Correspondence and Rigidity Results on Asymptotically Anti-de Sitter Spacetimes

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Section 1

Introduction
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Physical Motivations

Correspondence Principles

Outstanding problem in theoretical physics:

- Reconciling *Einstein’s theory of gravity* with *quantum field theories*.

Influential research direction:

- **AdS/CFT correspondence**
- (AdS: Anti-de Sitter)
- (CFT: Conformal field theory)

**AdS/CFT \( \Rightarrow \) holographic principle:**

- Gravitational theory on spacetime encoded in some theory on its boundary (of one less dimension).

Original paper*:

- 12154 12201 12381 12869 13156 13278 citations.


† Data from http://inspirehep.net/record/451647/citations.
In AdS context, little rigorous mathematics for:

- Positive statements of this principle.
- Precise formulations of this principle.

In particular, in dynamical (non-static) settings.

**Main (long-term) questions:**

1. Can rigorous statements toward holographic correspondences be formulated?
2. Can these statements be proved?
3. Can one understand the mechanisms behind such a correspondence?
Gravity described by Einstein’s theory of general relativity.

- **Spacetime**: \((n + 1)\)-dimensional *Lorentzian manifold* \((M, g)\).
- \(g\): *Lorentzian metric*, with *signature* \((-++, \ldots, +)\).
- Gravity modelled by *curvature* of \((M, g)\).

Gravity and matter coupled via the **Einstein equations**:

\[
\text{Ric}_g \frac{1}{2} \text{Sc}_g g + \Lambda g = T.
\]

- No matter \((T \equiv 0) \Rightarrow \text{Einstein-vacuum equations (EVE)}:

\[
\text{Ric}_g = \frac{2\Lambda}{n-1} g.
\]
Anti-de Sitter (AdS) spacetime:
- Maximally symmetric solution of EVE...
- ... with negative cosmological constant $\Lambda = \frac{-n(n-1)}{2}$.
- Lorentzian analogue of hyperbolic space.

Globally represented as $(\mathbb{R}_t \times \mathbb{R}_x^n, g)$, with
\[ g := (1 + r^2)^{-1}dr^2 - (1 + r^2)dt^2 + r^2 \gamma. \]
- $r > 0, \omega \in \mathbb{S}^{n-1}$: Polar coordinates on $\mathbb{R}^n$.
- $\gamma$: Round metric for unit sphere $\mathbb{S}^{n-1}$. 

Maximally symmetric solution of EVE...
Consider “inverted radius” $\rho := r^{-1}$:

$$g = \rho^{-2} \left[ (1 + \rho^2)^{-1} d\rho^2 - (1 + \rho^2) dt^2 + \gamma \right].$$

- $\rho^2 g$ smooth at $\rho = 0$ ($r = \infty$).

$\rho$ is a boundary defining function.

- Conformal boundary $\mathcal{J} := \{\rho = 0\}$ of AdS:

$$\mathcal{J} \simeq \mathbb{R}_t \times S^{n-1}, \tilde{g} := -dt^2 + \gamma.$$

Asymptotically AdS (aAdS):

- Spacetime with “similar conformal boundary”.
Introduction

The Main Question

The Correspondence Question

Question (0’)

Is there some correspondence between:

- aAdS solution of EVE ("gravitational dynamics").
- Data prescribed at conformal boundary $\mathcal{I}$.
  - (Ideally: boundary metric, stress-energy tensor.)

Attempt 1: Formulate this in terms of PDEs:

- Given: "Cauchy" data on conformal boundary $\mathcal{J}$.
- Question: Solve for unique solution of EVE in interior?
Ill-Posedness

**Bad news:** This problem is generally ill-posed.

- AdS $\approx$ cylinder in Minkowski spacetime:
  
  $$C := \{(t, x) \in \mathbb{R}^{1+n} \mid |x| < 1\}.$$ 

- EVE $\approx$ wave equation.

- Wave equations ill-posed with Cauchy data on $C$.

**For EVE to be well-posed, one requires:**

- Initial data at $t = 0$.

- Dirichlet or Neumann data on $I$. 

The cylinder $C$. 

AdS cylinder in Minkowski spacetime:
A Unique Continuation Problem

Attempt 2: Formulate as unique continuation problem for PDEs.

- If a solution exists, then must it be unique?

Question (0)

Suppose two aAdS solutions of the EVE have the same “boundary-Cauchy” data on their conformal boundaries. Then, must these solutions be isometric?

- Is there a one-to-one correspondence between aAdS solutions of EVE and some space of “boundary-Cauchy” data?
Consider a model problem:

- Wave equation on fixed AdS/aAdS spacetime.

**Question (1)**

\[ (\Box_g + \sigma)\phi = G(\phi, \nabla\phi), \quad \sigma \in \mathbb{R}. \]

If \( \phi_1, \phi_2 \) have same **Dirichlet and Neumann data on the boundary** \( \mathcal{J} \), then is \( \phi_1 = \phi_2 \) **locally near** \( \mathcal{J} \)?

- **\( G \) linear**: \( \phi = 0 \) at \( \mathcal{J} \) \( \Rightarrow \) \( \phi = 0 \) near \( \mathcal{J} \)?
Why the Wave Equation?

Question (1): essential step toward Question (0).

- Wave equation: first linearization of EVE.
- EVE \rightarrow curvature satisfies nonlinear wave equation.

Question (1) also has applications to rigidity results.

Remark. Why $\square + \sigma$, not $\square$?

- $\sigma$ determines asymptotics of $\phi$ near $I$. 
Section 2

Results on AdS Spacetime
Some Intuition

Consider: \((\Box_g + \sigma)\phi = 0\).

- (Over)assume \(\phi\) depends only on \(\rho\) \(\Rightarrow\) 2nd-order ODE for \(\phi\).
- Frobenius method \(\Rightarrow\) two branches of solutions:

\[
\phi_\pm = \rho^{\beta_\pm} \sum_{k=0}^{\infty} a_k \rho^k, \quad \beta_\pm = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} - \sigma}.
\]

- (Breitenlohner–Freedman)

Thus, for \(\phi\) to vanish, we must eliminate both branches:

\[
\rho^{-\beta_+} \phi \to 0, \quad \rho \searrow 0.
\]

Q. Is this condition sufficient in general?

(A. Almost, for physically relevant \(\sigma\).)
The Main Theorem

Theorem (Holzegel–S.; 2015)

Suppose $\phi$ is a $C^2$-solution of

$$|(\Box_g + \sigma)\phi| \leq \rho^{2+p}(|\partial_t \phi| + |\partial_\rho \phi| + |\nabla_{S^2} \phi|) + \rho^p|\phi|,$$

where $\sigma \in \mathbb{R}$ and $p > 0$. Assume the vanishing condition

$$|\rho^{-\beta} + \phi| + |\nabla_{t,\rho,S^2} (\rho^{-\beta} + 1) \phi| \to 0, \quad \text{if } \sigma \leq \frac{n^2 - 1}{4} \left( \beta_+ \geq \frac{n + 1}{2} \right),$$

$$|\rho^{-\frac{n+1}{2}} \phi| + |\nabla_{t,\rho,S^2} (\rho^{-\frac{n-1}{2}} \phi)| \to 0, \quad \text{if } \sigma \geq \frac{n^2 - 1}{4} \left( \beta_+ \leq \frac{n + 1}{2} \right),$$

as $\rho \searrow 0$, on a sufficiently large time interval

$$t \in [0, t_0], \quad t_0 > \pi.$$

Then, $\phi$ vanishes in the interior of AdS, near $J \cap \{0 < t < t_0\}$. 
Some Remarks

1. First such correspondence result in dynamical, non-analytic setting.

2. The **sufficiently large time interval assumption** is new.
   - Clearly necessary for *global* uniqueness.
   - Surprisingly, seems necessary even for *local* uniqueness.

3. **Vanishing condition** optimal when $\sigma \leq \frac{n^2 - 1}{4}$.
   - $\sigma = \frac{n^2 - 1}{4}$: conformal mass.
   - **Q.** (Open) Can we do better for $\sigma > \frac{n^2 - 1}{4}$?

4. Result also holds for (appropriately defined) tensor waves.
   - Useful for future applications to EVE.
Theorem (Holzegel–S.; 2015)

On pure AdS, main theorem extends to global uniqueness result.

- Can show $\phi$ vanishes on all of $\{0 < t < t_0\}$.

Theorem (Holzegel–S.; 2015)

When well-posedness theory* exists for $\square_g + \sigma$:

- For $\phi$ with finite (twisted) $H^2$-energy...
- ... if $\phi$ has vanishing Dirichlet and Neumann data on $J \cap \{0 < t < t_0\}$...
- ... then $\phi = 0$ near $J \cap \{0 < t < t_0\}$.

* See (Warnick; 2013).
Pose as \textbf{unique continuation (UC) problem:}

\begin{itemize}
  \item \textbf{Problem (Unique Continuation)}
  \begin{align*}
    \Box_g \phi - a^\alpha \nabla_\alpha \phi - V\phi &= 0. \\
    \text{If } \phi, d\phi \text{ vanish on a hypersurface } \Sigma, \ldots \\
    \ldots \text{then must } \phi \text{ vanish on one side of } \Sigma?
  \end{align*}
\end{itemize}

In our context: \( \Sigma = I \).
Analytic, linear wave equations:
- Holmgren’s theorem $\Rightarrow$ UC
- Assuming analyticity is too strong

Non-analytic: Crucial criterion is pseudoconvexity.
- (Hörmander, Lerner–Robbiano) $\Sigma := \{f = 0\}$ pseudoconvex $\Rightarrow$ UC, $\Sigma$ to $\{f > 0\}$.
  - Purely local result (neighborhood of $p \in \Sigma$).
- (Alinhac, Alinhac–Baouendi) $\Sigma$ not pseudoconvex $\Rightarrow$ ...
  - ... then $\exists a^\alpha, V$ for which UC from $\Sigma$ to $\{f > 0\}$ does not hold.
Pseudoconvexity

**Definition**

\[ \Sigma := \{ f = 0 \} \text{ is pseudoconvex (w.r.t. } \Box_g \text{ and } \text{sgn } f) \leftrightarrow \ldots \]

- \( -f \) is convex in tangent null directions to \( \Sigma \).

**Rough interpretation:**

- Any null geodesic hitting \( \Sigma \) tangentially...
- ... will lie in \( \{ f < 0 \} \) nearby.

**Definition**

\( \Sigma \) is zero pseudoconvex \( \Leftrightarrow \) \( \Sigma \) is ruled by null geodesics.
Examples: Zero Pseudoconvexity

Zero pseudoconvex case is complicated:

- Depends on geometry of spacetime near $\Sigma$.
- Result may be local, semi-global, or global.

1. (Global) $\Sigma = \text{timelike hyperplane in } \mathbb{R}^{n+1}$:
   - No local UC (Alinhac–Baouendi).
   - Global UC from all of $\Sigma$ (Kenig–Ruiz–Sogge).

2. (Semi-global) $\Sigma = \text{null infinity of } \mathbb{R}^{n+1}$:
   - UC from $>\frac{1}{2}$ of $\mathcal{J}^{\pm}$ (Alexakis–Schlue–S.).

3. (Local) $\Sigma = \text{null infinity of Schwarzschild}$:
   - UC from $\mathcal{J}^{\pm}$ near $t^0$ (Alexakis–Schlue–S.).
The Conformal Boundary

**Bad news:** The AdS conformal boundary $\mathcal{I}$ is zero pseudoconvex.

**OK news:** Cylinders $\{\rho = \rho_0\}$, $\rho_0 > 0$, are (inward) pseudoconvex.

- $\Rightarrow$ UC from $\mathcal{I}$ inward, provided $\phi$ decays as $t \to \pm\infty$.
- (Since region $\{0 < \rho < \rho_0\}$ has boundary $t = \pm\infty$.)

**Question**

*Can we “bend” the hypersurfaces $\{\rho = \rho_0\}$ back toward $\mathcal{I}$, so that:*

1. They remain pseudoconvex.
2. They intersect $\mathcal{I}$ after a finite time interval.
A Pseudoconvex Foliation

Fix $y > 0$, and consider level sets of

$$f := \frac{\rho}{\sin(yt)}, \quad 0 < t < y^{-1}\pi.$$ 

\[ \text{Lemma} \]

If $y < 1$, then $\{f = f_0\}$ is pseudoconvex for $f_0 \ll 1$.

- $\Rightarrow$ Time interval $[0, t_0 := y^{-1}\pi]$ has length $> \pi$.
- $\Rightarrow$ “sufficiently long time interval assumption”.

Level sets of $f$. 

Time interval $[0, t_0 := y^{-1}\pi]$ has length $> \pi$. 

"sufficiently long time interval assumption".
Carleman Estimates

Main analysis tool for UC: Carleman estimates.

- Weighted spacetime integral estimate with free parameter.
- Pseudoconvexity + Carleman estimate + standard argument $\Rightarrow$ UC.

Carleman estimate roughly of the form

$$\|w_{\lambda,f}(\Box g + \sigma)\phi\|_{L^2(f<f_0)}^2 \gtrsim \lambda \|w_{\lambda,f}\nabla\phi\|_{L^2(f<f_0)}^2 + \lambda^3 \|w_{\lambda,f}\phi\|_{L^2(f<f_0)}^2.$$

- $\lambda \gg 1$: free chosen constant.
- $w_{\lambda,f}$: weight depending on $f, \lambda$.

Some technical difficulties:

1. Zero pseudoconvexity $\Rightarrow$ dealing with degenerating weights.
2. Infinite domains $\Rightarrow$ infinite volume $\Rightarrow$ integrability issues.
Short Time Intervals

Conjecture

*UC does not generally hold if* $t_0 < \pi$.

Special property of AdS geometry:

- ∃ family of future null geodesics from $I \cap \{t = 0\}$...
- ... *which refocus at* $I \cap \{t = \pi\}$.

Idea: Counterexamples via geometric optics.

- Similar to (Alinhac–Baouendi).
- Solutions concentrated near such a geodesic...
- ... arbitrarily close to zero data for $\phi$. 
Section 3

Results on aAdS Spacetimes
Main goal: UC problem for the EVE.

- *For EVE, the geometry itself is the unknown.*
- Results on AdS spacetime must be robust.
- Methods must apply to general aAdS spacetimes.

(Holzegel–S.; 2015) aAdS spacetimes with static conformal boundary.

- In context of solving the EVE (as initial-boundary problem)...
- ... one also encounters non-static conformal boundaries.
### Definition of aAdS

#### Step 1: Construction of conformal boundary and spacetime.
- **Conformal boundary:** $J := \mathbb{R}_t \times S$.
  - $S$: $(n - 1)$-dimensional manifold—cross-section of $J$.
- **Spacetime** (near boundary): $\mathcal{M} := (0, \rho_0)_\rho \times J$.

#### Step 2: Construction of aAdS metric.
- Adopt **Fefferman–Graham (FG)** gauge.
- Expand remaining $(t, S)$-components of $g$ about $J$.
  \[
g = \rho^{-2}\{d\rho^2 + [\tilde{g}_{ab} + \bar{g}_{ab}\rho^2 + O(\rho^3)]dx^a dx^b}\].
- **Conformal boundary:** $(J, \hat{g})$. 
Features of Construction

1. Allow general boundary topology and geometry: $(\mathcal{J}, \bar{g})$.
   - *Example:* Can consider planar AdS.

2. No loss of generality in choosing FG gauge.
   - Can change coordinates from more general gauge to FG...

3. For *Einstein-vacuum* spacetimes in FG gauge:
   - EVE connects $\bar{g}$ to geometry of conformal boundary.
   - $-\bar{g}$ is precisely the *Schouten tensor* $\hat{P}$ for $(\mathcal{J}, \hat{g})$. 
Results on aAdS Spacetimes

Extending Unique Continuation

The Pseudoconvexity Criterion

Question

Do the previous results on AdS extend to aAdS spacetimes?

- In particular, can we still find pseudoconvexity near $I$?

Lemma (Pseudoconvexity Criterion)

Suppose the following conditions hold:

1. $-\bar{g}$ satisfies a (pseudo-)positivity condition:

   
   $$-\bar{g} - \zeta \bar{g} \geq c > 0$$

   for some function $\zeta$.

2. $|\mathcal{L}_{\partial_t} \bar{g}|$ is sufficiently small (depending on $\bar{g}$).

Then, there is a pseudoconvex foliation near $I$...

- ... spanning a sufficiently long time interval on $I$. 

An Additional Difficulty

Previous choice of $f := \rho / \sin(yt)$ generally fails.

\[ \sin(yt) \] shows up in AdS computations as solution of

\[ \psi'' + y^2 \psi = 0. \]

- aAdS setting: Nonzero $L_{\partial_t} \tilde{g} \Rightarrow$ extra $\psi'$-term.
- Idea: Replace $\sin(yt)$ by function $\psi$ resembling damped oscillator.

Lemma

Assuming the pseudoconvexity criterion:

- For an appropriate $\psi$, the level sets of $f := \rho / \psi$ are pseudoconvex for $f \ll 1$. 

The Main Theorem

Theorem (Holzegel–S.; 2016)

Let \((\mathcal{M}, g)\) be an aAdS spacetime satisfying the pseudoconvexity criterion. Suppose \(\phi\) is a \(C^2\)-solution of

\[
|\Box_g + \sigma|\phi| \leq \rho^{2+p}|\partial_t \phi| + |\partial_\rho \phi| + |\nabla_{S^2} \phi| + \rho^p|\phi|,
\]

where \(\sigma \in \mathbb{R}\) and \(p > 0\), and suppose that

\[
|\rho^{-\beta} + \phi| + |\nabla_{t,\rho,S^2}(\rho^{-\beta} + 1 \phi)| \to 0, \quad \text{if } \sigma \leq \frac{n^2 - 1}{4},
\]

\[
|\rho^{-\frac{n+1}{2}} \phi| + |\nabla_{t,\rho,S^2}(\rho^{-\frac{n+1}{2}} \phi)| \to 0, \quad \text{if } \sigma \geq \frac{n^2 - 1}{4},
\]

as \(\rho \searrow 0\), on a sufficiently large time interval \(0 \leq t \leq t_0\). Then, \(\phi\) vanishes in the interior of AdS, near \(J \cap \{0 < t < t_0\}\).
Some Remarks

Pseudoconvexity criterion directly connects:
- Pseudoconvexity (and hence UC)
- Asymptotics on conformal boundary ($\mathcal{L}_{\partial_t} \bar{g}$ and $\bar{g}$).

For vacuum spacetimes:
- $\mathring{R}ic(v, v) \geq c > 0$ for null $v$.
- Static boundary: positive Ricci curvature on $S$.

Many, but not all, well-known aAdS spacetimes satisfy this criterion.
- AdS, Schwarzschild-AdS, and Kerr-AdS.
- Planar AdS ($\mathbb{R} \times S^{n-1} \rightarrow \mathbb{R} \times \mathbb{R}^{n-1}$) fails the criterion.
Some More Remarks

How to interpret the pseudoconvexity criterion?

\[-\bar{g} - \zeta \bar{g} \geq c > 0.\]

- Consider the truncated metric
  \[\tilde{g} = \rho^{-2}[d\rho^2 + (\hat{g}_{ab} + \bar{g}_{ab}\rho^2)dx^a dx^b].\]

- If pseudoconvexity criterion holds, then...
  - \(\tilde{g}\)-null geodesics from \(\mathcal{I}\) remaining close to \(\mathcal{I}\)...
  - ... will return to \(\mathcal{I}\) within some time \(t_0\).

The pseudoconvexity criterion is gauge-dependent.

- Not invariant under change of \(\rho\).
- \((\text{Work in progress})\) Is there a gauge-independent statement?
- \((\text{Work in progress})\) Or, is there an “optimal” gauge?
Applications to Vacuum Spacetimes

1. **Linearised EVE**: (Holzegel–S.; 2015) applies to *linearised* EVE about AdS.

2. **Rigidity of AdS**: If the Weyl curvature $W$ vanishes at $I$, ...
   - ... then the spacetime must be pure AdS.

3. **Rigidity of Kerr-AdS**: Is there an analogous result?

4. **Extension of symmetry**: Given (appropriately defined) symmetry of $(J, \hat{g})$:
   - Can it be extended to the spacetime?

5. **Correspondence for EVE**: (Original question)
The Vacuum Setting

Assume FG expansion for (vacuum) $g$:

$$\rho^2 g = d\rho^2 + [g^{(0)}_{ab} + g^{(2)}_{ab} \rho^2 + \cdots + g^{(n)}_{ab} \rho^n + O(\rho^{n+1})] dx^a dx^b.$$ 

- $g^{(0)} = \hat{g}$, $g^{(2)} = \bar{g}$.
- All coefficients below $\rho^n$ determined by $g^{(0)}$.
- $g^{(n)}$: connected to stress-energy tensor.

For full nonlinear problem (5):

- **Goal**: Prove $g^{(0)}$ and $g^{(n)}$ determine $g$ (near $J$).
- (On a **large enough time interval**.)
- (Work in progress) Previous linear results are a key step.
Symmetry extension (4): more immediately tractable problem

- **Assume:** aAdS, symmetry at conformal boundary.
  - Killing vector field $Z$ with $\mathcal{L}_Z g^{(0)} = 0$ and $\mathcal{L}_Z g^{(n)} = 0$.
  - Vanishes on large enough time interval.

- **Goal:** Symmetry extends to aAdS bulk (near $I$).
  - Extend $Z$ to *spacetime* Killing field.

*(Work in progress):*

- $n = 3$: Pieces in place.
- $n > 3$: Some issues.
Key Steps

1. \textit{Guess} an extension of $Z$.

2. Derive \textit{closed} system of:
   - Wave equations (for $\mathcal{L}_ZW$).
   - Transport equations (for $\mathcal{L}_Zg$).
   - Adapt (Ionescu–Klainerman, Alexakis–Ionescu–Klainerman), (Alexakis–Schlue)

Apply UC results (from $J$) to this system.

3. \textit{Prove} $\mathcal{L}_Zg^{(0)} = 0$ and $\mathcal{L}_Zg^{(n)} = 0 \Rightarrow \ldots$
   - \ldots Solution of system vanishes at $J$ ...
   - \ldots at sufficient rate for UC.
Thank you for your attention.