Uniqueness Theorems for Waves from Infinity

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Section 1

Introduction
Wave Equations

Wave equations:

\[ \Box \phi = \ldots, \quad \Box := -\partial^2_t + \Delta_x, \quad \phi : \mathbb{R}_t \times \mathbb{R}_x^n \to \mathbb{R}. \]

- Prototypical examples of hyperbolic PDE.
- Models many physical phenomena (vibrating strings, electromagnetism, gravitation).

Cauchy problem:

\[ \Box \phi = \ldots, \quad \phi|_{t=0} = \phi_0, \quad \partial_t \phi|_{t=0} = \phi_1. \]

- **Well-posedness**: \( \phi_0 \) and \( \phi_1 \) “nice enough” \( \Rightarrow \exists \) unique solution \( \phi \) “depending continuously” on \( \phi_0 \) and \( \phi_1 \).
Introduction

The General Problem

Uniqueness of Waves

Suppose \( \{ t = 0 \} \) replaced by another hypersurface \( \Sigma \):
- Cauchy problem may be ill-posed.
- But, can one still show uniqueness of solutions?

**Main interest:** \( \Sigma \) is “portion of infinity”.

**Question**

*When is a wave determined by radiation data at infinity?*

- *Linear and nonlinear waves?*
- *Waves on curved backgrounds?*

*Applications of results and techniques to other problems?*
Unique Continuation

Formal framework: as **unique continuation** problem.
- Classical, well-studied problem in PDEs.

**Problem (Unique Continuation (UC))**

*Consider second-order linear hyperbolic PDO on* $\Omega \subseteq \mathbb{R}^{n+1}$:

$$
\mathcal{L} := \Box g + a^\alpha \nabla_\alpha + V, \quad \Box g := g^{\alpha \beta} \nabla^2_{\alpha \beta}.
$$

- $\phi$: *solution of* $\mathcal{L} \phi = 0$ *on* $\Omega$.
- $\Sigma$: *hypersurface in* $\Omega$.

*If* $\phi, d\phi = 0$ *on* $\Sigma$, *then must* $\phi$ *vanish on one side of* $\Sigma$?*
Local and Global Uniqueness

Various ways to view UC problem as “local” or “global”.

(1) Domain:
- Local (DL): UC from \( \Sigma \) to some neighborhood.
- Global (DG): UC from \( \Sigma \) to all of one side of \( \Sigma \).

(2) Hypersurface:
- Local (HL): UC from neighborhood of any \( p \in \Sigma \).
- Global (HG): UC from all of \( \Sigma \).
The Analytic Theory

Cauchy-Kovalevskaya: $\mathcal{L}$ analytic and $\Sigma$ noncharacteristic
$\Rightarrow \exists$ unique power series solutions (HL, DL).
  - Does not exclude other non-analytic solutions.

Holmgren’s theorem: Same setting
$\Rightarrow$ solution unique in class of distributions (HL, DL).
  - Cannot deal with non-analytic equations.
  - Cannot deal with nonlinear equations (Métivier).
The Non-Analytic Theory

The non-analytic setting is more complicated.
- Theory often described using microlocal analysis.
- Here, we describe from geometric viewpoint.

(Hörmander, Lerner–Robbiano) Main criterion is \textit{pseudoconvexity}:
- $\Sigma := \{ f = 0 \}$ is \textit{pseudoconvex}
  $\Rightarrow$ UC for $\mathcal{L}$ from $\Sigma$ to $\{ f > 0 \}$ (HL, DL).
- Main tool of proof: \textbf{Carleman estimates}.
- Carleman estimate $+ \text{standard argument} \Rightarrow \text{UC}$. 
Remarks on Optimality

Pseudoconvexity is the “correct” notion:

- (Alinhac) $\Sigma$ not pseudoconvex $\Rightarrow \exists a^\alpha, V$ s.t. UC does not hold (HL) across $\Sigma$ for this $\mathcal{L}$.
- Note: Cannot freely choose $a^\alpha$ and $V$, and must complexify.

Remark

Want results for time-dependent wave equations:

- For time-independent equations, there are notable improvements (Tataru, Robbiano–Zuily, Hörmander).
- More generally, for PDE with some analytic components.
Pseudoconvexity

Definition (Lerner–Robbiano)

\[ \Sigma := \{ f = 0 \} \text{ is pseudoconvex (wrt. } \Box_g \text{ and } \text{sgn } f) \text{ iff } \nabla^2 f(X, X) < 0 \text{ on } \Sigma, \text{ whenever } g(X, X) = Xf = 0, \]

- \(-f\) convex in tangent null directions.
- Any null geodesic (bicharacteristic) hitting \( \Sigma \) tangentially lies in \( \{ f < 0 \} \) nearby.

Pseudoconvexity is a conformally invariant property.
Carleman Estimates

UC results often proved using **Carleman estimates**:

- Weighted $L^2$-estimates with extra positive parameter(s).

\[ \| e^{-F_\lambda \Box} \phi \|_{L^2}^2 \gtrsim \lambda \| e^{-F_\lambda \nabla} \phi \|_{L^2}^2 + \lambda \| e^{-F_\lambda} \phi \|_{L^2}^2. \]

- $L^2$-norms over some spacetime domain.
- $\lambda \gg 1$: constant.
- $F_\lambda = F_\lambda(f)$: some reparametrization of $f$.

Carleman estimates have applications in many other areas:

- Inverse problems, control theory.
Example and Application

Example

**Timelike cylinder:** $\Sigma := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \mid |x| = 1\} \subseteq \mathbb{R}^{n+1}$.

- $\Sigma$ is inward pseudoconvex
- $\Rightarrow$ Carleman estimate $\Rightarrow$ UC from $\Sigma$ into interior (HL, DL).
- Vanishing on all of $\Sigma$ $\Rightarrow$ vanishing in full interior (HG, DG).

General relativity: Rigidity of black hole spacetimes:

- (Alexakis–Ionescu–Klainerman) An asymptotically flat, vacuum, stationary black hole spacetime (plus other conditions) must be Kerr.
- Previous results (Hawking + Carter–Robinson) assumed analytic spacetime.
Zero Pseudoconvexity

Definition

$\Sigma$ is zero pseudoconvex $\iff \Sigma$ is ruled by null geodesics (bicharacteristics).

- Also conformally invariant property.
- “Infinity” will tend to be zero pseudoconvex.

**Bad news:** Zero pseudoconvex $\Rightarrow$ (HL), only (HG), or no UC.

Example

**Timelike hyperplane:** $\Sigma := \{ x_n = 0 \} \subseteq \mathbb{R}^{n+1}$

- (Alinhac–Baouendi) No (HL) UC.
- (Kenig–Ruiz–Sogge) (HG, DG) UC, assuming some global regularity.
Complications of Degeneracy

Carleman estimates often become more complicated.

- Degenerating pseudoconvexity leads to decaying weights.
- Sometimes, “delicate balancing of weights” required.

Thus, UC results required more restrictions on PDE:

- Decay needed for $a^\alpha$, $V$.
- Sometimes $a^\alpha$ must be zero.
Section 2

Asymptotically Flat Spacetimes
Minkowski Spacetime

Begin with simplest case, **Minkowski spacetime:**

\[
\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}_x^n.
\]

(Lorentzian) geometric structure: **Minkowski metric,**

\[
g_M := -dt^2 + (dx^1)^2 + \cdots + (dx^n)^2.
\]

- Setting of special relativity.
- Wave operator intrinsic to this geometry:

\[
\Box = g_M^{\alpha\beta} \nabla^2_{\alpha\beta} = -\partial_t^2 + \Delta_x.
\]
Q: What exactly do we mean by infinity?

- Explicitly realised via conformal transformation.

**Minkowski: Penrose compactification.**

- Compress “distances” via conformal transformation:
  \[ \tilde{g}_M = \Omega^2 g_M, \quad \Omega = \left(1 + |t - r|^2\right)^{-\frac{1}{2}} \left(1 + |t + r|^2\right)^{-\frac{1}{2}}. \]

- \((\mathbb{R}^{n+1}, \tilde{g}_M)\) imbeds into *Einstein cylinder*, \(\mathbb{R} \times \mathbb{S}^n\).

- Boundary of \(\mathbb{R}^{n+1}\) interpreted as infinity.
**Asymptotic Flatness**

Minkowski infinity partitioned into timelike ($i^\pm$), spacelike ($i^0$), and null ($j^\pm$) infinities.

- Describes where geodesics terminate.

More generally, asymptotically flat (AF) spacetimes:

- Roughly, this describes when one “has an analogous model of infinity.”
- Captured by metric decay toward $g_M$. 

Compactified Minkowski spacetime.
Radiation Field

**Radiation field**: Data for wave $\phi$ at infinity.

- Physical intuition: *non-radiating waves must be trivial.*

Many such (HG, DG) results come from scattering theory:

- (Friendlander) $\square \phi = 0$ on $\mathbb{R}^{n+1}$:
  - Isometry between initial data and radiation data on (all of) $J^+$.
- Various generalizations: e.g., AF static spacetimes $M = \mathbb{R} \times X$.
- Q. Other (in particular, non-stationary) backgrounds?
Questions of Localisation

Question

*Can we localise (domain and hypersurface) UC results in any way?*

**Obstruction 1:** In $\mathbb{R}^{3+1}$, the following solve $\Box \phi = 0$:

$$\phi_k := \nabla^k x r^{-1} = O(r^{-1-k})$$

- Thus, need at least infinite-order vanishing for DL UC.

**Obstruction 2:** $J^\pm$ is null (characteristic) in the compactified picture.
- Conformal invariance $\Rightarrow J^\pm$ is zero pseudoconvex.
Recall **conformal inversion** on $\mathbb{R}^{n+1}$:

$$
\Psi(t, x) = \frac{c}{t^2 - |x|^2} \cdot (t, x).
$$

- $\Psi$ is a conformal isometry:
  $$
  \Psi^* g_M = f^{-2} \cdot g_M, \quad f := \frac{1}{4} (r^2 - t^2).
  $$
- Identifies “half of $J^\pm$" with $\mathcal{N}_0$.

Thus, inward UC from $\frac{1}{2} J^\pm$ $\iff$ outward UC from $\mathcal{N}_0$. 
Hyperbolic Strong UC

**Strong UC**: UC from a point.
- Elliptic PDE (Aronszajn, Carleman): \( \infty \)-order vanishing at \( r^2 = 0 \Rightarrow \) vanishing on \( r^2 \ll 1 \).

Hyperbolic analogue: replace \( r^2 \) in \( \mathbb{R}^n \) by
\[
(x^1)^2 + \cdots + (x^n)^2 - (x^0)^2 = r^2 - t^2 = 4f.
\]
- **Q**: Vanishing at \( f = 0 \Rightarrow \) vanishing for \( 0 < f \ll 1 \)? (UC from \( \mathcal{N}_0 \) to exterior)
- **A**: Yes (can derive Carleman estimates analogous to elliptic case).
A Preliminary Result

**Proposition (Ionescu–Klainerman)**

Assume $\phi$ satisfies $\Box \phi + V \phi = 0$.

Then, $\phi$ vanishing to $\infty$-order on $\mathcal{N}_0 \Rightarrow (DL, HG)$ UC to exterior.

**Corollary**

Assume $\phi$ satisfies $\Box \phi + V \phi = 0$.

- $V$ satisfies certain decay assumptions near infinity.

$\phi$ vanishing to $\infty$-order on $\frac{1}{2} J^\pm \Rightarrow (DL, HG)$ UC to interior.
Generalized Hyperbolic SUC

**Note:** Above results cannot handle first-order terms $a^\alpha \nabla_\alpha$.
- Level sets of $f$ are zero pseudoconvex.

**Theorem:** (Alexakis–Schlue–S.) Using geometric techniques:
- *Carleman estimates generalized to “geometrically warped” cones (replace $g_M$ by other $g$), if level sets of $f$ are pseudoconvex.*

Moreover, full pseudoconvexity $\Rightarrow$ UC (DL, HG) for full $\mathcal{L}$ (with $a^\alpha$).

This gives general answer to hyperbolic SUC.
A Warped Inversion

**Idea:** Consider “a bit more than half of $J^\pm$”.

- Hyperboloids terminating at its boundary are (inward) pseudoconvex.

**Problem:** No natural conformal inversion $\Psi$ adapted to $(\frac{1}{2} + \varepsilon)J^\pm$ and “red” hyperboloids.

- Can still use conformal factor: define “warped” conformal inversion $\bar{g}_M := f_{\varepsilon}^{-2} g_M$.

Reduces to UC from “warped” cone.

- Conformal invariance $\Rightarrow$ level sets of $f$ are now pseudoconvex.
Results, Zero Mass

**Theorem (Alexakis–Schlue–S.; 2013)**

Assume $\Box \phi + a^\alpha D_\alpha \phi + V \phi = 0$ near $J^\pm$ in $\mathbb{R}^{n+1}$.

- $a^\alpha$, $V$ decays sufficiently toward $J^\pm$.

If $\phi$, $\nabla \phi$ vanish to infinite order on $(\frac{1}{2} + \varepsilon)J^\pm$, then $\phi$ vanishes in the interior near $(\frac{1}{2} + \varepsilon)J^\pm$.

Carleman estimate applies to large family of geometries $\Rightarrow$ need not consider just $g_M$.

**Theorem (Alexakis–Schlue–S.; 2013)**

The above also holds for (zero-mass) perturbations of Minkowski spacetime $(\mathbb{R}^{n+1}, g := g_M + \delta)$. 

UC (HG, DL) from $(\frac{1}{2} + \varepsilon)J^\pm$ to shaded region.
Asymptotically Flat Spacetimes

The Main Results

Results, Positive Mass

Improvements for AF spacetimes with positive mass:

- Leading-order pseudoconvexity comes from positive mass, not position of hyperboloids.
- Positive mass “bends null geodesics” toward $\iota^0$.

Theorem (Alexakis–Schlue–S.; 2013)

The previous results hold for a large class of $\pm$-mass spacetimes, except we need only assume $\infty$-order vanishing on an arbitrarily small portion of $I^\pm$ near $\iota^0$ (HL, DL). Examples include Schwarzschild, (all) Kerr, and many dynamical spacetimes.
An Application

One main motivation is a rigidity problem in general relativity.

**Theorem (Alexakis–Schlue, 2015)**

*Roughly: Time-periodic, asymptotically flat, positive-mass, vacuum spacetimes must be stationary (at least near infinity).*

Analogous past results (Bičák–Scholtz–Tod, Papapetrou) required spacetime to be analytic.
Section 3

Global Results on $\mathbb{R}^{n+1}$
Removing Infinite-Order Vanishing

**Q:** Can we remove $\infty$-order vanishing assumption?

- Counterexamples $(\nabla_x^k r^{-1}, \ n = 3)$ blow up at $r = 0$.
- Positive result must see $r = 0 \Rightarrow$ need (DG) UC result.

Return to basic case: UC from $\frac{1}{2}J^\pm$ in $\mathbb{R}^{n+1}$.

Must overcome two (related) difficulties:

1. Push Carleman estimate to $r = 0$.
   (Careful choosing and matching of weights.)
2. Stamp out the source of $\infty$-order vanishing requirement.
   (We focus on this point.)
Carleman to Uniqueness, I

For $a > 0$, we have, roughly, the Carleman estimate:

$$
\int_{\{f_0 < f < f_1\}} f^{2a} u^2 \lesssim a^{-1} \int_{\{f_0 < f < f_1\}} f^{2a} |\Box u|^2 - \int_{\{f = f_0\}} f^{2a} G(u, du) + \int_{\{f = f_1\}} f^{2a} G(u, du),
$$

- Neglected various decaying weights.

Suppose, for simplicity, that $\Box \phi = 0$.

- Apply estimate to $u = \chi \cdot \phi$, where:
  - $\chi = 0$ near $f = f_0$, and $\chi = 1$ on $f > f_\ast$. 

Carleman to Uniqueness, II

\[ \int_{\{f=f_0\}} f \text{ vanishes by } \chi. \]
\[ \int_{\{f=f_1\}} f \text{ vanishes when } f_1 \nearrow \infty \text{ (if } \phi \text{ vanishes to } > 2a \text{-order}). \]

\[ \int_{\{f_0 < f < \infty\}} f^{2a} \chi^2 \phi^2 \lesssim a^{-1} \int_{\{f_0 < f < \infty\}} f^{2a} |\Box(\chi \phi)|^2. \]

Reduce domain on LHS, use \( \Box \phi = 0 \) on RHS:

\[ \int_{\{f_* < f < \infty\}} f^{2a} \phi^2 \lesssim a^{-1} \int_{\{f_0 < f < f_*\}} f^{2a} (|\nabla \chi \cdot \nabla \phi|^2 + |\Box \chi \cdot \phi|^2), \]

Since \( f > f_* \) on LHS, and \( f < f_* \) on RHS,

\[ \int_{\{f_* < f < \infty\}} \phi^2 \lesssim a^{-1} \int_{\{f_0 < f < f_*\}} (|\nabla \chi \cdot \nabla \phi|^2 + |\Box \chi \cdot \phi|^2). \]
Carleman to Uniqueness, III

\[ a \uparrow \infty \Rightarrow \phi \equiv 0 \text{ when } f > f^*. \]

- Need $\infty$-order vanishing for $\phi$ (for $\int_{\{f=f_1\}}$ to vanish).
- Limit $a \uparrow \infty$ is necessary, due to cutoff $\chi$.

**Key:** Avoid using cutoff.

- Weight $f^{2a}$ vanishes on null cone $f = 0$.
- If Carleman estimate can be pushed to $f = 0$, then $\int_{\{f=f_0\}}$ vanishes naturally, without using $\chi$.

Thus, would need $\varepsilon$-order vanishing, for some $\varepsilon > 0$. 
A Linear Result

Theorem (Alexakis–S.; 2014)

Assume $\Box \phi + V \phi = 0$ in the full exterior $\mathcal{D}$ of the null cone about the origin in $\mathbb{R}^{n+1}$.

- $V$ decays sufficiently near infinity.
- $\|V\|_{L^\infty(\mathcal{D})} \leq C(\varepsilon)$.

If radiation field on $\frac{1}{2} J^\pm$ vanishes to $\varepsilon$-order, then $\phi$ vanishes on all of $\mathcal{D}$.

Remark

$L^\infty$-assumption for $V$ is necessary.
**Nonlinear Carleman Estimates**

**Q:** Can we, in some cases, remove $L^\infty$-assumption for $V$?

**Main idea:** Derive Carleman estimate for $\Box_V := \Box + V$, rather than $\Box$.  
- Bound $\Box_V \phi$, rather than $\Box \phi$, from below by $\phi$ and $\nabla \phi$.

More generally, **nonlinear** Carleman estimates for

$$
\Box_{V,p} \phi := \Box \phi + V|\phi|^{p-1} \phi, \quad p \geq 1.
$$

Removes $L^\infty$-assumption for $V$ satisfying a monotonicity condition.
- When $V \equiv c$: depends on $p$ and sign of $V$.  


A Nonlinear Result

Theorem (Alexakis–S.; 2014)

Assume \( \phi \) satisfies on \( \mathcal{D} \) one of

\[
\Box \phi + |\phi|^{p-1} \phi = 0, \quad 1 \leq p < 1 + \frac{4}{n-1},
\]

\[
\Box \phi - |\phi|^{p-1} \phi = 0, \quad p \geq 1 + \frac{4}{n-1}.
\]

If radiation field on \( \frac{1}{2} J^\pm \) vanishes to \( \varepsilon \)-order, then \( \phi \) vanishes on all of \( \mathcal{D} \).

- (May need slightly more vanishing in the “+” case.)

The result generalizes to certain operators of the form \( \Box \phi + V|\phi|^{p-1} \phi \). provided \( V \) satisfies certain monotonicity conditions.
Carleman Estimates Revisited

“Global” Carleman estimates can be localized to other regions within $D$.

- Useful property: estimates do not see $f = 0$.

Consider focusing, subconformal NLW,

$$\square \phi + |\phi|^{p-1} \phi = 0, \quad 1 < p < 1 + \frac{4}{n-1}.$$

Nonlinear Carleman estimate yields

$$\int_{\text{shaded}} w_1 |\phi|^{p+1} \leq \int_{\text{shaded}} w_2 |\square \phi + |\phi|^{p-1} \phi|^2 + \int_{\text{red}} G(\phi, d\phi)$$

$$= \int_{\text{red}} G(\phi, d\phi).$$

$C^\sigma := \{(-t, x) \mid 0 < r < \sigma t\}$, $0 < \sigma < 1$, is a past time cone.
Another Estimate

Consider the sum of two such Carleman estimates:
- Centered at $Q$ and $Q'$.

This yields the following estimate:
\[
\int_{\text{shaded}} |\phi|^{p+1} \lesssim \int_{\text{red}} G(\phi, d\phi).
\]
- Controls $\phi$ in interior of $C$ entirely by $\phi$ on boundary of $C$.

We consider one unexpected application.
Subconformal Blowup Rate

(Merle–Zaag, 2005) Suppose a solution $\phi$ of

$$\Box \phi + |\phi|^{p-1}\phi = 0, \quad 1 < p < 1 + \frac{4}{n-1}$$

blows up at $(0, 0)$:

- If $(0, 0)$ is noncharacteristic, then $\exists \, \varepsilon > 0$ such that $\forall \, 0 < t \ll 1$,
  $$\varepsilon \leq t^{\frac{2}{p-1} - \frac{n}{2}} \|\phi(-t)\|_{L^2(B(0,t))} + t^{\frac{2}{p-1} - \frac{n}{2} + 1} \|\nabla_{t,x} \phi(-t)\|_{L^2(B(0,t))}.$$

- Moreover, given any $\sigma \in (0, 1)$, we have that $\forall \, 0 < t \ll 1$,
  $$t^{\frac{2}{p-1} - \frac{n}{2}} \|\phi(-t)\|_{L^2(B(0,\sigma t))} + t^{\frac{2}{p-1} - \frac{n}{2} + 1} \|\nabla_{t,x} \phi(-t)\|_{L^2(B(0,\sigma t))} \leq K_\sigma.$$

Essentially, blow-up rate of $H^1$-norm in past null cone from singularity.

- But, no information about how $H^1$-norm is “distributed”.
Using our time cone estimate, we can prove:

- $H^1$-norm cannot concentrate in smaller time cone.
- Significant action near null cone.

**Theorem (Alexakis–S.; 2014)**

Suppose $\phi \in C^2$ is as before, with blow-up at $(0, 0)$. If

$$\limsup_{t_* \to 0} |t_*|^{2-n+\frac{4}{p-1}} \int_{\{\sigma_0 |t_*| < r < \sigma_1 |t_*|\}} (|\nabla_{t,x} \phi|^2 + t_*^{-2} \phi^2)|_{t=t_*} < \delta,$$

then

$$\limsup_{t_* \to 0} |t_*|^{1-n+\frac{4}{p-1}} \int_{\{|t| \simeq |t_*|, \ r < \sigma_0 |t_*|\}} (|\nabla_{t,x} \phi|^2 + t_*^{-2} \phi^2) \lesssim \delta.$$

This also extends to some equations $\Box \phi + V|\phi|^{p-1} \phi$. 
Section 4

Asymptotically Anti-de Sitter Spacetimes
Anti-de Sitter Spacetime

**Anti-de Sitter (AdS) spacetime:**
- Max. symmetric solution of Einstein vacuum equations with negative cosmological constant.
- Globally represented as manifold \((\mathbb{R}^{n+1}, g)\),

\[
g = (1 + r^2)^{-1} dr^2 - (1 + r^2) dt^2 + r^2 \gamma_{S^{n-1}}.
\]

AdS conformally embeds into half of \(\mathbb{R} \times S^n\).
- \(\text{AdS} \simeq \mathbb{R} \times \text{(hemisphere)}\).
- Boundary \(\mathcal{I} \simeq \mathbb{R} \times S^{n-1}\) is timelike—can be thought of as “infinity”.

Compactified AdS, mod \(S^{n-1}\).
The Popularity of AdS

Asymptotically AdS (aAdS): spacetimes with “similar infinity structure”.

**AdS/CFT correspondence (holographic principle):** Roughly, some relationship between gravitational dynamics of aAdS spacetimes and conformal field theory at \( \mathcal{I} \).

- Important topic in theoretical physics.
- Original paper (Maldacena, 1999) has > 10000 citations.
- Almost no rigorous mathematical formulation or results.

Model problem: scalar waves on fixed AdS (aAdS) backgrounds.

- Full Einstein vacuum equations (work in progress).
Waves on AdS

Model for waves on AdS: waves on timelike cylinder.
- Well-posed problem: initial value + boundary value.
- Can also pose UC problem like for cylinder.

Question

Suppose \((\Box_g + \sigma)\phi + (a^\alpha \nabla_\alpha + V)\phi = 0\) on AdS. If \(\phi\) has zero Dirichlet and Neumann data on \(\mathcal{I}\), then does \(\phi = 0\) in the interior near \(\mathcal{I}\) (DL)?
- \(\sigma \in \mathbb{R}\), and \(a^\alpha, V\) decay.
- (HL) or (HG)?
**Main observation:** Can find pseudoconvex foliations like in figure (in blue) only when $t_0 > \pi$.

- Expect (HG) UC from segments of $J$ with time length $> \pi$.

**Conjecture:** (HL) UC when length $< \pi$ does not hold.
The Positive Result

Theorem (Holzegel–S.; 2015)

(Roughly) Suppose $\phi$ is sufficiently regular and solves

$$(\Box_g + \sigma)\phi + (a^\alpha \nabla_\alpha + V)\phi = 0 \text{ on AdS, with } \sigma, a^\alpha, V \text{ as before.}$$

If $\phi$ satisfies both vanishing “Dirichlet” and “Neumann” conditions on a segment $0 \leq t \leq t_0$, $t_0 > \pi$ of $\mathcal{I}$, then $\phi = 0$ in the interior near $\mathcal{I} \cap \{0 < t < t_0\}$.

Theorem has been generalized in several ways:

- To many aAdS spacetimes.
- To tensor waves (e.g., spin-1 and spin-2).
Applications

**Goal:** Apply result to study UC for Einstein vacuum equations.

Works in progress on vacuum aAdS spacetimes (joint with G. Holzegel):

1. **Rigidity of AdS:** Weyl curvature $W$ vanishes at $I \Rightarrow$ spacetime is AdS.
2. **Extension of symmetries:** Symmetries of $W$ on $I$ extend to interior.
3. **Rigidity of Schwarzschild-AdS and Kerr-AdS.**

**Conjecture**

*In the aAdS setting, does “Dirichlet” and “Neumann” data for Einstein vacuum equations determine the spacetime?*

- “Correspondence” between data at infinity and interior dynamics?