Statistical Modelling and Estimation
Mock Exam Question

March 18, 2009

1. Consider the model

\[ Y_{ij} = \mu + \tau_i + \epsilon_{ij}, \]

where \( i = 1, 2, j = 1, \ldots, n_i, \tau_1 + \tau_2 = 0, E(\epsilon_{ij}) = 0, V(\epsilon_{ij}) = \sigma^2, \]
\( V(\epsilon_{2j}) = 4\sigma^2 \) and all \( \epsilon_{ij} \) are independent.

(a) Show how this can be written in the form of the general linear model.

(b) If all \( n \) observations have \( i = 2 \), find a least squares estimator of \( \beta \).

(c) Show that \( \tau_2 \) is not estimable.

(d) Show that \( \mu + \tau_2 \) is estimable.

2. In an experiment, data \( y_1, \ldots, y_6 \) have been collected using two explanatory variables \( X_1 \) and \( X_2 \), from the following design.

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The proposed model is

\[ E(Y_i) = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_{11}(3x_{1i}^2 - 2) + \beta_{22}(3x_{2i}^2 - 2), \]

with \( Y_i \)s uncorrelated and \( V(Y_i) = \sigma^2 \).
(a) Express the model in matrix form, i.e. write down the design matrix $\mathbf{X}$ and the vector of parameters $\beta$.

(b) Find the rank of the matrix $\mathbf{X}$.

(c) Find a generalized inverse of $\mathbf{X}'\mathbf{X}$.

(d) Hence find a least squares estimator $\hat{\beta}$ of $\beta$.

(e) Show that $\beta_{11}$ is not estimable.

(f) Show that $\theta = \beta_{11} + \beta_{22}$ is estimable and find its least squares estimator $\hat{\theta}$.

(g) Find the variance of $\hat{\theta}$.

(h) Write down the form of a $100(1 - \alpha)$% confidence interval for $\theta$. 