MTH 5105 – Differential and Integral Analysis

(6 Jan 9)

Revision: $D \subset \mathbb{R}$ domain (e.g. interval or all of $\mathbb{R}$)

Definition (a) $f : D \to \mathbb{R}$ is continuous at $a \in D$ if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

(b) $f : D \to \mathbb{R}$ is continuous if f is continuous at all $a \in D$.

(c) $f : D \to \mathbb{R}$. We say $f(x)$ tends to the limit $L \in \mathbb{R}$ as x tends to $a \in D$, $\lim_{x \to a} f(x) = L$, if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D : 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Theorem $f : D \to \mathbb{R}$ is continuous at $a \in D$ if and only if

(a) $\lim_{x \to a} f(x) = L$ exists and

(b) $f(a) = L$ (or, briefly, $\lim_{x \to a} f(x) = f(a)$)

Proof "$\Rightarrow$": Let f be continuous at $a \in D$. Then

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

But this means that $\lim_{x \to a} f(x) = f(a)$

"$\Leftarrow$": Let $\lim_{x \to a} f(x) = f(a)$. Then

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D : 0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

Additionally, for $x = a$, we have $|f(a) - f(a)| = 0$ trivially. $\square$

Theorem If $f: D \to \mathbb{R}$ is continuous at $a \in D$ and $b = f(a) \neq 0$ then $f(x) \neq 0$ nearby, i.e.

$$\exists \delta > 0 \; \forall x \in D : \; |x-a| < \delta \Rightarrow f(x) \neq 0$$

Proof f is continuous at a, and $b = f(a)$, so that

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in D : |x-a| < \delta \Rightarrow |f(x) - b| < \varepsilon$$

Now pick $\varepsilon = |b|$ so that $|f(x) - b| < |b|$.

Therefore $|b| > |f(x) - b| \geq \big| |f(x)| - |b| \big| \geq |b| - |f(x)|$

or, equivalently $|f(x)| > 0$.

Therefore, by choosing ε as we did, we have shown

$$\exists \delta > 0 \; \forall x \in D : |x-a| < \delta \Rightarrow |f(x)| > 0$$

$\square$

Reminder: $|a+b| \leq |a| + |b|$ triangle inequality Δ

Claim: $|a-b| \geq \big| |a| - |b| \big|$

Proof: show both (a) $|a-b| \geq |a| - |b|$ and (b) $|a-b| \geq |b| - |a|$

(a) is equivalent to $|a| \leq |a-b| + |b|$

but $|a| = |(a-b) + b| \leq |a-b| + |b|$ by Δ

(b) is equivalent to $|b| \leq |a-b| + |a|$

but $|b| = |(b-a) + a| \leq |b-a| + |a|$ by Δ

$\square$

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1. Differentiation

$D \subset \mathbb{R}$ domain without isolated points (to allow limits at all points of D)

<u>Definition 1</u>: (a) $f : D \to \mathbb{R}$ is <u>differentiable at $a \in D$</u> if the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

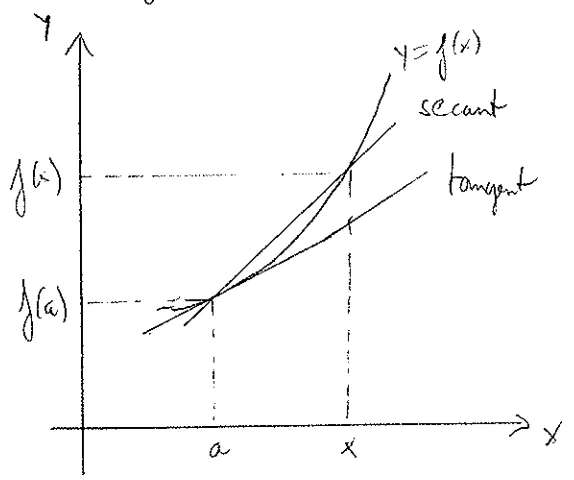
exists. $f'(a)$ is the <u>derivative of f at a</u>

(b) $f : D \to \mathbb{R}$ is <u>differentiable</u> if f is differentiable at all $a \in D$. The function

$f' : D \to \mathbb{R}$ given by $x \mapsto f'(x)$ is the

<u>derivative of f</u>

Geometric interpretation: the difference quotient $\dfrac{f(x) - f(a)}{x - a}$ is the slope of the secant through the points $(a, f(a))$ and $(x, f(x))$ and the limit $f'(a)$ is the slope of the tangent at $(a, f(a))$ of the graph of f



Examples

1) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is differentiable at every $a \in \mathbb{R}$:

$$\frac{f(x) - f(a)}{x - a} = \frac{x^2 - a^2}{x - a} = x + a \quad \text{and}$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} (x + a) = 2a$$

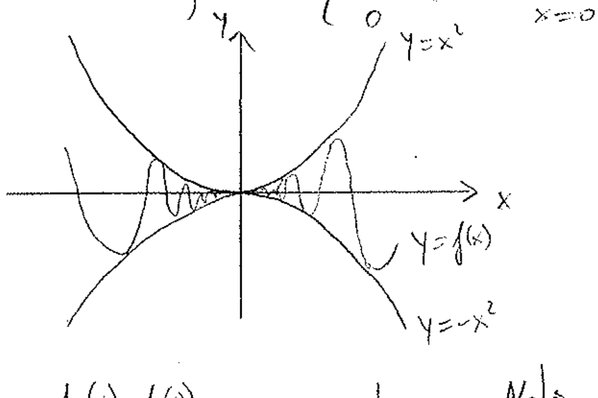
The derivative is $f' : \mathbb{R} \to \mathbb{R}$, $x \to 2x$.

2) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$ is not differentiable at $a = 0$:

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

and $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist.

3) $f : \mathbb{R} \to \mathbb{R}$, $x \to \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at $a = 0$



unclear, as f "wobbles" near zero

$$\frac{f(x) - f(0)}{x - 0} = x \sin \frac{1}{x}$$

Noting that $\left|\sin \frac{1}{x}\right| \leq 1$, we have

$$\left| \lim_{x \to 0} x \sin \frac{1}{x} \right| = \lim_{x \to 0} \left| x \sin \frac{1}{x} \right| \leq \lim_{x \to 0} |x| = 0$$

and therefore $f'(0) = \lim\limits_{x \to 0} x \sin \frac{1}{x} = 0$

<u>Lemma 2</u> $f: D \to \mathbb{R}$ is differentiable at a if and only if there exist $s, t \in \mathbb{R}$ and $r: D \to \mathbb{R}$ such that

(1) $f(x) = s + t(x-a) + r(x)(x-a)$ for all $x \in D$

(2) $\lim_{x \to a} r(x) = 0$

<u>Remark</u> These properties say that $f(x)$ can be approximated by a linear function $y = s + t(x-a)$ for x close to a.

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<u>Proof</u> "$\Rightarrow$" Let f be differentiable at a. We define $r: D \to \mathbb{R}$ by

$$r(x) = \begin{cases} \dfrac{f(x)-f(a)}{x-a} - f'(a) & x \neq a \\ 0 & x = a \end{cases}$$

For $x \neq a$ it follows that

$$f(x) = f(a) + f'(a)(x-a) + r(x)(x-a)$$

For $x = a$, this identity holds as well, as it reduces to $f(a) = f(a)$, Therefore (1) holds with $s = f(a)$ and $t = f'(a)$. To show (2), we compute $\lim_{x \to a} r(x) = f'(a) - f'(a) = 0$

"$\Leftarrow$" Inserting $x = a$ into (1) gives $f(a) = s$, so that (1) gives $f(x) = f(a) + t(x-a) + r(x)(x-a)$, and therefore

$$\frac{f(x)-f(a)}{x-a} = t + r(x).$$ Now (2) implies that the limit

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = t + \lim_{x \to a} r(x) = t$$ exists. $\square$

Remark: If $f(x) = s + t(x-a) + r(x)(x-a)$ with $\lim_{x\to a} r(x) = 0$

then f is differentiable at a with $s = f(a)$ and $t = f'(a)$

The equation of the tangent at a of the graph of f is therefore

$$y = f(a) + f'(a)\,(x-a)$$

Theorem 3 If $f : D \to \mathbb{R}$ is differentiable at a then f is continuous at a.

Proof: By Lemma 2,

$$f(x) = s + t(x-a) + r(x)(x-a)$$

with $\lim_{x\to a} r(x) = 0$. Therefore $\lim_{x\to a} f(x) = s = f(a)$ $\square$

Remark: $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$ is continuous at 0 but not differentiable.

The converse of Theorem 3 is therefore not true.

Theorem 4 Let $f, g : D \to \mathbb{R}$ be differentiable at $a \in D$ and $c \in \mathbb{R}$.

Then $f+g$, cf, fg, f/g (if $g(a) \neq 0$) are differentiable at a. We have

(a) $(f+g)' = f' + g'$

(b) $(cf)' = cf'$

(c) $(fg)' = f'g + fg'$ product rule

(d) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ quotient rule

<u>Proof</u> (a) easy (b) special case of (c) with $g(x) = c$ constant function

(c) $$\frac{f(x)g(x) - f(a)g(a)}{x-a} = \frac{f(x)-f(a)}{x-a}\, g(x) + f(a)\, \frac{g(x)-g(a)}{x-a},$$ and

as f and g are differentiable at a and g is continuous at a by Thm 3,

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a)$$

(d) g continuous at a by Thm 3. $g(a) \neq 0$, therefore $g(x) \neq 0$ nearby, i.e.
$\exists \delta > 0\ \forall x \in D : |x-a| < \delta \Rightarrow g(x) \neq 0$. Therefore $\frac{f(x)}{g(x)}$ is defined near a,
and $$\frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} = \frac{1}{g(x)g(a)}\left(\frac{f(x)-f(a)}{x-a}\, g(a) - f(a)\, \frac{g(x)-g(a)}{x-a}\right)$$

the limit as $x \to a$ exists on the right-hand-side, and therefore

$$(f/g)'(a) = \frac{1}{g^2(a)}\left(f'(a)g(a) - f(a)g'(a)\right)$$ $\square$