Correspondence Properties for Waves on Asymptotically Anti-de Sitter Spacetimes

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Section 1

Introduction
Anti-de Sitter Spacetime

Anti-de Sitter (AdS) spacetime:

- Maximally symmetric solution of Einstein vacuum equations (EVE).
- With negative cosmological constant $\Lambda$.
  - For convenience, fix $\Lambda := -3$.

Globally represented as manifold $(\mathbb{R}^4, g)$, with

$$g = (1 + r^2)^{-1} dr^2 - (1 + r^2) dt^2 + r^2 \hat{\gamma}.$$  

- $\hat{\gamma}$: round metric on $S^2$.
- Generalises directly to higher dimensions.
Consider “inverted radius” $\rho := r^{-1} \Rightarrow$

$$g = \rho^{-2}[(1 + \rho^2)^{-1}d\rho^2 - (1 + \rho^2)dt^2 + \hat{\gamma}].$$

- $\rho$ is a “boundary defining function” $\Rightarrow$ can think of “$\mathcal{I} := \{\rho = 0\}$” as AdS infinity.
- $\mathcal{I} \cong \mathbb{R} \times S^2$ has Lorentzian structure:

$$\hat{g} := -dt^2 + \hat{\gamma}.$$ 

Spacetimes that “have same asymptotic infinity $\mathcal{I}$” called asymptotically AdS (aAdS).
Motivations

Question (∞)

Does “geometric boundary data” prescribed at AdS infinity determine interior dynamics of EVE?

- If two aAdS vacuum spacetimes have identical “Dirichlet and Neumann data” at infinity, then must they be isometric?
- (If not globally, then at least locally near infinity?)

In other words: Is there some correspondence between boundary data at infinity and interior gravitational dynamics?
Difficulties

**Bad news:** Initial value problems for hyperbolic equations generally ill-posed on timelike hypersurfaces (such as $I$).

- Thus, may not expect to solve EVE.
- However, can still ask whether existing solutions are unique.

Also, the EVE are highly nonlinear.

This is work in progress:

- Expect to prove various positive results.
A Linear Model Problem

EVE is hard $\Rightarrow$ consider first a model problem.

“(Very) poor man’s linearisation” of EVE: scalar wave equation

$$\Box \phi + \sigma \phi = 0, \quad \sigma \in \mathbb{R}$$

on fixed AdS (or aAdS) spacetime.

- Consider analogous problem for scalar wave equation.
- Recently completed work.
The Main Problems

Question (1)

\( \phi_1, \phi_2: \text{solutions on AdS of} \)

\[(\Box_g + \sigma)\phi + a^\alpha \nabla_\alpha \phi + V\phi = 0, \quad \sigma \in \mathbb{R}, \quad a^\alpha, V \text{ decay}, \]

\text{with same Dirichlet and Neumann data on AdS infinity. Does} \ \phi_1 = \phi_2 \text{ near infinity?}

- \text{Equivalently: If } \phi \text{ solves the above, then does } \phi \text{ vanishing at } J \Rightarrow \phi \text{ vanishes near } J? \n
Question (2)

\text{Can we generalise solution to Question (1) so it can be applied to solve Question (\infty)?}
Section 2

Results for Scalar Waves
Results for Scalar Waves

Analytic Theory, Simplified

To get basic idea, assume:

- $a^\alpha$ and $V$ vanish ($\Rightarrow (\Box_g + \sigma)\phi = 0$).
- $\phi$ depends only on $\rho$.

$\Rightarrow$ 2nd-order ODE for $\phi$:

$$\rho^2(1 + \rho^2)\partial^2_\rho \phi - 2\rho \partial_\rho \phi + \sigma \phi = 0.$$ 

Frobenius method $\Rightarrow$ two branches of solutions:

$$\phi_{\pm} = \rho^{\beta_{\pm}} \sum_{k=0}^{\infty} a^\pm_k \rho^k, \quad \beta_{\pm} = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \sigma}.$$ 

- Agrees with Breitenlohner-Freedman ($5/4 < \sigma < 9/4$).
Removal of Analyticity

Analytic theory $\Rightarrow$ for $\phi$ to vanish, must eliminate both branches:

$$\rho^{-\beta} + \phi \to 0, \quad \rho \searrow 0.$$ 

**Goal:** Remove analyticity assumptions.

1. Consider non-analytic $\phi$ (depending on all variables).
2. Consider non-analytic $a^{\alpha}$, $V$.
3. (Later) Consider other non-analytic metrics $g$.

**Question:** Similar results if $\phi$, $a^{\alpha}$, $V$ are only $C^\infty$?
Main Theorem, I

Theorem (Holzegel, S.; 2015)

Suppose \( \phi \) is a \( C^2 \)-solution of

\[
|(\Box_g + \sigma)\phi| \leq \rho^{2+p}(|\partial_t \phi| + |\partial_\rho \phi| + |\nabla_{S^2} \phi|) + \rho^p|\phi|,
\]

where \( \sigma \in \mathbb{R} \) and \( p > 0 \). Suppose that

\[
|\rho^{-\beta} \phi| + |\nabla_{t,\rho, S^2}(\rho^{-\beta} \phi)| \to 0, \quad \text{if } \sigma \leq 2 \ (\beta_+ \geq 2),
\]

\[
|\rho^{-2} \phi| + |\nabla_{t,\rho, S^2}(\rho^{-1} \phi)| \to 0, \quad \text{if } \sigma \geq 2 \ (\beta_+ \leq 2),
\]

as \( \rho \downarrow 0 \), on a sufficiently large time interval

\[
0 \leq t \leq t_0, \quad t_0 > \pi.
\]

Then, \( \phi \) vanishes in the interior of AdS, near \( I \cap \{0 < t < t_0\} \).

Furthermore, the results extend to \( (n + 1) \)-dimensional AdS spacetime for any \( n \) (with natural modifications to \( \beta_\pm \), ranges of \( \sigma \), etc.).
Remarks: Comparisons

Comparisons with the analytic theory:

1. \( \sigma \leq 2 \): vanishing condition is optimal.
2. \( \sigma > 2 \): require more vanishing than expected.
   - **Question:** Can this be improved?
3. We also require vanishing conditions for \( \nabla \phi \).
   - Analytic case: redundant information, not needed.
4. **New:** “Sufficiently large time interval” assumption.
Remarks: Local and Global Uniqueness

Result is “local”: only show $\phi$ vanishes near $\mathcal{I} \cap \{0 < t < t_0\}$.

- AdS: can use global geometric properties to show “global” uniqueness (i.e., $\phi$ vanishes on $\{0 < t < t_0\}$).
- Does not extend to general aAdS spacetimes.

Remark

Global uniqueness: $t_0 \geq \pi$ necessary by finite speed of propagation.

- More surprisingly, this also seems necessary for local uniqueness.
Remarks: Bounded Potentials

Question

What if lower-order terms $a^\alpha$ and $V$ decay less?

- In particular, $V$ only bounded?

Proposition (Holzegel, S.; 2015)

Suppose $\phi$ is a $C^2$-solution of

$$|\Box_g \phi| \leq \rho^2 (|\partial_t \phi| + |\partial_\rho \phi| + |\nabla_{S^2} \phi|) + |\phi|. \quad (\rho = 0)$$

Suppose $\phi$ and $\nabla \phi$ vanish to infinite order as $\rho \downarrow 0$, on a sufficiently large time interval $0 \leq t \leq t_0$, $t_0 > \pi$.

Then, $\phi$ vanishes in the interior of AdS, near $I \cap \{0 < t < t_0\}$.

Again, the results extend to $(n + 1)$-dimensional AdS spacetime for any $n$. 
Well-Posedness

**Goal:** Connect results to (non-analytic) local well-posedness theory.
- Connect vanishing conditions to zero Dirichlet and Neumann data.

**Theorem (Warnick)**

Let $5/4 < \sigma < 9/4$. Then:
- $\Box_g + \sigma$ propagates a “twisted $H^1$-energy” $E^1(t)$.
  - Roughly, like the $H^1$-norm, but $\nabla$ replaced by $\rho^\beta - \nabla \rho^{-\beta -}$.
- Similarly defined “twisted $H^2$-energy”, $E^2(t)$, is also propagated.

Assuming one of the following boundary conditions,

$$
\rho^{-\beta -} \phi \to 0 \ (\text{Dirichlet}), \quad \rho^{-2+2\beta -} \partial_\rho (\phi \rho^{-\beta -}) \to 0 \ (\text{Neumann}),
$$

then $(\Box_g + \sigma)\phi = 0$ is well-posed in the twisted $H^1$-norm.
Main Theorem, II

Main idea: Given extra regularity, Dirichlet and Neumann conditions ⇒ vanishing assumptions in first theorem.

Theorem (Holzegel, S.; 2015)

Suppose $\phi$ is a $C^2$-solution of

$$|(|\Box_g + \sigma|\phi| \leq \rho^{2+p} (|\partial_t \phi| + |\partial_\rho \phi| + |\nabla_{S^2} \phi|) + \rho^p |\phi|,$$

where $5/4 < \sigma < 9/4$ and $p > 0$. Suppose

- $\phi$ satisfies both vanishing Dirichlet and Neumann conditions.
- $\phi$ has finite twisted energy $E^2(t)$.

on a time interval $0 \leq t \leq t_0$ ($t_0 > \pi$). Then, $\phi = 0$ near $I \cap \{0 < t < t_0\}$.

Again, analogous results hold in other dimensions.
Section 3

Ideas Behind the Proof
Unique Continuation

View question as **unique continuation (UC)** problem—a classical problem in PDEs:

- Suppose \((\Box g + a^\alpha \nabla_\alpha + V)\phi = 0\) on a domain.
- Suppose \(\phi, d\phi = 0\) on hypersurface \(\Sigma\).

Must \(\phi\) vanish (locally) on one side of \(\Sigma\)?

**Cauchy-Kovalevskaya**: \(g, a^\alpha, V\) analytic \(\Rightarrow\) can solve for unique power series solutions (if \(\Sigma\) noncharacteristic).

**Holmgren’s theorem**: Solution unique in class of distributions.
Non-Analytic Theory

However, this is not a satisfactory answer:

- Cannot deal with non-analytic $g$, $a^\alpha$, $V$.
- Cannot deal with \textit{nonlinear} wave equations.

In the non-analytic setting, UC depends on geometry near $\Sigma$.

- (Hörmander, Lerner-Robbiano) Main criterion is \textit{pseudoconvexity}:
  - $\Sigma := \{ f = 0 \}$ pseudoconvex $\Rightarrow$ UC from $\Sigma$ to $\{ f > 0 \}$.

- (Alinhac) $\Sigma$ not pseudoconvex $\Rightarrow \exists \ a^\alpha, V$ for which UC from $\Sigma$ to $\{ f > 0 \}$ does not hold.
Pseudoconvexity

**Definition (Lerner-Robbiano)**

$\Sigma := \{ f = 0 \}$ is pseudoconvex (w.r.t. $\Box_g$ and $\text{sgn} \, f$) iff

$$\nabla^2 f(X, X) < 0 \text{ on } \Sigma, \text{ if } g(X, X) = Xf = 0.$$  

($-f$ convex on $\Sigma$ in tangent null directions.)

Visually: any null geodesic hitting $\Sigma$ tangentially will lie in $\{ f < 0 \}$ nearby.

**Definition**

If $\Sigma$ ruled by null geodesics, it is called zero pseudoconvex.
Applications in Relativity

UC results have been applied in general relativity:

1. Rigidity of Kerr black holes: (Alexakis-Ionescu-Klainerman; Carter-Robinson, Hawking)
2. UC for waves from infinity in asymptotically flat spacetimes: (Alexakis-Schlue-S.)
3. Non-existence of time-periodic spacetimes: (Alexakis-Schlue; Papapetrou, Bičák-Scholtz-Tod)
Example: Pseudoconvexity

Consider the finite timelike cylinder

$$\Sigma := \{ t_0 < t < t_1, |r| = r_0 \} \subseteq \mathbb{R}^{n+1}.$$

- $\Sigma$ is pseudoconvex (w.r.t. $\Box$, inward).
- $\Rightarrow$ (local) UC from $\Sigma$ into the interior.
Examples: Zero Pseudoconvexity

Example

Zero pseudoconvex—depends on geometry near $\Sigma$:

1. **Timelike hyperplane in $\mathbb{R}^{n+1}$: no local UC** (Alinhac-Baouendi).

2. **Null infinity of $\mathbb{R}^{n+1}$: UC from “at least half of $\mathcal{J}^\pm$”** (Alexakis-Schlue-S.).

3. **Null infinity of Schwarzschild: UC from $\mathcal{J}^\pm$ near $\iota^0$** (Alexakis-Schlue-S.).

(2) demonstrates non-locality in $\Sigma$. 

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AdS Infinity

AdS infinity, $\mathcal{I}$, is zero pseudoconvex.

- Must examine geometry near $\mathcal{I}$ more carefully.

However, cylinders \{\(\rho = \rho_0\)\} are pseudoconvex.

- \(\Rightarrow\) UC from $\mathcal{I}$ to the interior, provided \(\phi\) decays as $t \to \pm \infty$.
- Extra decay needed, since region $\{0 < \rho < \rho_0\}$ has boundary $t = \pm \infty$.

Of course, extra decay condition is undesirable.
Improving Pseudoconvexity

**Question**

*Can we “bend” the hypersurfaces \( \{ \rho = \rho_0 \} \) back toward \( \mathcal{I} \), so that:

1. They remain pseudoconvex (inward).
2. They intersect \( \mathcal{I} \) after a finite time interval.*

More precisely, fix \( y > 0 \), and consider level sets of

\[
 f := \frac{\rho}{\sin(yt)}, \quad 0 < t < y^{-1}\pi.
\]
**Improved Pseudoconvexity**

**Lemma**

*If \( y < 1 \), then \( \{ f = f_0 \} \) is pseudoconvex (inward) for \( f_0 \ll 1 \).*

- i.e., *time interval* \([0, t_0 := y^{-1}\pi]\) *must have length* \( > \pi \).

**Remark**

Lemma \( \Rightarrow \) “sufficiently long time interval” assumption in main theorems.

- \( \Rightarrow \) Optimism that some UC from \( J \cap \{0 < t < y^{-1}\pi\} \) holds.
Carleman Estimates

As usual, prove UC via a **Carleman estimate**.

- Carleman estimate + standard argument $\Rightarrow$ UC.

Carleman estimate is roughly of the form

$$\|e^{-F_\lambda(f)(\Box_g + \sigma)}\phi\|_{L^2(f<f_0)}^2 \gtrsim \lambda \|e^{-F_\lambda(f)}D\phi\|_{L^2(f<f_0)}^2 + \lambda^3 \|e^{-F_\lambda(f)}\phi\|_{L^2(f<f_0)}^2.$$  

- $\lambda \gg 1$: constant.
- $F_\lambda(f)$: reparametrisation of $f$.

Some technical difficulties:

1. Infinite domains $\Rightarrow$ infinite volume $\Rightarrow$ integrability issues.
2. Zero pseudoconvexity $\Rightarrow$ have to balance decaying weights.
Proof of Carleman Estimate

Carleman estimate can be thought of as an energy estimate for $\Box_g$, but:

1. We want boundary terms to vanish.
2. We want bulk terms to be positive.

Objective (1) from vanishing assumptions for $\phi$ as $\rho \searrow 0$.

Objective (2) achieved using a positive commutator:

- Consider wave equation not for $\phi$, but for $\psi = e^{-F_\lambda(f)} \phi$.
- Multiplier method: integrate by parts

\[
\int_{f < f_0} \Box_g \psi (S \psi + h \psi), \quad S^\alpha := \nabla^\alpha f.
\]
Carleman Estimates, Continued

To ensure bulk terms are positive:

1. Bulk terms containing derivative of $\phi$ tangent to level sets of $f$:
   - Positive only when level sets of $f$ are pseudoconvex.

2. Bulk terms containing $\phi$ and normal derivatives:
   - Use freedom to choose reparametrization $F_\lambda(f)$:
     \[ F_\lambda(f) = \kappa \log f + \lambda p^{-1} f^p, \quad \kappa \in \mathbb{R}, \quad p > 0. \]

Precise vanishing assumption needed for $\phi$ depends on $\kappa$:
- $\Rightarrow$ Must carefully optimise $\kappa$ in Carleman estimate.
Conjecture

UC (or at least the Carleman estimate) does not generally hold if \( t_0 < \pi \).

Observation: Family of future null geodesics from \( J \cap \{ t = 0 \} \) which bend back to \( J \).

- Any such geodesic hits \( J \) at time \( \pi \).

Geometric optics solutions near these geodesics:

- Support of \( \phi \) arbitrarily close to \( J \).
- But, \( \phi \) vanishes on \( J \cap \{ \varepsilon < t < \pi - \varepsilon \} \).
Comparison with Timelike Cylinders

Can contrast with cylinder $C = \{ r = r_0 \}$ in $\mathbb{R}^{n+1}$ (inwardly pseudoconvex).

- Timespan of analogous null geodesics depends on angle made with $C$.

Observation also important for studying linear waves in AdS (Holzegel-Luk-Smulevici-Warnick).

- Explains loss of derivatives in energy decay.
Section 4

Toward the Einstein Equations
Tensor Waves

**Problem:** EVE is tensorial, not scalar.
- Scalarising tensorial quantities $\Rightarrow$ angular frames degenerate.

**Solution:** Generalise results to spherical tensorial waves.
- Application: treat linearised EVE about AdS (L-EVE).

**Corollary (Holzegel, S.; 2015)**

Suppose a Weyl field $W = W_{\alpha\beta\gamma\delta}$ on AdS spacetime satisfies L-EVE. If $W$ vanishes to sufficient order at $\mathcal{J}$, then $W$ vanishes in the interior.
Asymptotically AdS Spacetimes

To deal with EVE, we must handle other aAdS spacetimes.

Theorem (Holzegel, S.; 2015)

Results extend to large subclass of aAdS spacetimes:

- General boundary topology allowed (replace $S^2$ by another surface).
- Metrics (in Fefferman-Graham gauge) of form
  \[ g = \rho^{-2} \left\{ d\rho^2 + \left[ \mathring{g}_{ab} + \tilde{g}_{ab}\rho^2 + O(\rho^3) \right] dx^a dx^b \right\}. \]

  - \( \mathring{g} \) is static, and \( \tilde{g} \) satisfies a positivity condition.

Moreover, for vacuum spacetimes:

- Condition for \( \tilde{g} \Leftrightarrow \) positive curvature on sections of \( J \).
Nonstatic Spacetimes

Static assumption on boundary metric \( \hat{g} \) too restrictive.

**Q:** Can we remove this assumption?

**A:** Yes (work in preparation with G. Holzegel).

- Main idea: alter level sets of \( f = \rho / \sin(yt) \).
- \( \sin(yt) \) arises in computations as harmonic oscillator \((\psi'' + y^2\psi = 0)\).
- Roughly: replace \( \sin(yt) \) by function resembling damped oscillator.
Rigidity of AdS

Question

*Can we apply previous results to treat EVE itself?*

- Wave nature of EVE: curvature satisfies nonlinear wave equation.
- Caveat: background spacetime depends on wave.

Simplest case: rigidity result for AdS.

(Work in preparation with G. Holzegel).

- Assume aAdS spacetime, with Weyl curvature $W$ vanishing on $\mathcal{J}$.
- Then, spacetime must be AdS (at least near $\mathcal{J}$).
Extension of Symmetries

Question

For vacuum aAdS spacetimes, are symmetries on \( \mathcal{J} \) inherited inside?

- **Metric level:** no (e.g., Kerr-AdS).
- **What about at curvature level?**

Conjecture (Work in progress with G. Holzegel)

If \( \mathcal{L}_{\partial_t} W \) vanishes on \( \mathcal{J} \), then spacetime is stationary near \( \mathcal{J} \).

- **Similar results expected to hold for other symmetries (spherical, axial).**
Rigidity of Kerr-AdS

Extension of spherical symmetry $\implies$ rigidity result for Schwarzschild-AdS.
- Can we push this further?

Question

*Rigidity result for Kerr-AdS? (Work in progress with G. Holzegel)*
- *Does not yet follow from extension of axial symmetry.*
Future Work

Conjecture (Question (∞))

*If two aAdS vacuum spacetimes have same Dirichlet and Neumann data on $I$, then they must be isometric near $\bar{I}$.*

- *In other words: there exists a correspondence between vacuum aAdS spacetimes and boundary data.*

Question

*What about existence of solutions to EVE from boundary data?*

- *For which Dirichlet + Neumann data on $\bar{I}$ is there a solution to EVE?*

*In other words: for what space of boundary data do we have the above correspondence?*