PERMUTATION MODELS

SAM TARZI

ABSTRACT. Permutation Group Theory provides a tool for studying symmetry. The ubiquity of the latter concept ensures that the former permeates many branches of natural enquiry. We give a descriptive account of an application of Permutation Groups to set theory - specifically to the construction of so-called Fraenkel-Mostowski (FM) models of set theory, which allow a negation of the Axiom of Choice (AC). This account leans heavily on [21].

1. Brief History

In 1922 Fraenkel endeavoured to prove the independence of AC from the other axioms of set theory. To do so, he used the other axioms that had been adopted by Zermelo (excluding AC), starting from the empty set and countable distinct pairs of objects $(a_n, \bar{a}_n)$, none of which is a set, and which were subsequently called atoms or urelements. Intuitively, atoms were intended to be indistinguishable. The construction leads to a set for which no 'choice set' exists.

For forty years before the method of forcing was discovered, FM models were used for proofs of independence of various axioms of set theory from the other axioms.

This innovation may have seemed potentially problematic, given the philosophical debates regarding what a set is and is not. Yet three years later Pauli discovered his Exclusion Principle, according to which electrons are indistinguishable and the properties of a collection of them is invariant under interchange of any two of them. So no selection principle can exist for sets of electrons; AC is false for electronic sets.

2. Permutations & the Axiom of Choice

In the 1930s, Mostowski continued the study of such models, illustrating the non-implications between various weak versions of AC by careful choice of group and 'support structure', for example creating models in which every set can be linearly ordered but AC fails. First-order set theory is the theory of ϵ, the binary set membership relation.

Date: 1 July 2000, revised 3 March 2003.

If G is a group acting on a set so as to preserve this relation we say that one of the group elements g is an *ϵ-automorphism* if for every set x, $xg = \{yg : y \in x\}$. In Zermelo-Fraenkel (ZF) set theory, the axioms of foundation and extensionality together imply that there can be no non-trivial ϵ-automorphisms of the universe, so it must be the new ingredient, the atoms, that are permuted. This can be proved by induction on rank. To distinguish the true empty set from the atoms, adjoin a unary predicate symbol $U(x)$ to express 'x is an atom'.

To negate AC in FM models, we need to find a symmetrical family of sets such that any choice function for it must be asymmetrical and so excluded. The symmetry argument allows the construction of models where the set of atoms cannot be well-ordered.

To form an FM model, we require a model $\mathcal{M}$ of FM+AC, a set or class of atoms $U(x)$, a permutation group of the atoms G, a filter $\mathcal{F}$ of subgroups of G closed under conjugacy. The group G acts on members $x \in \mathcal{M}$ by respecting the membership relation $xg = \{yg : y \in x\}$. The setwise and pointwise stabilisers are defined respectively as

$$G_{\{x\}} := \{g \in G : xg = x\}$$

and

$$G_{(x)} := \{g \in G : yg = y \text{ for all } y \in x\}.$$

The normal filter of subgroups of G is defined such that for all $H, K \leq G$, and $g \in G$,

$$H \in \mathcal{F} \text{ and } H \leq K \Rightarrow K \in \mathcal{F}$$

$$H, K \in \mathcal{F} \Rightarrow H \cap K \in \mathcal{F}$$

$$H \in \mathcal{F} \Rightarrow gHg^{-1} \in \mathcal{F}$$

The resulting FM model $\mathcal{N}$ consists of members of $\mathcal{M}$ which are hereditarily symmetric with respect to $\mathcal{F}$, i.e.

$$\mathcal{N} := \{x \in \mathcal{M} : x \subseteq \mathcal{N} \text{ and } G_{\{x\}} \in \mathcal{F}\}$$

(definition by transfinite induction.) $\mathcal{N}$ is certainly transitive, meaning that any member of a member of $\mathcal{N}$ is itself in $\mathcal{N}$, and from this extensionality (slightly weakened to say that if two sets are *non-empty* and have the same members then they are equal), and foundation hold. Either this modification or the introduction of the predicate allows other empty sets, other than the 'real empty set'. Note that "X is non-empty" is not the same as "$X \neq \emptyset$"; they are the same only if X is a set. Our ability to index is lost in going from $\mathcal{M}$ to $\mathcal{N}$. That AC fails in $\mathcal{N}$ (for suitable choice of group and filter), is due to the fact that many sets will have been put into $\mathcal{N}$ because they are 'symmetrical enough' (i.e. their stabiliser lies in $\mathcal{F}$), but that any well-ordering for them (or choice function) would have to be asymmetrical, and so must be absent. We cannot fit a square peg into a round hole.

A set is said to be *amorphous* if it is infinite but is not the disjoint union of two infinite sets; for example U as defined above is amorphous in $\mathcal{N}$. Amorphous sets are *Dedekind finite*, that is have no countably infinite subset. The presence of Dedekind finite cardinals means that AC cannot hold. A set is said to be *strictly amorphous* if it has no non-trivial partition. To construct such a set in a model, take U to be infinite and G its permutation group in $\mathcal{M}$. Let the filter generated by the stabilisers of the members of U be

$$\mathcal{F} := \{H \leq G : \exists A \subseteq U \big(A \text{ finite and } H \geq G_{(A)}\big)\}$$

Since $g^{-1}G_{(A)}g = G_{(Ag)}$, $\mathcal{F}$ is closed under conjugacy. This defines the model.

Notions such as Dedekind-finiteness are more than just a curiosity. For example, in addition to carrying a non-trivial structure, a strictly amorphous set has a Dedekind-finite power set. Plotkin [19] shows that in a certain permutation model of a countable $\aleph_0$-categorical structure, the set of all atoms is Dedekind-finite, whilst Hodges [13] uses the Dedekind finite copy of the countable atomless Boolean algebra to simulate finiteness in commutative algebra. Both AC and the *well-ordering theorem* (asserting that every infinite set can be well ordered) are each equivalent to the proposition that every infinite cardinal is an aleph.

There may be ways of recognising amorphous sets when they arise, for example, in the subjectivist Bayesian approach to statistics, where the notion of exchangeability is an important concept. It is not always possible to extend a finite sequence of exchangeable random variables to a countable such sequence. There is a theorem of de Finetti that requires an infinite exchangeable sequence of random variables and is false for finite sequences. So it will always be false for an amorphous set of random variables. See [9] for a discussion and a geometric interpretation of this theorem.

Theorem 2.1. *$|U|$ is Dedekind Finite in $\mathcal{N}$.*

Proof. Assume $|U|$ is not Dedekind Finite to get a contradiction. So there exists a function (in $\mathcal{N}$) from ω into U, f say. By definition of $\mathcal{N}$ we have $G_{\{f\}} \in \mathcal{F}$ and so $G_{(A)} \leq G_{\{f\}}$ for some finite $A \subseteq U$. There is some $u \in U$ such that $u \notin A$ but is in the range of f, that is $f(u) = u$, say. Let $g \in G$ switch u and v for some distinct $v \in U$ with $v \notin A$

and leave everything else fixed. Then $g \in G_{(A)}$ and so $f(u) = v$ which contradicts f a function. □

U will be the strictly amorphous set in $\mathcal{N}$, so first we check that U *does* lie in $\mathcal{N}$. Each member of U lies in $\mathcal{N}$ since it has no members itself, and its stabiliser was explicitly put into the filter. So U is in $\mathcal{N}$ since all its members are, because its stabiliser equals G. It is at the *next* step that many members of $\mathcal{M}$ have been omitted. For if $X \subseteq U$ lies in $\mathcal{N}$ and is infinite, there must be a finite subset A of U such that $G_{(A)} \leq G_{\{X\}}$. Since X is infinite,there is $x \in X - A$, and as $G_{(A)}$ acts transitively on $U - A$, X must contain the whole of $U - A$, so is cofinite. U is amorphous in $\mathcal{N}$ because it is infinite, but not the disjoint union of two infinite sets. The existence of an amorphous set clearly contradicts AC, so AC is false in $\mathcal{N}$. A stronger statement is also true of U, i.e. it is strictly amorphous - in addition any partition into infinitely many pieces, all except finitely many pieces are singletons. The fact that U is amorphous in $\mathcal{N}$ corresponds to the transitivity of $G_{(A)}$ on $U - A$, the fact that it is strictly amorphous corresponds to its primitivity there.

Truss [22] has achieved a partial classification of amorphous sets, (see also [7] [8]), motivated by Azriel Levy's question regarding what is the "smallest" notion of infinite set. Truss concludes that strictly amorphous sets satisfy this notion and that *bounded amorphous* sets, those whose gauge is finite and greater than 1, are a slight generalisation. The four categories are:

1. the bounded amorphous sets
2. the amorphous sets of projective type over a bounded field
3. the amorphous sets of projective type over an unbounded field
4. the unbounded amorphous sets not of projective type

To analyse the properties of models, a 'support structure' is needed. A set x is said to *support* set y if $G_{(x)} \leq G_{\{y\}}$, i.e. any element which fixes every element of x also fixes y. By assigning minimal supports to all members of the model, we can relate the structure of general sets to that of the family of finite subsets of U. The support structure consists of a mechanism for relating properties of arbitrary sets in the model to the behaviour of G and $\mathcal{F}$. The key point which makes the reduction from the class of *all* sets to arbitrary families of non-empty sets work, is that any family all of whose members are supported by one fixed set can be well-ordered in the model. This is because a set can be well-ordered in an FM model *iff* its *pointwise* stabiliser lies in $\mathcal{F}$. Now, by definition, any set in our model has a support which is a finite subset of U. What we want is to be able to make a simultaneous choice of supports for all sets. Since any set has many supports, for example if A supports x and

$B \supseteq A$, then B also supports x, this is not always immediate. If x has a *minimal* support, then things work out well in general, and this is what happens here. We need that $G_{(A\cap B)} = \langle G_{(A)}, G_{(B)}\rangle$. To see this is enough to get minimal supports, let x be any set in $\mathcal{N}$. Then there is a finite $A \subseteq U$ such that $G_{(A)} \leq G_{\{x\}}$. Choose A of least possible cardinality. Then it is contained in any other support B for x, for as $G_{(A\cap B)} = \langle G_{(A)}, G_{(B)}\rangle$, $A \cap B$ is also a support for x, so by minimality of $|A|$, $A \subseteq B$.

Definability and choice are closely related. There is an alternative description of this model, comprising precisely those members of $\mathcal{M}$ which are hereditarily definable over U, with standard sets (i.e. those whose transitive closures contain no atoms) allowed as parameters. Here when we talk about definable *over* U, we mean that members of U may be employed as parameters of the definition. That all members of U, and U itself, are then in this model is immediate. Clearly, it is the same model as was previously defined in terms of permutations. The construction is more direct; for here we explicitly 'put into' the model exactly the sets we want — U, members of U, and unavoidably in view of the axioms of set theory, the finite and cofinite subsets of U. But *no other* subsets of U. However the models are easier to work with in terms of permutations. The great advantage of the FM method is its simplicity. In a ZF framework, we must work much harder to achieve the same effect, that is using definability.

3. Classical Independence Proofs

The *Law of Trichotomy* states that any two infinite cardinals $\mathfrak{a}$ and $\mathfrak{b}$ are comparable, that is, either $\mathfrak{a} < \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} > \mathfrak{b}$. The *Generalised Continuum Hypothesis* (GCH) states that, for every cardinal $\mathfrak{m}$, no cardinal lies strictly between $\mathfrak{m}$ and $2^{\mathfrak{m}}$. Two fascinating results worth mentioning are:

(a) Trichotomy implies AC (Hartogs, 1915)

and

(b) GCH implies AC (Sierpiński, 1947).

More on these can be found in [11] and [14].

There are also a huge number of independence questions between the many weak versions of AC, that is which version implies which other version. For example, Lauchli showed that the AC for families of finite non-empty sets implies the ordering principle (that any set can be linearly ordered). He showed that if there is a group G which preserves a linear ordering on a set Ω, then no member of G has a non-trivial finite cycle. The converse is false by *Cameron's Theorem* [2]:

Theorem 3.1. *Let* G *be a highly set-transitive but not highly transitive infinite permutation group on* Ω. *Then there is a linear or circular order on* Ω *which is preserved or reversed by* G.

We give part of Pincus proof of Lauchli's result based on the idea of universal homogeneous structures. Universal means that all finite structures embed into the countable structure as induced substructures, and homogeneous means that every isomorphism between finite induced substructures extends to an automorphism of the structure.

Theorem 3.2. *There is a permutation group* G *acting on a countably infinite set* Ω *which preserves no linear ordering on* Ω, *and such that no member of* G *has a non-trivial finite cycle.*

Proof. Consider structures $\langle \Omega; f_1, f_2, \dots \rangle$ where f_n is a choice function for n-element subsets of Ω. Suppose $\langle \dots \rangle$ is a universal homogeneous relational structure, i.e. obtained in a generic fashion as a countable union of finite substructures. Strictly speaking these are not first-order structures, but they can be easily replaced by equivalent first-order structures by using for each n an n-ary function f_n which chooses one of its arguments, and is symmetric under interchange of arguments. Let $G = Aut\langle \Omega; f_1, f_2, \dots \rangle$. Then it is immediate that no member of G can have a finite non-trivial cycle, since if A is a finite orbit of $g \in G$ with $|A| \geq 1$, then g cannot preserve $f_{|A|}$ on A. The rest of the proof comprises showing G cannot preserve a linear ordering on Ω, which we omit. $\square$

The universal-homogeneous structure can be constructed by Fraïssé's method using a suitable amalgamation class of structures.

Example 3.3. Let $U \cong (\mathbb{Q}, <)$. This is Mostowski's Ordered Model, in which the automorphism group consists of order-preserving permutations. We can use the model resulting from this structure to establish the stated FM consistencies. Let $\mathcal{M}$ be a model of FMC with U the countable set of atoms. Since our structure has a countable domain, we may suppose that U is indexed by it, and we get a natural action of the automorphism group G of the structure on U. The supports are taken to be finite, i.e. $\mathcal{F}$ is the filter of subgroups of G generated by $\{G_x : x \in U\}$. Let $\mathcal{N}$ be the resulting FM model. It can be shown that the Boolean Prime Ideal Theorem (BPIT) holds but AC does not. The BPIT is a weak version of AC which says that any boolean algebra has a prime ideal. Note that AC does not hold in $\mathcal{N}$, since any subset of U must be a finite union of intervals and points, having finite support A, and G acts transitively on the open intervals defined by A. This is the prototype of an *o-amorphous* set (i.e. amorphous and linearly ordered and the *only* subsets of U are finite unions of intervals with endpoints in $U \cup \{\pm\infty\}$), and we easily deduce that U has no countably infinite subset, and in particular, cannot be well-ordered.

4. The Jech-Sochor Embedding/Transfer Theorem

We say x is *symmetric* if $G_x \in \mathcal{F}$.

Theorem 4.1. *Let $\mathcal{B}$ be a permutation model,* A *its set of atoms,* α *an ordinal in $\mathcal{B}$. There exists a symmetric model $\mathcal{N}$ of ZF and an embedding $x \mapsto \tilde{x}$ of $\mathcal{B}$ in $\mathcal{N}$ such that*

$$\left(\wp^{\alpha}\left(\mathcal{A}\right)\right)^{\mathcal{B}} \text{ is } \in\text{-isomorphic to } \left(\wp^{\alpha}(\tilde{\mathcal{A}})\right)^{\mathcal{N}}$$

When FM models were first studied it was not known how to construct ZF models in which AC was false, so models which have no non-trivial ϵ-automorphisms were used. After the advent of forcing, most results obtained by FM methods could be transferred straight to ZF. The first general result that such a transfer could take place was the Jech-Sochor Transfer Theorem. It applies to statements of a certain special form 'boundable statements'. The basic idea is that 'totally indistinguishable' atoms should be replaced by some 'sufficiently indistinguishable' sets. These may be reals, sets of reals, sets of sets of reals, . . . , depending on the statement to be transferred. For example, because an amorphous set cannot be linearly ordered, it is hopeless to try to transfer the consistency of the existence of an amorphous set by replacing the atoms by reals, since any set of reals can be ordered. The next option is to try to represent them as a set of sets of reals, and this turns out to be good enough, for example if U is strictly amorphous. In the above example, it is actually simpler in some respects for the ZF case than for the FM case, because here there is a generic linear ordering on the structure, and if we take the atoms to be represented as reals, it may be taken as the usual ordering. In this case that is all that is needed.

There are some non-transferable FM consistencies. These seem mainly to be based on the statement 'the power set of any well-ordered set can be well-ordered' which implies AC in ZF, but not in FM. They are not so much non-transferable, rather their naturally transferred version is an appropriate weakening of the original.

5. The Status of AC

We conclude with some remarks about how AC is viewed today. In most branches of mathematics, for example topology and model theory, acceptance of the truth of AC is seen as essential for their future development; hence papers have appeared with titles such as 'Horrors of topology without choice' and 'Continuing horrors of topology without choice'. In model theory, this is due to the equivalence of the BPIT and the Compactness Theorem, the latter being of central importance to first-order logic. After the discovery of the method of forcing, FM models were never in the mainstream of set-theoretic research. Within set theory itself, most researchers prefer to work with models in which AC is true. But since the mid-sixties, there have been large and important schools in Poland and California that study set-theoretic models that contradict certain forms of AC. In particular, the Axiom of Determinacy/Determinateness (AD) asserts that if S is a set of countable sequences of zeros & ones, and if two players alternately choose terms, each zero or one, and thereby form an infinite sequence, then the game is determined, that is there is a strategy which will always ensure that the sequence is in S, or ensure that it is not. AD is a relative of the game-theoretic methods in model theory [12]. These are logical games and not von-Neumann-Nash type games — nor those of Conway. When restricted to certain collections of countable subsets of a set, AD does not contradict AC; most of the collections studied in *Descriptive Set Theory* (DST) are of this sort. Like so many areas in mathematics, DST has its origins around the turn of the century, but has been most intensively studied in the last thirty years. To underline its importance (and hence the importance of $\neg AC$ models) it should be pointed out that during this period, DST has been one of the most intensely studied parts of set theory.

We see no reason to believe in the truth of AC purely because, for example, it ensures that vector spaces have bases. The axiomatisation and definition of a vector space, either as a linear space over a field or in terms of abelian groups do not involve bases. Bases are only introduced later, as a useful generalisation of thinking about resolving Euclidean vectors in 2 and 3 dimensions, and because dimension is an invariant so is useful for classification and characterisation of the space and reducing it to irreducibles. Moreover, when matrices are introduced in school mathematics, addition of square matrices appears sensible; but the curious multiplication rule is only understood in university-level algebra courses, as defining transformations between coordinate systems, i.e. changes of basis vectors. Also, most modules have no dimensions, and even in those that do, that is free modules, different bases may have different cardinalities. This results in a proliferation of different dimensions in ring theory. Another example is that permutation groups may have bases of different cardinality [3]. Thus we are in-bred into thinking of basis elements as fundamental objects, whilst it is perhaps not so clear-cut that this is the case. Presumably, the failure of the 'Exchange Axiom' condition in matroid theory characterises structures wherein dimension is not an invariant. It would be interesting to know if 'dimension' as an invariant is the norm or the exception for mathematical structures in general, but this may be sensitive to the exact definition of dimension. Model theory presents a particularly sophisticated formalism for studying concepts such as bases and orthogonality. So we are led to conclude that the concept of space comes before that of a basis, just as the egg comes before the chicken — one is a primary concept and the other secondary, or derived. Therefore it is not so implausible that there exists a model of set theory in which a vector space without a basis can be built [14, p. 96].

On the other hand, there seems no reason to banish AC because it gives us conceptual difficulties. For example the Banach-Tarski paradox arises only if we assume AC — however that still leaves the physical impossibility of constructing sets of measure zero as an obstacle to forming a planet-sized object from a pea-sized object by successive doubling of size. Furthermore, AC is inconsistent with Quine's set theory, called New Foundations (***NF***), but becomes consistent in R. B. Jensen's slight modification ***NFU*** of ***NF***, obtained by weakening extensionality so as to admit urelements. Evidently, care must be taken to deal with choice principles in theories such as ***NF***, which do not abide by the "limitation of size" principle. If further evidence were needed that $\neg AC$ models cannot be dismissed as is so often done by the working mathematician, consider the quantum statistics of identical particles, mentioned in section 1 of this essay, (see [20]), noting that electrons are responsible for the whole of chemistry and therefore life. For a recent application of urelements to model real-life phenomena, see [10], where the authors assert the Indistinguishability Postulate, that permutations of indistinguishable elementary particles cannot be regarded as observable. They work with quasi-sets, whose theory is essentially equivalent to ZFU, and prove the Unobservability of Permutations Theorem:- Let x be a qset such that x is distinct from z, an m-atom, such that $z \in x$. If $w \equiv z$ and $z \notin x$, then $\exists w'$ such that

$$(x - z') \bigcup w' \equiv x.$$

They have two sorts of atoms, M-atoms for which the concept of identity is allowed, and m-atoms for which the weaker indistinguishability relation $\equiv$ is defined. Note the similarity between the given condition and the Exchange Lemma of combinatorial geometry.

Connes [6] points out how different the countable and uncountable versions of AC are. The uncountable version is used to construct non-standard versions of $\mathbb{Z}$ or $\mathbb{R}$. Robinson-Penrose tilings use two triangles of different shapes connected to the golden ratio, to tile the plane in an infinite number of non-congruent ways, that is in ways which are not isometric to each other. Only with the uncountable AC can we designate within each equivalence class of isometric tilings a unique tiling; therefore it is never possible in practice. The isometries are ergodic within the set of tilings. Two infinite tilings are different if there is no isometry mapping one to the other. Again because of the uncountable AC we cannot distinguish two Penrose tilings through a finite portion because any given universe will have an identical finite portion.

One possible bridging of the dichotomy that exists between proponents and opponents of AC is offered by Lavine [15] who suggests that the axiom of choice is evident for combinatorial collections and not, in general, evident for logical ones, and that the disagreements over the acceptance of AC have in fact been dissent over whether or not mathematical collections are combinatorial or logical ones. The opposing sides are seen to have different conceptions of set; for example, Zermelo and Hadamard were pro-AC, working as they did with the combinatorial concept, whilst Baire, Borel and Lebesgue belonged to the other side. For more on the history and applications of AC and permutation models, we refer the reader to [1] and [18]. One leading set-theorist, D.A. Martin writes [16] that "much of the traditional concern about the axiom of choice is probably based on a confusion between sets and definable properties. In many cases it appears unlikely that one can *define* a choice function for a particular collection of sets. But this is entirely unrelated to the question of whether a choice function *exists*. Once this kind of confusion is avoided, the axiom of choice appears as one of the least problematic of the set theoretic axioms".

The analogy that is usually made [5] is with non-Euclidean geometry. Different geometries are studied, some where the parallel postulate is true and others where it is false, and people do not concern themselves with the postulates' absolute truth or falsehood. The situation regarding AC is not dissimilar to that in intuitionistic or constructive mathematics, where forbidding the use of the *law of excluded middle*, $P \vee \neg P$ for proposition P, precludes so many mathematical proofs, that its adoption as mainstream mathematical practice is regarded by most researchers as too high a price to pay. Choosing to work in $\neg AC$ models, yields a similarly stringent barrier to the development of many parts of infinite mathematics. Illustrations of the restrictiveness of working in $\neg AC$ models abound. Consequently, results so obtained are often described as 'pathological', a term we prefer to reserve for medical diseases. However at times the opposite conclusion can be drawn. For example in ZFC, the factor group $\mathrm{Sym}(U)/\mathrm{FSym}(U)$ has cardinality $2^{|U|}$, and its normal subgroups form a chain. Indeed, the proper normal subgroups of $\mathrm{Sym}(U)$ are as follows:

- the trivial group;
- the alternating group;
- the bounded symmetric group $\mathrm{BSym}_\alpha(U)$ consisting of all permutations moving fewer than α points, for each infinite cardinal $\alpha \leq |U|$;
- $\mathrm{Sym}(U)$.

However in the absence of AC and as expounded in [4], the group $\mathrm{Sym}(U)/\mathrm{FSym}(U)$ can be isomorphic to any finite group. We are left to conclude that a room containing all of the known correct mathematical results derived by man thus far, would have to include $\neg AC$ results. Until and unless a universally accepted family of set-theoretic axioms is found (and in view of Gödel's Incompleteness Theorems this may be doomed even in principle), it will be a matter of individual taste, that is the specific details of the problem being attacked, whether to work in models of set theory in which AC is true or false. Given the independence of AC from ZF, such an alternative is always there. FM models provide a way of choosing the truth or otherwise of AC without being very far from ZF, the most widely accepted theory.

CAVEAT : (a) The FM Permutation models used for proving independence of choice are unrelated to the Rieger-Bernays Permutation Models, used for proving the independence of the axiom of foundation. (b) Categorical logic aims to unify semantics (truth) and syntax (provability). In the category of Boolean-valued (BV) sets, the BV models arise as permutation models. However, in this context, permutation models are BV functors which are first-order, and more 'basic' than is required for consideration of the normal set-theoretic universe $\mathcal{V}$ [17].

Acknowledgements I would like to thank Professor Peter Cameron for guidance, and Professor John Truss and Dr. Mirna Dzamonja for informative discussions on set theory.

If our question is directed simply to a yes or no, we are well advised to ... consider what we would gain as the answer is in the affirmative or in the negative. Should we then find that in both cases the outcome is sheer nonsense, there will be good reason ... to determine whether the question does not itself rest on a groundless presupposition ...

I. Kant, *Critique of Pure Reason*

References

[1] N. Brunner, *75 years of independence proofs by Fraenkel-Mostowski Permutation Models*, Math. Japonica, *43*, 1, 177-199, 1996.
[2] P. J. Cameron, *Permutation Groups*, Cambridge University Press, Cambridge, 1999.
[3] P. J. Cameron, *Bases in permutation groups*, in *Automorphisms of First-Order Structures*, edited by R. Kaye and D. Macpherson, Clarendon Press, Oxford, 1994.
[4] Peter J. Cameron and Sam Tarzi, *Group actions on amorphous sets*, submitted
[5] P. J. Cohen and R. Hersh, *Non-Cantorian Set Theory*, Scientific American, December 1967.
[6] A. Connes, A. Lichnerowicz, and M. P. Schützenberger, *Triangles of Thought*, AMS, 2001.
[7] P. Creed and J.K. Truss, *On o-Amorphous Set*, Annals of Pure and Applied Logic 101 (2000) 185-226.
[8] P. Creed and J.K. Truss, *On Quasi-Amorphous Sets*, Archive for Mathematical Logic 40, 581-596, 2001.
[9] P. Diaconis, *Finite forms of de Finetti's Theorem on exchangeability*, Synthese, *36*, 271-281, 1977.
[10] S. French and D. Krause, *The logic of quanta*, in *Conceptual foundations of quantum field theory*, edited by T. Y. Cao, Cambridge University Press, Cambridge, 1999.
[11] L. Gillman, *Two classical surprises concerning the axiom of choice and the continuum hypothesis*, Am. Math. Month, *109*, 544-552, 2002.
[12] J. Hintikka, *On the development of the model-theoretical tradition in logical theory*, Synthese, *77*, 1-16, 1988.
[13] W. Hodges, *Six impossible rings*, J. Algebra, *31*, 218-244, 1974.
[14] T. J. Jech, *The Axiom of Choice*, North-Holland Publishing Company, Amsterdam, 1973.
[15] S. Lavine, *Understanding the infinite*, Harvard University Press, Cambridge, Mass., and London, 1994.
[16] P. Maddy, *Realism in Mathematics*, Clarendon Press, Oxford, 1990.
[17] I. Moerdijk, private communication.
[18] G. H. Moore, *Zermelo's axiom of choice*, Springer, Berlin, 1982.
[19] J. M. Plotkin, *Generic Embeddings*, J. Symb. Logic, *34*, 388-394, 1969.
[20] G. Temple, *100 Years of Mathematics*, Duckworth, 1981.
[21] J. Truss, *Permutations and the axiom of choice*, in *Automorphisms of First-Order Structures*, edited by R. Kaye and D. Macpherson, Clarendon Press, Oxford, 1994.
[22] J.K. Truss, *The Structure of Amorphous Sets*, Annals of Pure and Applied Logic 73 (1995) 191-233.

School of Mathematical Sciences
Queen Mary
University of London
Mile End Road
London E1 4NS
UK