MTH5105 Differential and Integral Analysis 2010-2011

Exercises 3

There are two sections. Questions in Section 1 will be used for feedback. Questions in Section 2 are voluntary but highly recommended. Starred questions are more difficult than unstarred ones.

1 Exercises for Feedback

1) The functions sinh and cosh are given by

$$\sinh : \mathbb{R} \to \mathbb{R} , \qquad x \mapsto \frac{1}{2}(\exp(x) - \exp(-x)) ,$$
$$\cosh : \mathbb{R} \to \mathbb{R} , \qquad x \mapsto \frac{1}{2}(\exp(x) + \exp(-x)) .$$

   (a) Prove that sinh and cosh are differentiable and that $\sinh' = \cosh$ and $\cosh' = \sinh$.

   (b) Prove that the function
   $$f(x) = \cosh^2(x) - \sinh^2(x)$$
   is constant by considering $f'(x)$.
   What is the value of the constant?

   (c) Prove that sinh is invertible.

   (d) Prove that $\sinh(\mathbb{R}) = \mathbb{R}$. *Hint: show that* $\sinh(2x) > x$ *for* $x > 0$, *and mimic the proof of the statement that* $\exp(\mathbb{R}) = \mathbb{R}^+$.

   (e) Prove that $\text{arsinh} = \sinh^{-1}$ is differentiable, and that
   $$\text{arsinh}'(x) = \frac{1}{\sqrt{1+x^2}} .$$

2 Extra Exercises

2) (a) Find a bijective, continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f'(0) = 0$ and a continous inverse.

   (b) Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be differentiable and decreasing. Prove or disprove: If $\lim_{x\to 0} f(x) = 0$, then $\lim_{x\to 0} f'(x) = 0$.

3) Using the Intermediate Value Theorem, prove that a continuous function maps intervals to intervals.

4*) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 2$ as $x \to 0$. Does it follow that f is differentiable at 0? If yes, give a rigorous proof; if no, provide a counter-example.

The deadline is 5.00pm (strict) on Monday 7th February. Please hand in your coursework to the orange coursework box on the second floor. Coursework will be returned during the exercise class immediately following the deadline.

Thomas Prellberg, January 2011