g'(x) \qquad \square$$

Corollary Let $g : [a,b] \to \mathbb{R}$ be continuously differentiable and

let $f : g([a,b]) \to \mathbb{R}$ be continuous. Then
$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Proof f and $(f\circ g)g'$ are both continuous, hence R-integrable

f is continuous, hence has an antiderivative F. Thus

by Theorem 51, $(f\circ g)\,g'$ has antiderivative $F\circ g$

and

$$\int f(g(x))\,g'(x)\,dx = F(g(x))$$

By FTC

$$\int_a^b f(g(x))\,g'(x)\,dx = F(g(b)) - F(g(a))$$

again by FTC

$$= \int_{g(a)}^{g(b)} f(u)\,du$$

$\square$

9. Sequences and Series of Functions

$D \subset \mathbb{R}$ domain, functions generally $f : D \to \mathbb{R}$

Definition 52 (1) A sequence (f_n) of functions converges pointwise to a function f if

$$\forall x \in D \ \forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 : |f_n(x) - f(x)| < \varepsilon$$

(2) A sequence (f_n) of functions converges uniformly to a function f if

$$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 \ \forall x \in D : |f_n(x) - f(x)| < \varepsilon$$

Remark In (1) n_0 depends on x and ε

whereas in (2) n_0 depends on ε, but not on x.

In both cases, we have $f(x) = \lim_{n \to \infty} f_n(x)$. Note that this notation does not indicate whether the convergence is uniform or pointwise.

Uniform convergence implies pointwise convergence, but the converse is not true.

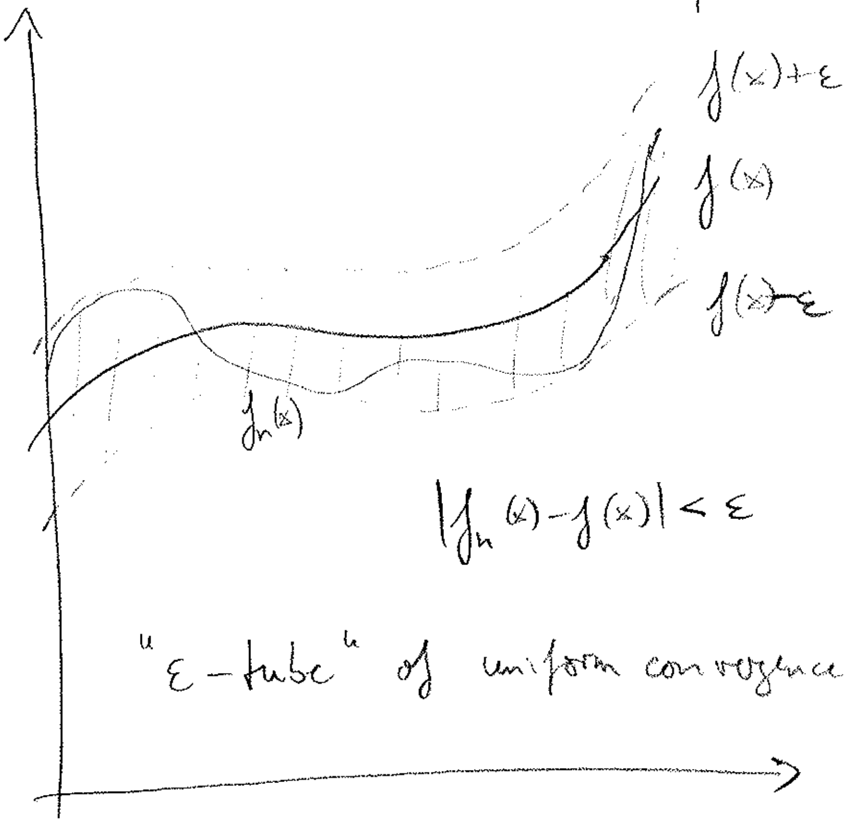
Examples

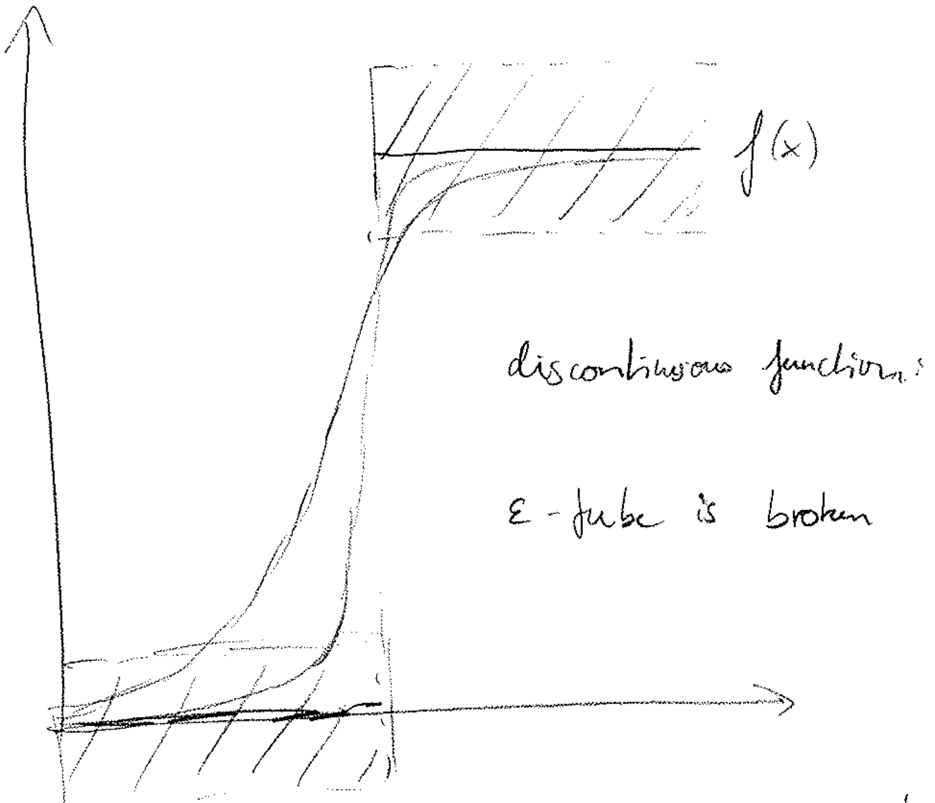
(1) $f_n : [0,1] \to \mathbb{R} \qquad x \mapsto x^n$

(2) $f_n : [0, \frac{1}{2}] \to \mathbb{R} \qquad x \mapsto x^n$

(3) $f_n : [0,2] \to \mathbb{R} \qquad x \mapsto \begin{cases} nx & 0 \leq x \leq \frac{1}{n} \\ 2 - nx & \frac{1}{n} < x \leq \frac{2}{n} \\ 0 & \frac{2}{n} < x \leq 2 \end{cases}$

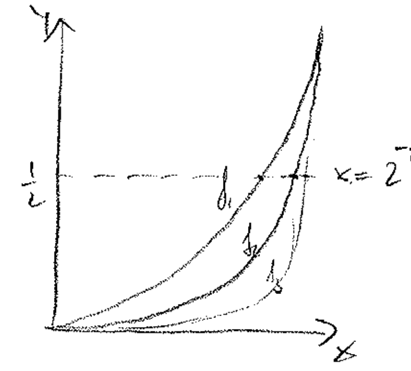
uniform convergence





convergence is non-uniform

(1)



$f_n : [0,1] \to \mathbb{R}$

$f_n(x) = x^n$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

Thus (f_n) converges pointwise to the discontinuous function

$$f : [0,1] \to \mathbb{R} \qquad f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

This convergence is <u>not</u> uniform: need to show

$\exists \varepsilon > 0 \ \forall n_0 \ \exists n \ge n_0 \ \exists x \in [0,1] : |f_n(x) - f(x)| \ge \varepsilon$:

Consider $x = 2^{-\frac{1}{n}}$ Then $|f_n(x) - f(x)| = |(2^{-\frac{1}{n}})^n - 0| = \frac{1}{2}$.

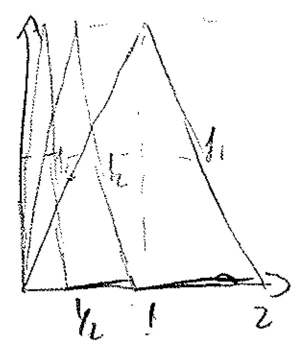
(2) Same function, but on $[0,\frac{1}{2}]$ instead of $[0,1]$

Now (f_n) converges to $f(x) = 0$ on $[0,\frac{1}{2}]$.

The convergence is uniform: given $\varepsilon > 0$, pick $n_0 > -\frac{\log \varepsilon}{\log 2}$ $\quad \left(\left(\frac{1}{2}\right)^{n_0} < \varepsilon \right)$

to get $|f_n(x) - f(x)| = |x^n - 0| \le \left(\frac{1}{2}\right)^n \le \left(\frac{1}{2}\right)^{n_0} < \varepsilon$ for all $n \ge n_0$

(3)



$f_n : [0,2] \to \mathbb{R}$ $\qquad f_n(x) = \begin{cases} nx & 0 \le x \le \frac{1}{n} \\ 2 - nx & \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \frac{2}{n} \le x \le 2 \end{cases}$

$\lim_{n\to\infty} f_n(x) = 0$, as $f_n(x) = 0$ for $n \ge \frac{2}{x}$.
(and $f_n(0) = 0$)

This convergence is <u>not</u> uniform: consider $x = \frac{1}{n}$. The $f_n(x) = 1$.

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Theorem 53 Let $f_n : D \to \mathbb{R}$ converge uniformly to $f : D \to \mathbb{R}$

If f_n are continuous at $a \in D$ then f is continuous at a.

Proof We need to show $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in D : |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

By assumption

(a) $\forall \varepsilon' > 0 \; \exists n_0 \; \forall n \geq n_0 \; \forall x \in D : |f(x) - f_n(x)| < \varepsilon'$

(b) $\forall \varepsilon'' > 0 \; \exists \delta > 0 \; \forall x \in D : |x - a| < \delta \Rightarrow |f_n(x) - f_n(a)| < \varepsilon''$

Now estimate

$$|f(x) - f(a)| \leq \underbrace{|f(x) - f_n(x)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_n(x) - f_n(a)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_n(a) - f(a)|}_{< \frac{\varepsilon}{3}};$$

First, given $\varepsilon > 0$, choose $\varepsilon' = \frac{\varepsilon}{3}$. By (a) there is an n_0 such that for all $n \geq n_0$ and all $x \in D$: $|f(x) - f_n(x)| < \frac{\varepsilon}{3}$ (and of course $|f(a) - f_n(a)| < \frac{\varepsilon}{3}$)

Now fix an $n > n_0$ and choose $\varepsilon'' = \frac{\varepsilon}{3}$. By (b) there is a $\delta > 0$ such that for all $x \in D$: $|x-a| < \delta \Rightarrow |f_n(x) - f_n(a)| < \frac{\varepsilon}{3}$.

Thus, given $\varepsilon > 0$, we have shown that there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ for } |x - a| < \delta \quad \square$$

Remark This implies that <u>under the assumption of uniform convergence</u> of continuous f we can exchange limits as follows:

$$\lim_{x \to a} \underbrace{\lim_{n \to \infty} f_n(x)}_{f(x)} = \lim_{n \to \infty} \underbrace{\lim_{x \to a} f_n(x)}_{f_n(a)}$$