MTH5105 Differential and Integral Analysis 2010-2011

Exercises 6

There are two sections. Questions in Section 1 will be used for feedback. Questions in Section 2 are voluntary but highly recommended. Starred questions are more difficult than unstarred ones.

1 Exercises for Feedback

1) (a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with bounded derivative. Show that f is uniformly continuous.
[Hint: Use that if $|f'(x)| \le M$ for all $x \in \mathbb{R}$ then $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}$ (similar to Exercise sheet 2).]
   (b) Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x$ and $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sin(x)$. Prove or disprove:
      (i) f is uniformly continuous.
      (ii) g is uniformly continuous.
      (iii) fg is uniformly continuous.
      (iv) $x \mapsto \begin{cases} g(x)/f(x) & x \neq 0 \\ 1 & x = 0 \end{cases}$ is uniformly continuous.

2 Extra Exercises

2) Let $f : (0, 1) \to \mathbb{R}$ be continuous. Show that
   a) f is uniformly continuous if $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist.
   *b) If f is uniformly continuous then $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist.

3) Let $\alpha \in \mathbb{R}$ and $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^\alpha & x \in \{1/k; k \in \mathbb{N}\} , \\ 0 & \text{else.} \end{cases}$$

For which values of α is f Riemann-integrable? If f is Riemann-integrable, what is the value of $\int_0^1 f(x)\, dx$?

4) Let $f : [a, b] \to \mathbb{R}$ be Riemann-integrable and $c \in \mathbb{R}$.
   (a) Given a partition P of $[a, b]$, show that

$$U(cf, P) - L(cf, P) \le |c|(U(f, P) - L(f, P)) .$$

   (b) Deduce from (a) that cf is integrable and $\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx$. *[This completes the proof of Theorem 7.4.]*

The deadline is 5.00pm (strict) on Monday 14th March. Please hand in your coursework to the orange coursework box on the second floor. Coursework will be returned during the exercise class immediately following the deadline.

Thomas Prellberg, March 2011