A $\mathbb{Z}/p$ Analogue for Unoriented Bordism

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Summary

\[ N^*(-) \mapsto H^*(-; \mathbb{Z}/2) \] (Thom). I set out to construct a
cohomology theory \( V^*(-) \), as closely analogous as possible to
\( N^*(-) \), mapping onto \( H^*(-; \mathbb{Z}/p) \) where \( p \) is an odd prime. We
first recall the following properties of \( N^*(-) \):

\[ N^*(\mathbb{Z}/2) \cong N^*[X] \], where we may choose \( X \) to be \( e_{\mathbb{Z}^2}(1) \) (the
\( N \)-Euler class of the universal \( \mathbb{Z}/2 \)-bundle). Multiplication
\[ \mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2 \] then induces a map
\[ N^*[X] \to N^*[Y, Z], X \mapsto F_h(Y, Z) \]
and Boardman, Quillen and others showed that \( (N^*, F_h) \) is iso-
morphic to the universal "\( \mathbb{Z}/2 \) formal group".

I consider ring theories \( \mathcal{h}^*(-) \) mapping onto \( H^*(-; \mathbb{Z}/p) \).
Such theories have \( \mathcal{h}^*(\mathbb{Z}/p) \cong \mathcal{h}^*[\beta, \gamma] \otimes E[\alpha, \beta] \). (\( E \) denotes an
exterior algebra). Corresponding to \( m: \mathbb{Z}/p \times \mathbb{Z}/p \to \mathbb{Z}/p \) I define
a "\( \mathbb{Z}/p \) formal group law" in a pair of variables \( \alpha, \beta \) and
calculate algebraically the universal \( \mathbb{Z}/p \) formal group in this
situation. I then set up a certain cobordism ring \( V^*(-) \) of
U-manifolds with singularities of type \( \mathbb{Z}/p, \mathbb{Z}/p \times \mathbb{Z}/p, \ldots \)
\[ \ldots, \mathbb{Z}/p^* \ldots \mathbb{Z}/p, \ldots \] with structure groups \( \mathbb{Z}/p \) and show
that a natural subtheory \( V^*(-) \) has \( V^*(X) \to H^*(X; \mathbb{Z}/p) \)
for all \( X \) and \( V^*(\mathbb{Z}/p) \) canonically isomorphic to the universal
\( \mathbb{Z}/p \) formal group (by defining a pair of "Euler classes"
\( \alpha \), \( \beta \) for \( \mathbb{Z}/p \)-bundles). The method can be used to
generate other cobordism ring theories.
Contents

1. Introduction 1

2. $N^*(-)$ and $\mathbb{Z}/2$ formal groups 4
   2.1 $\mathbb{Z}/2$ formal groups 4
   2.2 The ring splitting $\nu : H^*(-; \mathbb{Z}/2) \to N^*(-)$ 11
   2.3 The action of the Steenrod algebra $\mathcal{A}_2$ 13

3. $\mathbb{Z}/p$ formal groups 15
   3.1 $\mathbb{Z}/p$ theories 15
   3.2 The action of the Steenrod algebra $\mathcal{A}_p$ 16
   3.3 The inclusion $\mathbb{Z}/p \hookrightarrow S'$ 17
   3.4 $\mathbb{Z}/p$ formal groups 17
   3.5 The universal $\mathbb{Z}/p$ formal group 20

4. Geometric theories 35
   4.1 Quillen's approach to cobordism theories 35
   4.2 "Euler classes" for $\mathbb{Z}/p$-bundles 39
   4.3 U-manifolds with $\mathbb{Z}/p$-singularities: $^p\nu^*(-)$ 42
   4.4 Products of elements of $^p\nu^*$ 50
   4.5 Douady's manifolds with corners 52
   4.6 The bordism theories $^p\nu^*(-)$ 57
   4.7 The exact triangle 61
   4.8 The mock-bundle approach to cobordism theories 67

5. Calculation of $^p\nu^*$ 68
   5.1 $^p\nu^*$ as an abelian group 68
   5.2 $^p\nu^*$ as a $U^*$-module 72
6. $\omega V^*(-)$ and its $\mathbb{Z}/p$ formal group

6.1 Problems in using the exact triangle
6.2 The "$\mathbb{Z}/p$ homology of spectra" exact triangle
6.3 $H^*(M^{\omega V}; \mathbb{Z}/p)$ as an abelian group
6.4 $H^*(M^{\omega V}; \mathbb{Z}/p)$ as a module over $\mathcal{A}_p$
6.5 The cobordism theories $(\omega)w^*(-)$
6.6 The relation of $\omega w^*(-)$ to $\omega V^*(-)$
6.7 The $\mathbb{Z}/p$ formal group $(\omega) V^*(B\mathbb{Z}/p)$
6.8 Realisation of the universal $\mathbb{Z}/p$ formal group $V^*(B\mathbb{Z}/p)$

7. Operations in $\omega V^*(-)$ and further speculation

7.1 Steenrod operations
7.2 Landweber-Novikov operations
7.3 Further speculation

7.4 A remark on the $\mathbb{Z}/2$ case

References
1. Introduction

Throughout this work all manifolds are $C^\infty$ and all group actions are smooth.

Let $X$ be a CW complex and let $N^\ast(-)$ denote the cobordism theory dual to the bordism theory of unoriented manifolds, $N_{\text{u}}(-)$. Thom (41) showed that the natural map $\mu:N^\ast(X) \to H^\ast(X;\mathbb{Z}/2)$ is an epimorphism of rings.

Later, using geometric Steenrod operations in $N^\ast(-)$, Quillen (33) and tom Dieck (6) proved that $N^\ast(-)$ is characterised by the algebraic property that $N^\ast(B\mathbb{Z}/2)$ is canonically isomorphic to the "universal $\mathbb{Z}/2$-formal group". A consequence is that there is a natural ring isomorphism $N^\ast(X) \cong H^\ast(X;\mathbb{Z}/2)$ and a canonical splitting of $\mu$ as a map of rings, both first observed by Boardman (4) (see Quillen (33)). In Chapter 2 all these results are examined in detail to motivate analogous $\mathbb{Z}/p$ results later.

The object of this thesis is to consider a corresponding problem for $\mathbb{Z}/p$ (for primes $p \neq 2$), in particular to find a multiplicative cobordism theory $V^\ast(-)$ mapping onto $H^\ast(-;\mathbb{Z}/p)$ such that $V^\ast(B\mathbb{Z}/p)$ is canonically isomorphic to the "universal $\mathbb{Z}/p$-formal group". As $H^\ast(B\mathbb{Z}/p;\mathbb{Z}/p) \cong P[\beta]\otimes E[\alpha]$ ($P$ denotes poly. E ext. algebras over $\mathbb{Z}/p$), in Chapter 3 I define a notion of "$\mathbb{Z}/p$-formal group" involving both $\alpha'$ and $\beta'$ variables; then, using methods based on those of Lazard (25) and Fröhlich (17), I calculate algebraically the "universal $\mathbb{Z}/p$-formal group" and its ground ring, which I denote $A_V$.

The next stage is to build a geometric theory $V^\ast(-)$ such that $V^\ast(B\mathbb{Z}/p)$ is canonically isomorphic to this universal formal group. The theory must have a natural geometric pair
in Chapter 4, we set up a geometric theory \( \mathcal{U}^*(-) \) of U-manifolds with \( \mathbb{Z}/p \)-singularities having such a natural pair of Euler classes \( (\alpha_v^*, \beta_v^*) \) for each \( \mathbb{Z}/p \)-bundle. The requirements of a commutative multiplication lead us to define a series of cobordism groups \( (n)_{\mathcal{U}}^*(-) \), with multiplications \( (n)_{\mathcal{U}}^*(-) \times (m)_{\mathcal{U}}^*(-) \rightarrow (m+n)_{\mathcal{U}}^*(-) \), and finally a multiplicative theory \( (\omega)_{\mathcal{U}}^*(-) \). The elements of \( (n)_{\mathcal{U}}^* \) may be pictured as U-manifolds with singularities, where the links of points are \( S^{*-1} \), \( S^{*-2} \times \mathbb{Z}/p \), \( S^{*-3} \times \mathbb{Z}/p \times \mathbb{Z}/p \), ..., \( S^{*-n} \times \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p \) and the singularity strata have structure groups \( \mathbb{Z}/p \Sigma_1 \), \( \mathbb{Z}/p \Sigma_2 \), \( \mathbb{Z}/p \Sigma_3 \), ..., \( \mathbb{Z}/p \Sigma_n \) on their normal bundles. Alternatively, by "cutting along the singularities" one may picture elements of \( (n)_{\mathcal{U}}^* \) as U-manifolds with faces and corners of index \( \leq n \), having free \( \mathbb{Z}/p \)-actions on each face, the actions being "orthogonal" at corners, in the sense that on a corner of index \( r \) they combine locally to give a free action of \( (\mathbb{Z}/p)^r \) and hence globally to give an action of \( \mathbb{Z}/p \Sigma_r \). (\( \Sigma_r \) carries the global "labelling data" of the faces).

As might be expected, the theories most amenable to calculation are \( (1)_{\mathcal{U}}^*(-) \) and \( (\omega)_{\mathcal{U}}^*(-) \). In Chapter 5, using techniques derived from those of Conner and Floyd (10) I calculate \( (1)_{\mathcal{U}}^* \) as an abelian group and as a \( \mathbb{U}^* \)-module. (A consequence of this calculation, not proved here, is that there exist CW complexes \( X \) with \( (1)_{\mathcal{U}}^*(X) \rightarrow H^*(X; \mathbb{Z}/p) \) not epimorphic, in contradiction to a conjecture of Sullivan (40).) However the theory we are really interested in is \( (\alpha)_{\mathcal{U}}^*(-) ; (\beta)_{\mathcal{U}}^*(-) \) induce a \( \mathbb{Z}/p \)-formal group structure on \( (\omega)_{\mathcal{U}}^*(\mathbb{BZ}/p) \) and thus we have a canonical map \( \phi: A_{\mathcal{U}} \rightarrow (\omega)_{\mathcal{U}}^* \). In Chapter 6 I show that \( \phi \) is injective, i.e. that \( (\omega)_{\mathcal{U}}^*(\mathbb{BZ}/p) \) "carries" the universal \( \mathbb{Z}/p \)-
-formal group law, and that as a consequence we may geometrically realise the $\mathbb{Z}/p$-analogue $V^*(-)$ of $N^*(-)$ as a naturally included "sub-theory" of $^{(\infty)}V^*(-)$.

The reason that $V^*(-)$ has to be constructed indirectly, via $^{(\infty)}V^*(-)$, is that while $\mathbb{Z}/2^{\infty}\cdots\mathbb{Z}/2$ is isomorphic to $S^{n-1}$ and thus is homogeneous, so that we may extend the structure group $\mathbb{Z}/2^{\infty}$ to $O(n)$, $\mathbb{Z}/p^{\infty}\cdots\mathbb{Z}/p$ has no such homogeneity. I conjecture that $^{(\infty)}V^*(-)$ is to $V^*(-)$ (multiplicatively) as $\left\{\mathbb{Q}CP^\infty,-\right\}$ is to $\left\{\mathbb{Q}CP^\infty,-\mathbb{Q}CP^\infty\right\}$ (additively), in the following sense: Segal (36) characterises $\left\{\mathbb{Q}CP^\infty,-\right\}$ as the minimal representable functor containing the group of formal sums of line bundles. He shows that $\mathbb{Q}CP$ is a direct factor of $\mathbb{Q}CP^\infty$, so that $\left\{\mathbb{Q}CP^\infty,-\mathbb{Q}CP^\infty\right\}$ is a split summand of $\left\{\mathbb{Q}CP^\infty,-\mathbb{Q}CP^\infty\right\}$. I conjecture that $^{(\infty)}V^*(-)$ is the universal theory containing the commutative group of formal products of $\mathbb{Z}/p$-Euler classes, in some appropriate sense, and that $V^*(-)$ should be a split subring of $^{(\infty)}V^*(-)$.

In Chapter 7 I sketch briefly the construction of operations in $^{(\infty)}V^*(-)$ and $V^*(-)$ and outline some further possible lines of development. The method of introducing join-type singularities with wreath product structure groups appears to be a useful method of generating ring theories.

For convenience, cohomology theories $h^*(-)$ will mostly be pictured here using Quillen's notion (33) of cobordism of $h$-oriented maps of manifolds or else in mock-bundle terms. For general results on mock-bundles I refer to Rourke and Sanderson (35) for manifolds with singularities see Sullivan (40), Baas (7), Stone (38) and for manifolds with corners see Douady (16); for algebraic results on formal groups see Fröhlich (17) and for applications in algebraic topology see Quillen (33), tom Dieck (6), Bühlichkeit, Shirshnokho and Novikov (8), Adams (1) etc.
2. $N^*(-)$ and $\mathbb{Z}/2$-formal groups

We examine Quillen's (33), Bröcker & tom Dieck's (6),
and Boardman's results about $N^*(-)$ and the universal $\mathbb{Z}/2$-
formal group, as a preliminary to our construction of a $\mathbb{Z}/p$
analogue.

2.1 $\mathbb{Z}/2$-formal groups

Let $b$ be a finite CW complex and $\gamma : E \to B$ be a real $n$-dim.
vector bundle.

**Definition 2.1.1** The Thom space $T(\gamma)$ of $\gamma$ is the disc
bundle $D(\gamma)$ of $\gamma$, with its boundary sphere bundle $S(\gamma)$
identified to a point.

For each $\gamma$ there is a canonical class $t_n(\gamma) \in \tilde{N}^n(T(\gamma))$
called the Thom class of $\gamma$ (see Bröcker & tom Dieck (6)).

**Remark 2.1.2** For $B$ a manifold we may use Quillen's
description of $N^*(B)$ as cobordism classes of maps of manifolds
$f : M \to B$ carrying certain normal structure ((33) or see 4.1)
and then the manifold $B$ embedded as the zero section in $D(\gamma)$
represents the class $t_n(\gamma) \in \tilde{N}^n(D(\gamma), S(\gamma)) \cong \tilde{N}^n(T(\gamma))$.

**Definition 2.1.3** Let $i$ be the inclusion of the zero section
$i : B^+ \hookrightarrow T(\gamma)$ ($B^+$ denotes $B$ with a disjoint base point added)
The $N$-Euler class $e_N(\gamma)$ of $\gamma$ is defined to be $e_N(\gamma) = i^*t_n(\gamma)$ in
$N^*(B)$.

For the canonical line bundle $\xi_n$ over $\mathbb{R}P^n$, $T(\xi_n) \cong \mathbb{R}P^{n+1}$ and
t_n(\xi_n) may be represented by the embedded submanifold $\mathbb{R}P^n \to \mathbb{R}P^{n+1}$.
Thus $e_N(\xi_n) \in N^*(\mathbb{R}P^n)$ may be represented by $\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^n$.

**Proposition 2.1.4** Let $i : \mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$.

Then $(1) \quad i^*e_N(\xi_n) = e_N(\xi_n)$

so $\lim_{n \to \infty} e_N(\xi_n)$ defines a unique element $e_N(\xi) \in \lim_{n \to \infty} N^*(\mathbb{R}P^n)$.
(11) $N^k(\text{RP}^n) \cong N^k[X]_{X^*} \quad \text{where} \quad X = e_n(\mathfrak{f}_n) \tag{11}$

so $N^k(\text{RP}^{n+1}) \rightarrow N^k(\text{RP}^n)$ and as a result we have

$N^k(\text{RP}^\infty) \cong \lim_{\leftarrow n} N^k(\text{RP}^n) \quad \text{(all} \quad \lim_{\leftarrow n} N^k(\text{RP}^n) = 0 \quad \text{)}$

(see Bröcker & tom Dieck (6))

Proof: Bröcker & tom Dieck (6)

\[ \text{Corollary 2.1.5} \quad N^k(\text{RP}^\infty) \cong N^k[\![X]\!] \quad \text{(formal power series ring)} \]

where $X = e_n(\mathfrak{f})$ and $N^k(\text{RP}^\infty)$ is topologised by the

skeletal filtration $\text{RP} \rightarrow \text{RP} \rightarrow \cdots \text{of RP}^\infty; N^k[\![X]\!]$ is
topologised by the ideals generated by $X, X^2, X^3, \ldots$

Also $N^k(\text{RP}^\infty \times \text{RP}^\infty) \cong N^k[\![Y, Z]\!]$

where $Y = e_n(\mathfrak{f}'); Z = e_n(\mathfrak{f})$ (\mathfrak{f}' pull-back of \mathfrak{f} under projection
onto 1st factor $\text{RP}^\infty \times \text{RP}^\infty \rightarrow \text{RP}^\infty$, \mathfrak{f} similarly for 2nd factor)

Proof: Bröcker & tom Dieck (6)

$\mathbb{B}/2 \cong \text{RP}^\infty$ (\text{RP}^\infty classifies line bundles and thus $Z/2$-bundles)

Multiplication $m: \mathbb{Z}/2 \times \mathbb{Z}/2 \rightarrow \mathbb{Z}/2$ is a group homomorphism, as $\mathbb{Z}/2$
is abelian, and so $m$ induces:

$m: B(\mathbb{Z}/2 \times \mathbb{Z}/2) \cong B(\mathbb{Z}/2 \times \mathbb{Z}/2) \rightarrow B(\mathbb{Z}/2)$ \quad \text{(thinking of $B(\mathbb{Z}/2$ as $\text{RP}^\infty$)}

this classifies the $\bigotimes$-bundle over $\text{RP}^\infty \times \text{RP}^\infty$

$m^*: N^k[\![X]\!] \rightarrow N^k[\![Y, Z]\!]$ is an $N^k$-algebra morphism sending

$1 \in N^k$ to $1 \in N^k$ and so $m^*$ is determined by its value on $X$.

$m^*: X \mapsto F_n(Y, Z) = Y + Z + \sum_{\begin{smallmatrix} i \leq n \times j \leq n \\ i + j \leq n \end{smallmatrix}} a_{ij} Y^i Z^j \quad (a_{ij} \in N^k)$

$F$ satisfies certain properties from the group structure of $\mathbb{Z}/2$:-

1. $F(0, X) = F(X, 0) = X$ \quad \text{- identity}
2. $F(X, F(Y, Z)) = F(F(X, Y), Z) \quad \text{- associativity}$
3. $F(X, Y) = F(Y, X) \quad \text{- commutativity}$
4. $F(X, X) = 0 \quad \text{- order 2}$

\[ \text{Definition 2.1.6} \quad \text{for} \ R \ \text{a} \ \text{commutative algebra over} \ \mathbb{Z}/2, \ \text{call} \]

an $F \in R[\![X, Y]\!]$ satisfying (1), (2), (3) & (4) a $\mathbb{Z}/2$-\textit{formal group law} over $R$. 

Among pairs \((R, F)\) there is a universal pair \((L, F_L)\) such that for each \((R, F)\) there is a unique algebra homomorphism \(\phi: L \to R\) sending each coefficient \(l_{ij}\) of \(F_L\) to the corresponding coefficient \(a_{ij}\) of \(F\).

(It is clear that such an \(L\) exists—it is a quotient of the algebra \(\mathbb{Z}/2[a_{ij}; \text{all } i, j \geq 0]\) by the relations of \((1), (2), (3), (4)\).)

**Definition 2.1.7** A logarithm for \(F\) is a series \(l_F(x) = x + a_1 x^2 + a_2 x^3 + \ldots \in R[[x]]\) such that \(l_F(F(x, y)) = l_F(x) + l_F(y)\).

**Theorem 2.1.8 (Lazard)**

Every \(\mathbb{Z}/2\)-formal group law has a logarithm. This logarithm may be chosen with \(a_i = 0\) for all \(i = 2^i - 1\) and is unique in this form. We therefore have:

\[
L \cong \mathbb{Z}/2[l_i; i \neq 2^i - 1]
\]

\(l_F = x + l_2 x^2 + l_4 x^3 + l_5 x^4 + \ldots\)

and so \(F_L(x, y) = l_F^{-1}(l_F(x) + l_F(y))\) (\(l^{-1}\) means inverse series, i.e. \(l^{-1}(l(x)) = x\)).

The unique \(\phi: L \to R\) sending \(F_L\) to \(F\) is generated by sending the coefficients of \(l_{F_L}\) to the corresponding coefficients of the canonical \(l_F\).

**Proof** Fröhlich (14) or see 3.5.4, which applies for \(p = 2\).

**Theorem 2.1.9 (Quillen & tom Dieck)**

\(N^*_{\leq L}\). The formal group law \(F_N\) is isomorphic to \(F_L\).

**Proof**

(An outline of a proof only will be given here; it will be in the spirit of Boardman's work (44) and all details are explicit or implicit there.)

By the theorem of Thom (41) (or see Rourke (34)), the Thom spectrum \(\mathbb{Q}\) for \(N^*(-)\) is equivalent to a product of copies of \(K(\mathbb{Z}/2)\) (Eilenberg-Maclane spectra).
Thus we have a canonical ring isomorphism:

$$N^*(X) \cong \bigwedge_{\mathcal{A}_2} \text{Hom}_{\mathbb{Z}/2} \left[ H^*(\mathbb{M}_0; \mathbb{Z}/2), H^*(X; \mathbb{Z}/2) \right]$$

where $\mathcal{A}_2$ denotes the $\mathbb{Z}/2$-Steenrod algebra, and where the ring structure on the last group is made from cup product on $H^*(X; \mathbb{Z}/2)$ together with the comultiplication on $H^*(\mathbb{M}_0; \mathbb{Z}/2)$ induced by the product map of spectra $\mathbb{M}_0 \times \mathbb{M}_0 \to \mathbb{M}_0$.

So $N^*(X) \cong \bigwedge_{\mathbb{Z}/2} \left[ H^*(\mathbb{M}_0; \mathbb{Z}/2), H^*(X; \mathbb{Z}/2) \right] \cong H^*(X; \mathbb{Z}/2) \otimes H^*(\mathbb{M}_0; \mathbb{Z}/2)$.

$B$ is called the Boardman map. ($\ ^\wedge$ denotes completion with respect to neighbourhoods $\otimes H^*(X) \otimes H^*(\mathbb{M}_0)$; see Boardman (4) for details of $\otimes$). To prove 2.1.9 we shall need the following:

**Lemma 2.1.10 (Boardman)** There is an algebra isomorphism $H^*(\mathbb{M}_0; \mathbb{Z}/2) \cong \mathbb{Z}/2[a_1, a_2, \ldots]$ under which the Boardman map sends:

$$e_N(t) \longrightarrow f(t) = t + a_1 t^2 + a_2 t^3 + \ldots$$

$N^*(BO(1)) \cong H^{**}(BO(1); \mathbb{Z}/2) \otimes \mathbb{Z}/2[a_1, a_2, \ldots]$.

**Proof of lemma**

$BO(1) = \mathbb{R}Z/2$ so $H^*(BO(1); \mathbb{Z}/2) \cong \mathbb{Z}/2[t]$

$BO(1) \times \cdots \times BO(1) \to BO(n)$ induces a "splitting map":

$$H^*(BO(n); \mathbb{Z}/2) \hookrightarrow H^*(BO(1) \times \cdots \times BO(1); \mathbb{Z}/2)$$

with image all symmetric functions in $t_1, \ldots, t_n$. This injection sends:

Stiefel-Whitney class $w_i \mapsto$ $i$ th elem. sym. fn. in $t_1, \ldots, t_n$.

The maps $BO(m) \times BO(n) \to BO(m+n)$ "adding bundles", correspond under the splitting to:

$$H^*(BO(m+n); \mathbb{Z}/2) \xrightarrow{m^*} H^*(BO(m); \mathbb{Z}/2) \otimes H^*(BO(n); \mathbb{Z}/2)$$

$$H^*(BO(1) \times \cdots \times BO(1); \mathbb{Z}/2) \xrightarrow{m^*} H^*(BO(1) \times \cdots \times BO(1)) \otimes H^*(BO(1) \times \cdots \times BO(1))$$

$$t_i \otimes 1 \quad (1 \leq n)$$

$$t_i \otimes t_j \quad (1 > n)$$

From the Stiefel-Whitney classes $w_i$ we obtain a basis for $H^*(BO; \mathbb{Z}/2)$ as follows:
The evaluation map \( e: H^*(BO(n); \mathbb{Z}/2) \otimes H_*(BO(n); \mathbb{Z}/2) \to \mathbb{Z}/2 \) has dual \( e': \mathbb{Z}/2 \to H_*(BO(n); \mathbb{Z}/2) \hat{\otimes} H^*(BO(n); \mathbb{Z}/2) \).

The image of \( l \) under the dual to the evaluation map is called a universal element by Boardman (see (4) for details of this approach to duality).

Consider the element

\[
u_n = (1 + a_1 t_1 + a_2 t_1^2 + \ldots)(1 + a_1 t_2 + a_2 t_2^2 + \ldots) \in H^*(BO(1)^n; \mathbb{Z}/2) \hat{\otimes} \mathbb{Z}/2[a_1, a_2, a_3, \ldots]
\]

(\( a_i \) variable in dim. 1)

\( u_n \) may be viewed as a universal element giving an isomorphism of some subspace \( A_n \) of \( \mathbb{Z}/2[a_1, a_2, \ldots] \) to the dual \( B_n' \) of a subspace \( B_n \) of \( H^*(BO(1)^n; \mathbb{Z}/2) \).

\[
u_n = 1 + a_1(t_1 + \ldots t_n) + a_2(t_1^2 + \ldots + t_n^2) + a_1(t_2 + t_3 + \ldots + t_n t_{n-1} + \ldots)
\]

It is clear that each polynomial in the \( a_i \)'s of total degree \( \leq n \) is represented in \( u_n \) with a different non-zero coefficient. Thus \( A_n \cong \text{subspace of } \mathbb{Z}/2[a_1, a_2, \ldots] \) of polynomials of degree \( \leq n \).

Also each coefficient is a symmetric polynomial in \( t_1, \ldots, t_n \), so that \( B_n \subset H_*(BO(n); \mathbb{Z}/2) \subset H_*(BO(1)^n; \mathbb{Z}/2) \). But \( A_n \) is the same size as \( H_*(BO(n); \mathbb{Z}/2) \) in each dimension, so \( B_n \cong H_*(BO(n); \mathbb{Z}/2) \), and \( u_n \) gives an isomorphism \( A_n \cong H_*(BO(n); \mathbb{Z}/2) \).

In the limit \( n \to \infty \), \( BO(n) \to BO \), and \( u \) gives an isomorphism \( \mathbb{Z}/2[a_1, a_2, \ldots] \cong H_*(BO; \mathbb{Z}/2) \); further, from the form of \( u = (1 + a_1 t_1 + a_2 t_1^2 + \ldots)(1 + a_1 t_2 + \ldots) \ldots \), this is an isomorphism of algebras (where the algebra structure on \( H_*(BO; \mathbb{Z}/2) \) is dual to the coalgebra structure on \( H^*(BO; \mathbb{Z}/2) \), given by (1)).

The Thom isomorphism \( H^*(BO(n); \mathbb{Z}/2) \cong H^{*+n}(MO(n); \mathbb{Z}/2) \) is multiplication by \( t_1 t_2 \ldots t_n \). Applying this to the element \( u_n \) we
get an element:

\[ u' = (t + a_1 t^2 + a_2 t^3 + \ldots)(t + a_1 t^2 + \ldots) \ldots (t + a_i t^3 + \ldots) \]

\[ H^{**}(M\Omega(1); \mathbb{Z}/2) \hat{\otimes} \mathbb{Z}/2[a_1, a_2, a_3, \ldots] \]

\[ (H^{**}(M\Omega(n)) \hookrightarrow H^{**}(BO(n)) \hookrightarrow H^{**}(BO(1) \vee \ldots \vee BO(1))) \]

In the limit as \( n \to \infty \) \( u' \) gives an algebra isomorphism:

\[ \mathbb{Z}/2[a_1, a_2, a_3, \ldots] \cong H_{*+n}(M\Omega; \mathbb{Z}/2) \quad (\cong \lim_{\to} H_{*+n}(M\Omega(n); \mathbb{Z}/2)) \]

(The multiplication on \( H_{*}(M\Omega; \mathbb{Z}/2) \) comes from \( M\Omega \times M\Omega \to M\Omega \),
which in turn comes from the maps \( BO(m) \times BO(n) \to BO(m+n) \).)

The identity map \( \epsilon^{M\Omega, M\Omega} \) gives an element \( \epsilon \in N^{0}(M\Omega) \)

\( (\cong \lim_{\to} N^{0}(M\Omega(n))) \). Under the Boardman map \( B: N^{*}(M\Omega) \hookrightarrow \]

\[ H^{*}(M\Omega; \mathbb{Z}/2) \hat{\otimes} H_{*}(M\Omega; \mathbb{Z}/2) \]

maps to the identity map in

\[ \text{Hom}_{\mathbb{Z}/2}[H^{*}(M\Omega; \mathbb{Z}/2), H^{*}(M\Omega; \mathbb{Z}/2)] \]

i.e. to the universal element in

\[ H^{*}(M\Omega; \mathbb{Z}/2) \hat{\otimes} H_{*}(M\Omega; \mathbb{Z}/2) \], which in terms of our bases is \( u' \).

The identity map \( \epsilon^{M\Omega(n), M\Omega(n)} \) gives rise to an element

\( \epsilon_{n} \in N^{0}(M\Omega(n)) ; (\epsilon = \lim_{\to} \epsilon_{n}) \). Under the Boardman map

\[ B: N^{*}(M\Omega(n)) \hookrightarrow H^{*}(M\Omega(n); \mathbb{Z}/2) \hat{\otimes} H_{*}(M\Omega; \mathbb{Z}/2) \]

\( \epsilon_{n} \) maps to the

restriction of \( u' \), i.e. \( \epsilon_{n} \) maps to the element

\[ u'_{n} \in H^{*}(M\Omega(n)) \hat{\otimes} \mathbb{Z}/2[a_1, a_2, \ldots] \]

Thus, in particular, \( \epsilon_{n} \) represents an element of \( N^{1}(M\Omega(1)) \)
which maps to:

\[ t + a_1 t^2 + a_2 t^3 + \ldots \in H^{*}(M\Omega(1); \mathbb{Z}/2) \hat{\otimes} \mathbb{Z}/2[a_1, a_2, \ldots] \]

But: \( \epsilon_{n} \) is just the Thom class \( \epsilon \in N^{1}(T(\xi)) \) of the canonical
line bundle \( \xi \) over \( BO(1) \) so \( \epsilon = \epsilon_{n}(\xi) \) (where \( 1: BO(1) \hookrightarrow M\Omega(1) \).
Thus, by the naturality of \( B \), we have:

\[ e_{n}(\xi) \xrightarrow{B} f(t) = t + a_1 t^2 + a_2 t^3 + \ldots \]

\[ N^{1}(BO(1)) \quad H^{*}(BO(1); \mathbb{Z}/2) \hat{\otimes} \mathbb{Z}/2[a_1, a_2, a_3, \ldots] \]

completing the proof of the lemma 2.1.10.

We now return to the proof of the theorem 2.1.9:-
Let \( m \) be the multiplication map \( BO(1) \times BO(1) \to BO(1) \) (\( BO(1) = \mathbb{Z}/2 \))

Then \( m^* : H^*(BO(1); \mathbb{Z}/2) \to H^*(BO(1) \times BO(1); \mathbb{Z}/2) \)

sends \( t \to t' + t'' \)

\[ m^* : N^*(BO(1)) \to N^*(BO(1) \times BO(1)) \]

sends \( e_n(y) \to F_n(e_n(y'), e_n(y'')) \)

Thus using \( m^* \) and the naturality of \( B \),

\[ B : F_n(e_n(y'), e_n(y'')) \to f(t' + t'') \]

Write \( X = e_n(y'), Y = e_n(y''), X_i = \text{image } X = f(t'), Y_i = f(t'') \).

Then \( B : F_n(X, Y) \to f[f^{-1}(X_i) + f^{-1}(Y_i)] \)

\[ N^*[[X, Y]] \to \mathbb{Z}/2[a_1, a_2, \ldots][[X, Y]] \]

Let \( f^{-1}(X) = x + b_1 x^2 + b_2 x^3 + \ldots \) be the series formally inverse to \( f \), so that \( b_i = a_i + \text{composite terms} \)

\[ b_1 = a_1 + \ldots \]

i.e. \( b_i \) are poly. gens for \( \mathbb{Z}/2[a_1, a_2, \ldots] \)

Consider the map \( \mathbb{Z}/2[a_1, a_2, \ldots] \to L \)
given by

\[ \begin{cases} 
  b_{2i-1} & \to 0 \\
  b_i & \to 1 \end{cases} \]

otherwise

Then \( \mathcal{E} B : N^* \to \mathbb{Z}/2[a_1, a_2, \ldots] \to L \)

sends \( F_n(X, Y) \to l_{-1}(1(X_i) + 1(Y_i)) = F_n(X, Y) \)

\[ N^*[[X, Y]] \to L[[X, Y]] \]

So \( \mathcal{E} B : N^* \to L \) splits the canonical \( \phi : L \to N^* \) (recall from 2.1.8 that \( \phi \) is the unique ring map sending \( F_n \) to \( F_n \)). Thus \( \phi \) is injective; to complete the theorem it only remains to show \( \phi \) is epimorphic and we do this by showing \( N^* \) and \( L \) are the same size in each dimension:

\[ H^*(M\Omega; \mathbb{Z}/2) \text{ free over } \mathcal{A}_+ \implies H^*(M\Omega; \mathbb{Z}/2) \cong T^* M\Omega \otimes S_2 \cong N^* \otimes S_2 \]

(where \( S_2 \) is the dual to \( \mathcal{A}_+ \); Milnor (27) shows \( \mathcal{S}_2 \cong \mathbb{Z}/2[S_1, \lambda_2, \ldots] \)

\( \lambda_i \) in dimension \( 2^{i-1} \).

But \( H^*(M\Omega; \mathbb{Z}/2) \cong \mathbb{Z}/2[a_1, a_2, a_3, \ldots] \) so \( N^* \) is the same size as \( L \cong \mathbb{Z}/2[l_2, l_4, \ldots, l_i, \ldots] \) (i.e. \( 2^{i-1} \)).
2.2 The ring splitting $\nu: \mathbb{H}^*(-; \mathbb{Z}/2) \to N^*(-)$

Proposition 2.2.1 Any stable multiplicative natural transformation $\nu: \mathbb{H}^*(-; \mathbb{Z}/2) \to N^*(-)$ is determined by its value on the space $B\mathbb{Z}/2$.

Proof Let $\nu: \mathbb{H}^*(B\mathbb{Z}/2; \mathbb{Z}/2) \to N^*(B\mathbb{Z}/2)$

send $t \mapsto f(e_n(t))$ (f a power series over $N^*$)

Since $\nu$ is multiplicative $f$ determines $\nu$ on $\mathbb{H}^*(B\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2[t]$

Let $K^{2n}$ be the 2n-skeleton of the Eilenberg-MacLane space $K(\mathbb{Z}/2, n)$. Then:

$$\mathbb{H}^*(K^{2n}; \mathbb{Z}/2) \hookrightarrow \mathbb{H}^*(B\mathbb{Z}/2 \times \cdots \times B\mathbb{Z}/2; \mathbb{Z}/2)$$

(this injection is induced by the map classifying the $\mathbb{Z}/2$-cohomology class $t_1t_2 \cdots t_n \in \mathbb{H}^*(B\mathbb{Z}/2 \times \cdots \times B\mathbb{Z}/2; \mathbb{Z}/2)$; see Steenrod & Epstein (37))

Since the Atiyah-Hirzebruch spectral sequence $\mathbb{H}^*(-; N^*) \Rightarrow N^*(-)$ collapses we have the same inclusion:

$$N^*(K^{2n}) \hookrightarrow N^*(B\mathbb{Z}/2 \times \cdots \times B\mathbb{Z}/2)$$

Thus, by multiplicativity and naturality, $\nu$ is determined on $\mathbb{H}^*(K^{2n}; \mathbb{Z}/2)$ and so on the first $n$ cohomology groups of all complexes of dimension $< 2n$. But $\nu$ is stable so this determines it on all finite dimensional complexes; however $\mathbb{H}^*(X; \mathbb{Z}/2) \cong \lim \mathbb{H}^*(X_n; \mathbb{Z}/2)$ for $X$ CW and $X_n$ n-skeleton of $X$ (see Bröcker & tom Dieck (6)) and as the spectral sequences collapse, $N^*(X) \cong \lim N^*(X_n)$. Thus $\nu$ is determined on all CW $X$.  

Corollary 2.2.2 If $\nu: \mathbb{H}^*(-; \mathbb{Z}/2) \to N^*(-)$ is as in 2.2.1, and $\nu: t \mapsto f(e_n(t))$, then $f$ is a logarithm of $F_n$.

Proof Let $\gamma$ be any $\mathbb{Z}/2$-bundle over a CW $X$ and let $w(\gamma): \mathbb{H}^1(X; \mathbb{Z}/2)$ denote its 1st Stiefel-Whitney class. Then by the naturality of $\nu$, $\nu: w(\gamma) \mapsto f(e_n(\gamma)) \in N^1(X)$
In particular let \( X = \mathbb{Z}/2 \times \mathbb{Z}/2 \) and \( \gamma' = \gamma' \circ \gamma'' \).

Then \( \gamma : w(\gamma' \circ \gamma'') \mapsto f(e_N(\gamma' \circ \gamma'')) \in N'(\mathbb{Z}/2 \times \mathbb{Z}/2) \)

But \( w(\gamma' \circ \gamma'') = t' + t'' \circ H'(\mathbb{Z}/2 \times \mathbb{Z}/2; \mathbb{Z}/2) \) and \( \nu \) preserves addition

so \( \gamma : t' + t'' \mapsto f(e_N(\gamma')) + f(e_N(\gamma'')) \)

i.e. \( f(e_N(\gamma' \circ \gamma'')) = f(e_N(\gamma')) + f(e_N(\gamma'')) \) - \( f \) is a logarithm of \( F_N \).

**Proposition 2.2.3** The stable multiplicative natural transformations \( \gamma : H^*(-; \mathbb{Z}/2) \rightarrow N^*(-) \) are in 1-1 correspondence with the logarithms of \( F_N \); the canonical \( \gamma \) is given by the canonical logarithm. (see 2.1.8)

**Proof** Let \( h^*(-) = H^*(-; \mathbb{Z}/2) \otimes R \) (\( R \) an algebra over \( \mathbb{Z}/2 \))

A sequence of elements \( u_n \in h^*(MO(n)) \) determines a stable natural transformation \( N^*(-) \rightarrow h^*(-) \) if the map \( SMO(n) \rightarrow MO(n+1) \)

sends \( u_{n+1} \rightarrow u_n \). The transformation is multiplicative if the maps \( MO(m) \times MO(n) \rightarrow MO(m+n) \) send \( u_{m+n} \rightarrow u_m \cdot u_n \).

Put \( u_i = t \circ r_i + t^2 \circ r_i + t^3 \circ r_i + \ldots \in h^*(MO(1)) = H^*(-; \mathbb{Z}/2) \otimes R \)

Then the stability and multiplicativity conditions above determine a unique sequence of elements \( u_n \). (using the injections \( h^*(MO(m+n)) \rightarrow h^*(MO(m)) \otimes h^*(MO(n)) \) etc. We omit the details-they are straightforward)

Thus each series \( t \circ r_i + t^2 \circ r_i + t^3 \circ r_i + \ldots \) determines a unique stable multiplicative natural transformation \( N^*(-) \rightarrow h^*(-) \). In particular, put \( R = L \) and let the series be \( l'(t) \) for \( l \) any logarithm of \( F \). We then get a natural transformation \( N^*(-) \rightarrow H^*(-; \mathbb{Z}/2) \otimes L \) which sends the formal group law \( F_N \) to the group law \( l'(1(-) + l(-)) \) i.e. to \( F_L \).

Thus the natural transformation is an isomorphism on the point rings and so gives an isomorphism on all \( X \).

We have proved each logarithm determines a ring isomorphism \( N^*(-) \rightarrow H^*(-; \mathbb{Z}/2) \otimes L \) and thus a ring splitting \( \gamma : H^*(-; \mathbb{Z}/2) \rightarrow N^*(-) \). 2.2.1 and 2.2.2 complete the proposition.
2.3 The Action of the Steenrod Algebra $A_2$

We use Milnor's description (27) of the action of $A_2$ on $H^*(\mathbb{Z}/2; \mathbb{Z}/2)$. Let $S_2$ be dual Hopf algebra to $A_2$.

Theorem 2.3.1 (Milnor)

$S_2 = \mathbb{Z}/2 \langle \lambda_0, \lambda_1, \ldots, \lambda_i, \ldots; \lambda_i \text{ dim. } 2^i - 1 \rangle$

The action $H^*(\mathbb{Z}/2; \mathbb{Z}/2) \otimes A_2 \rightarrow H^*(\mathbb{Z}/2; \mathbb{Z}/2)$ is described by giving the dual coaction:

$$\theta : H^*(\mathbb{Z}/2; \mathbb{Z}/2) \rightarrow H^*(\mathbb{Z}/2; \mathbb{Z}/2) \otimes S_2$$

which is a ring homomorphism generated by:

$$t \mapsto t \otimes 1 + t^1 \lambda_1 + t^2 \lambda_2 + \ldots$$

$$(H^*(\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2[t])$$

Proof: Milnor (27)

Remark 2.3.2 The formal group $H^*(\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2[t]$ has law $F_H(t', t'') = t' + t''$ and $S_2$ gives "the most general coaction preserving $F_H"$. To be precise: $(S_2, \theta)$ is universal among pairs $(A, f)$ where $A$ is a $\mathbb{Z}/2$-algebra and $f : t \mapsto t \otimes 1 + t^1 \otimes a_1 + t^2 \otimes a_2 + \ldots$ is a ring homomorphism $H^*(\mathbb{Z}/2; \mathbb{Z}/2) \rightarrow H^*(\mathbb{Z}/2; \mathbb{Z}/2) \otimes A$ which preserves the formal group law $F_H$. (It is clear that if $f(t' + t'') = f(t') + f(t'')$ (mod 2) then $f$ must be of form (1), so that $(S_2, \theta)$ is indeed universal as claimed.) Of course any coalgebra $A$ corresponding to an algebra of additive operations on $H^*(-; \mathbb{Z}/2)$ must preserve $F_H$, by naturality and the argument of 2.2.2, so this says $A_2$ is "as large as it can be".

In fact any power series transformation preserving any $\mathbb{Z}/2$-formal group law must be of form (1), so that for example $(S_2, \theta)$ is also universal among pairs $(A, f)$ where $A$ is a $\mathbb{Z}/2$-algebra and $f : e_N(\tilde{y}) \mapsto e_N(\tilde{y}) \otimes 1 + (e_N(\tilde{y})) \otimes a_1 + \ldots$ is an $N^*$-algebra homomorphism preserving the $\mathbb{Z}/2$-formal group law $F_N$. In this case $A_2$ corresponds to the tom Dieck Steenrod operations on $N^*(-)$. (12)
When we have defined "\( \mathbb{Z}/p\)-formal group laws" in Chapter 3 it will be clear that \( \mathfrak{F}_p \) (the dual to \( \mathfrak{G}_p \)) is "the most general coaction preserving such a law" in just the same way.
3. Z/p-Formal Groups

3.1 Z/p-Theories

Definition 3.1.1 (Rourke (34)) A cohomology theory $h^*(-)$ is called a Z/p-theory if and only if it satisfies the following three conditions:

1. It is connected. (i.e. $h_i(pt.) = 0$ for $i < 0$)
2. It is a ring theory.
3. It has $h_0(pt.) \neq Z/p$.

(Note that (2) & (3) $\Rightarrow h^*(X)$ is a Z/p-algebra for all $X$.)

$H^*(-)$ is further said to be commutative (in the sense of Milnor (27)) if $a \cdot b = (-1)^{dim_a \cdot dim_b} b \cdot a$

(e.g. $H^*(-; Z/p)$ is a commutative Z/p-theory; $N^*(-)$ and $H^*(-; Z/2)$ are commutative Z/2-theories.)

$H^*(BZ/p; Z/p) = P[\beta] \otimes E[\alpha]$ (see Steenrod and Epstein (37); we use $P$ to denote polynomial, $E$ exterior, algebras over $Z/p$; $\alpha \in H^1(BZ/p; Z/p)$ $\beta \in H^2(BZ/p; Z/p)$.)

Since we are looking for a Z/p-theory $V^*(-)$, universal in some sense among Z/p-theories mapping onto $H^*(-; Z/p)$, we examine theories $h^*(-)$ with $h^*(BZ/p) \cong h^* \otimes H^*(BZ/p; Z/p)$ i.e.

with $h^*(BZ/p) \cong h^*[\beta] \otimes E[\alpha]$ (this may be regarded as a "free" power series ring in $\alpha$, $\beta$, over $h^*$, as the commutativity condition ensures $\alpha \beta = 0$.)

In order to have a natural formal group structure defined on $h^*(BZ/p)$ we shall require $h^*(-)$ to be a theory having, for the universal Z/p-bundle $\xi$ a "canonical pair of Z/p-Euler classes" $(\alpha_1(\xi) \in h^1(BZ/p))$ $(\beta_1(\xi) \in h^2(BZ/p))$ (see Definition 3.3.1) analogous
to $e_{n}(\xi)\in N'(BZ/2)$ in the $Z/2$ case.

**Theorem 3.1.2 (Rourke)** For any $Z/p$-theory $h^*(-)$ the natural map $\mu:h^*(X)\to H^*(X;Z/p)$ is epimorphic for all $X$ if and only if it is epimorphic $h^*(BZ/p)\to H^*(BZ/p;Z/p)$.

**Proof** Rourke (34) (The main ingredient is a splitting principle similar to that used in 2.2.1. This is that the map $BZ/p\to BZ/p\to K(Z/p,3n)$ classifying $\alpha_\beta\alpha_\ldots\beta$ gives an injection of the $Z/p$-cohomology of the 4n-skeleton of $K(Z/p,3n)$)

**Corollary 3.1.3 (Rourke)** If $h^*(-)$ is a $Z/p$-theory and there are classes $\alpha_\beta h^i(BZ/p)$, $\beta_\beta h^i(BZ/p)$ such that $\mu(\alpha_\beta) = \alpha \in H^i(BZ/p;Z/p)$, and $\mu(\beta_\beta) = \beta \in H^i(BZ/p;Z/p)$, then $\mu:h^*(X)\to H^*(X;Z/p)$ is epimorphic for all $X$.

**Proof** Immediate.

This gives a very useful easy test for whether a $Z/p$-theory $h^*(-)$ maps onto $H^*(-;Z/p)$.

### 3.2 The Action of the Steenrod Algebra $\mathcal{A}_p(p\neq 2)$

We use Milnor's description (27) of the action of $\mathcal{A}_p$ on $H^*(BZ/p;Z/p)$. Let $S_p$ = dual Hopf algebra to $\mathcal{A}_p$.

**Theorem 3.2.1 (Milnor)**

$S_p = P[I, 1, \ldots, I; \text{dim. } 2p^i - 2] \otimes E[T, \tau, \tau_1, \ldots; \tau; \text{dim. } 2p^i - 1]$

The action $H^*(BZ/p;Z/p)\otimes S_p \to H^*(BZ/p;Z/p)$ is described by giving the dual coaction:

$\Theta: H^*(BZ/p;Z/p) \to H^*(BZ/p;Z/p)\otimes S_p$

which is a ring homomorphism generated by:

- $\alpha \mapsto \alpha I + \beta_1 \tau + \ldots + \beta_i \tau_i + \ldots$
- $\beta \mapsto \beta I + \beta_1 \xi + \ldots + \beta_i \xi_i + \ldots$

**Proof** Milnor (27).

**Remark 3.2.2** We shall define "$Z/p$-formal group law" in such a way that this is the most general coaction preserving one. (c.f. 2.3.2.)
3.3 The Inclusion $\mathbb{Z}/p \hookrightarrow S^1$

To motivate a part of the definition of a "$\mathbb{Z}/p$-formal group" we examine a special property of $H^*(-; \mathbb{Z}/p):$

Take any group inclusion $\mathbb{Z}/p \hookrightarrow S^1$ and regard it as standard from now on in this work.

\[ H^*(BS^1; \mathbb{Z}/p) \cong P[\delta] \quad (\delta \in H^2(BS^1; \mathbb{Z}/p)) \]

\[ i^*: H^*(BS^1; \mathbb{Z}/p) \hookrightarrow H^*(BZ/p; \mathbb{Z}/p) \]

\[ \delta \mapsto \beta \quad \text{(for appropriate choice of $\delta$)} \]

Definition 3.3.1 Let $h^*(-)$ be a commutative $\mathbb{Z}/p$-theory containing classes $\alpha_\lambda, h^*(BZ/p), \beta_\lambda, h^*(BZ/p)$ satisfying the conditions of Corollary 3.1.3. Then if $\beta_\lambda$ is in the image of $i^*$ we call the triple $(h^*(-), \alpha_\lambda, \beta_\lambda)$ a representative $\mathbb{Z}/p$-theory. If $\beta_\lambda = i^*\delta_\lambda$ for $\delta_\lambda \in h^2(BS^1)$ then $h^*(BS^1) \cong P[\delta_\lambda]$ as $BS^1 \cong \mathbb{C}P^\infty$. Choosing a particular polynomial generator $\delta_\lambda$ of $h^*(BS^1)$ is equivalent to assigning to each complex line bundle a canonical "Euler class" $\delta_\lambda$.

3.4 Z/p-Formal Groups

We consider commutative (in the sense of 3.3.1) graded algebras $A^*$ over $\mathbb{Z}/p$. (with $A^n = 0$ for $n \neq 0$.) Let $\alpha$ be a formal variable (graded 1) and $\beta$ be a formal variable (graded 2).

As in the $\mathbb{Z}/2$ case we examine graded $A^*$-algebra homomorphisms $m^*$ sending 1 to 1:

\[ m^*: A^*[[\beta]] \otimes E[\alpha] \longrightarrow A^*[[[\beta], \beta]] \otimes E[\alpha, \alpha] \]

These are completely determined by:

\[ \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \mapsto F\left( \left( \begin{array}{c} \alpha_1 \\ \beta_1 \end{array} \right) \right) \equiv \left( \begin{array}{c} F_1\left( \left( \begin{array}{c} \alpha_1 \\ \beta_1 \end{array} \right) \right) \\ F_2\left( \left( \begin{array}{c} \alpha_1 \\ \beta_1 \end{array} \right) \right) \end{array} \right) \]

As in the $\mathbb{Z}/2$ case, we require $F$ to satisfy certain properties corresponding to the group structure of $\mathbb{Z}/p$ (c.f. 2.1.6).
Write $\gamma_i$ for $\left(\frac{\alpha_i}{\beta_i}\right)$, $i = 1, 2, 3$.

1. $F(0, \gamma) = F(\gamma, 0) = \gamma$ \quad \text{identity}
2. $F(\gamma_1, F(\gamma_2, \gamma_3)) = F(F(\gamma_1, \gamma_2), \gamma_3)$ \quad \text{associativity}
3. $F(\gamma_1, \gamma_2) = F(\gamma_2, \gamma_1)$ \quad \text{commutativity}
4. $F(F(\gamma, F(\gamma, F(\cdots))) = 0$ \quad \text{order p}

We have an additional property corresponding to 3.3 :-

Let $h^*(-)$ be a representative $\mathbb{Z}/p$-theory, so that we are given classes $\alpha_h, \beta_h$, with $h^*(BS') \hookrightarrow h^*(BS)$

$m: S' \times S' \rightarrow S'$ induces $m: B(S' \times S') \approx BS' \times BS' \rightarrow BS'$ (since $S'$ is abelian.)

This gives $m^*: h^*(BS') \rightarrow h^*(BS' \times BS')$

$$h^*[\gamma_h] \rightarrow h^*[\gamma_h' \gamma_h'']$$

say $\gamma_h' \rightarrow F(\gamma_h, \gamma_h'')$

Thus $F_2\left(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}\right) = i^*\left(F(\gamma_h', \gamma_h'')\right)$ \quad (since $i: \mathbb{Z}/p \hookrightarrow S'$ is a group homomorphism.)

so $F_2$ involves only $\beta_1, \beta_2$.

This motivates the property:

5. $F_2\left(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}\right)$ is independent of $\alpha_1$ and $\alpha_2$.

Definition 3.4.1 An $F = \left(\frac{F_1}{F_2}\right)$ satisfying (1), (2), (3), (4), & (5) is called a $\mathbb{Z}/p$-formal group law.

$F_1$ has the general form:

$$F_1\left(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}\right) = \alpha_1 + \alpha_2 + \alpha_1 \phi_0(\beta_1, \beta_2) + \alpha_2 \phi_0(\beta_2, \beta_1) + \phi_1(\beta_1, \beta_2) + \phi_2(\beta_1, \beta_2)$$

(Note that (3) gives immediately that $H_0'(\beta_1, \beta_2) = H_0(\beta_2, \beta_1)$.)

$F_2$ has the general form:

$$F_2\left(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}\right) = \beta_1 + \beta_2 + \phi_0(\beta_1, \beta_2).$$
Every representative $\mathbb{Z}/p$-theory $h^*(-)$ (see 3.3.1) carries a $\mathbb{Z}/p$-formal group law: $h^*(\mathbb{BZ}/p) \rightarrow h^*(\mathbb{BZ}/p \times \mathbb{BZ}/p)$ induced by
\[ h^*[\rho_1] \otimes E[\alpha_h] \]
the multiplication map: $\mathbb{Z}/p \times \mathbb{Z}/p \rightarrow \mathbb{Z}/p$.

**Definition 3.4.2**  A logarithm $l$ for $F$ is a pair of series
\[
\mathbf{l}^l \left( \begin{array}{c}
\alpha \\
\beta 
\end{array} \right) : \\
\mathbf{l}_1(\gamma) = \mathbf{l}_1(\gamma) = \alpha + \sigma_1 \beta + \gamma_2 \alpha \beta + \sigma_3 \beta^2 + \gamma_4 \alpha + \beta^2 + \ldots \\
\mathbf{l}_2(\gamma) = \mathbf{l}_2(\gamma) = \beta + x_2 \beta^2 + x_4 \beta^3 + x_6 \beta^4 + \ldots \\
such that \quad \mathbf{l}_i \left( \begin{array}{c}
\mathbf{F}_1(\gamma_i) \\
\mathbf{F}_2(\gamma_i) 
\end{array} \right) = \left( \begin{array}{c}
\mathbf{1}_i(\gamma_i) + \mathbf{1}_i(\gamma_i) \\
\mathbf{1}_i(\gamma_i) + \mathbf{1}_i(\gamma_i) 
\end{array} \right) \quad (i = 1, 2) \\
(\mathbf{l}_2 \text{ involves only } \beta \text{ because of (5).})
\]

**Proposition 3.4.3**  Let $h^*(-)$ be a representative $\mathbb{Z}/p$-theory, with formal group law $F_h$, and suppose the ring epimorphism
\[ \mu: h^*(-) \rightarrow h^*(-; \mathbb{Z}/p) \]
has a stable multiplicative natural splitting $\gamma: h^*(-; \mathbb{Z}/p) \rightarrow h^*(-)$. If $\gamma: h^*(\mathbb{BZ}/p; \mathbb{Z}/p) \rightarrow h^*(\mathbb{BZ}/p)$ is given by $\left( \begin{array}{c}
\alpha \\
\beta 
\end{array} \right) \mapsto f(\alpha_h) = \left( \begin{array}{c}
\mathbf{F}_1(\gamma) \\
\mathbf{F}_2(\gamma) 
\end{array} \right)$ then $f$ is a logarithm for $F_h$. (c.f. 2.2.2)

**Proof**  Because $H^*(\mathbb{pt}; \mathbb{Z}/p) \cong \mathbb{Z}/p$ is all in dimension 0, the $\mathbb{Z}/p$-formal group law on $h^*(\mathbb{BZ}/p; \mathbb{Z}/p)$ is given by:
\[ \alpha \mapsto \alpha' + \alpha'' \quad \beta \mapsto \beta' + \beta'' \]
Then precisely the same argument as 2.2.2 gives:
\[ f \left( \begin{array}{c}
\alpha_h^l \\
\beta_h^l 
\end{array} \right) = f(\alpha_h^l) + f(\alpha_h^l) \\
so f \text{ is a logarithm for } F_h. \]

**Remark 3.4.4**  As in 2.2.1 we can prove that any $\gamma$ is determined by its value on the space $\mathbb{BZ}/p$. (All that is needed is the splitting principle used by Rourke (34) or 3.1.2) For the universal theory $V^*(-)$, constructed in later chapters, we can show, as in 2.2.3, that each logarithm $f$ of $F_V$ determines a natural ring splitting $\gamma$ of $\mu: V^*(-) \rightarrow h^*(-; \mathbb{Z}/p)$. The canonical logarithm (see 3.5.1 below) gives a canonical splitting .
3.5 The Universal $\mathbb{Z}/p$-Formal Group

We use elaborations of methods of Lazard (25) to prove that every $\mathbb{Z}/p$-formal group law has a logarithm. (The proof is long and complicated). The structure of the universal law is an easy corollary.

**Theorem 3.5.1** Let $F$ be a $\mathbb{Z}/p$-formal group law over ground ring $A^*$, and suppose

$$F_1((\alpha_1'),(\alpha_2')) = \alpha_1 + \alpha_2 + \alpha_1H_0(\beta_1, \beta_2) + \alpha_1H_0(\beta_2, \beta_1) + H_1(\beta_1, \beta_2) +$$

$$+ \alpha_1\alpha_2H_2(\beta_1, \beta_2)$$

$$F_2((\beta_1'),(\beta_2')) = \beta_1 + \beta_2 + G_0(\beta_1, \beta_2)$$

Then (i) $H_2(\alpha_1', \beta_2) = 0$

(ii) $F$ has a logarithm, i.e. a pair of series $l = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$

$$l_1(\alpha_1') = \alpha + \sigma_1 \beta_1 + y_1 \alpha \beta_1 + \sigma_3 \beta_1^3 + y_4 \alpha \beta_1^4 + \ldots$$

$$l_2(\beta_2') = \beta + x_1 \beta_2^1 + x_2 \beta_2^2 + x_3 \beta_2^3 + x_4 \beta_2^4 + \ldots \quad \text{(with } x_i, y_i, \sigma_i \in A^*)$$

such that $l_1(F_1(\gamma_1', \gamma_2')) = l_1(\gamma_1') + l_1(\gamma_2')$ (1 = 1, 2)

($\gamma$ denotes $(\beta_2')$)

This logarithm is unique if we require that $\sigma_{2i+1} = 0$ for $i \equiv p^{-1}_2$, \(x_{2i} = 0\) for $i \equiv p^{-1}_2$

**Corollary 3.5.2** The universal $\mathbb{Z}/p$-formal group law $F^*_A$ lies over the ground ring $A^*_V$ where:

$$A^*_V = F[x_{2i}; i \equiv p^{-1}_2] \otimes E[\sigma_{2i+1}; i \equiv p^{-1}_2] \otimes F[y_{2i}; \text{ all } i > 0]$$

($x_{2i}$ has grading $-2i$)

$$\sigma_{2i+1} \quad \# \quad -(2i+1)$$

$$y_{2i} \quad \# \quad -2i$$

$F^*_A = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ is the (unique) solution of:

$$l_1(F_1(\gamma_1', \gamma_2')) = l_1(\gamma_1') + l_1(\gamma_2') \quad (1 = 1, 2)$$

where $l_1$ are as above.
To prove 3.5.1 we need various lemmas and corollaries. 3.5.3 is adapted from Lazard, (see Fröhlich (14) for similar results). 3.5.4 is Lazard's Theorem. 3.5.5, 6, 7 are proved using techniques based on Lazard's methods.

**Lemma 3.5.3** Let $A$ be an algebra over $\mathbb{Z}/p$, and $\Gamma(X,Y)$ a homogeneous polynomial of degree $n$ over $A$, then:

\[
\Gamma \text{ satisfies } (\star) \left\{ \begin{array}{l}
(1) \Gamma(X,0) = \Gamma(0,X) = 0 \\
(\it{i}) \Gamma(X,Y) + \Gamma(X+Y,Z) = \Gamma(X,Y+Z) + \Gamma(Y,Z) \\
(\it{ii}) \Gamma(X,Y) = \Gamma(Y,X) \\
(\it{iii}) \Gamma(X,X) + \Gamma(X,2X) + \ldots + \Gamma(X,(p-1)X) = 0
\end{array} \right.
\]

$\Gamma$ is of the form $a([X+Y]^n - X^n - Y^n)$ for some $a \in A$.

**Proof** (a) Functions of the form $a([X+Y]^n - X^n - Y^n)$ satisfy $(\star)$:-(i),(iii),(iv) are immediate.

1.h.s. of (ii) = $a([X+Y]^n - X^n - Y^n) + a([X+Y+Z]^n - [X+Y]^n - Z^n)
\]

= $a([X+Y+Z]^n - X^n - Y^n - Z^n)$ = r.h.s. of (i)

by symmetry.

(b) If $\Gamma$ satisfies $(\star)$ then it is of the form $a([X+Y]^n - X^n - Y^n)$:

Suppose $\Gamma = \sum_{i \in \mathbb{Z}_p} a_i X^i Y^{n-i}$

(1) gives $a_0 = a_n = 0$

(iii) gives $a_i = a_{n-i}$

Equating coefficients of $X^i Y^j Z^n - (i+j)$ in (ii) we obtain:

$a_{ij}(i+j) = a_i(n-1)$

These give sufficient relations, for:
Case 1: n not divisible by p

If \( k \) is not divisible by \( p \) either, we have (putting \( i=1, j=k-1 \)):

\[
a_k = \frac{1}{k(k-1)} a_i = \frac{1}{n} a_i.
\]

If \( k \) is divisible by \( p \), then \( n-k \) is not, so:

\[
a_k = a_{n-k} = \frac{1}{n-k} \left( \frac{n-1}{n-1-k} \right) a_i = \frac{1}{n} a_i.
\]

Thus \( \sum_{i=1}^{n-1} a_i x^i y^{n-i} = \sum_{i=1}^{n-1} \frac{1}{n} a_i x^i y^{n-i} = a_n [x^p - x^p - y^p] \) as required.

Case 2: \( n=p \)

Again \( a_k = \frac{1}{k(k-1)} a_i \) \((1 \leq k \leq n-1)\)

Thus \( \Gamma'(X,Y) = a_i \left( \frac{1}{p} [x^p - x^p - y^p] \right) \) (division by \( p \) having an obvious meaning here).

Now (iv) gives:

\[
a_i \left( \frac{1}{p} [2x] - x^p - [3x] - [2x^p - x^p + \ldots + [p x^p] - [(p-1)x^p - x^p] \right) = 0.
\]

i.e. \( a_i \left( \frac{1}{p} px^p \right) = 0 \) so \( a_i = 0 \).

Thus \( \Gamma'(X,Y) \equiv 0 \equiv a([x^p - x^p - y^p] \) as required.

Case 3: \( n=pq \) \((q \geq 1)\)

\[
a_{k+q-1} = \frac{(k+q-1)}{(p-1)} a_{n-k}.
\]

\( k=1 \) gives \( a_i = 0 \).

For any \( k \) not divisible by \( p \) we have: \( a_k = \frac{1}{k(k-1)} a_i = 0 \).

Thus \( \Gamma(X,Y) \equiv \Gamma'(X^p,Y^p) \) where \( \Gamma' \) is a homogeneous polynomial, which clearly satisfies (i), (ii), & (iii) (but not obviously (iv))

If \( q \) is not divisible by \( p \) we apply Case 1 to \( \Gamma' \) (only (i), (ii) & (iii) were needed in the proof of Case 1), and obtain:

\[
\Gamma(X,Y) \equiv \Gamma'(X^p,Y^p) \equiv a([x^p + y^p - x^p - y^p])
\]

\[= a([x^p - y^p - x^p]) \text{ as required.}
\]

If \( q \) is divisible by \( p \) we apply the above process to \( \Gamma' \) to get \( \Gamma'' \), and repeat until we reach a \( q \) not divisible by \( p \);
the only remaining case is if we eventually arrive at a \( \Gamma^{(r)} \) of degree \( q = p \). (i.e. when \( n = p^{r+1} \)). Then we have, by the

Case 2 argument:

\[
\Gamma(X, Y) = \Gamma^{(r+1)}(X^r, Y^r) = a_1 \left( \frac{1}{p^n} \left( \left[ X^r + Y^r \right]^r - X^r - Y^r \right) \right) + a_i \left( \frac{1}{p} \left( [X+Y]^r - X^r - Y^r \right) \right)
\]

and as in Case 2, \( \Gamma \) satisfying (iv) gives \( a_1 = 0 \) (whether or not \( \Gamma^{(r)} \) satisfies (iv).)

Thus \( \Gamma(X, Y) = 0 \equiv a \left( \left[ X+Y \right]^r - X^r - Y^r \right) \) as required.

**Corollary 3.5.4 (Lazard's Theorem)** Let \( A \) be an algebra over \( \mathbb{Z}/p \), and \( F_2(X, Y) \), a power series over \( A \) satisfying:

1. \( F_2(0, 0) = 1 \) - identity
2. \( F_2(F_2(X, Y), Z) = F_2(F_2(X, Y), Z) \) - associativity
3. \( F_2(X, Y) = F_2(Y, X) \) - commutativity
4. \( [F_2(X)] = F_2(F_2(X, X), X) \) - order \( p \)

Then there is a series \( l_2(X) = X + x_2X^2 + x_4X^4 + \ldots \in A[[X]] \) such that \( l_2(F_2(X, Y)) = l_2(X) + l_2(Y) \). \( l_2 \) is unique if we require that \( x_2 = 0 \) for \( i = p - 1 \).

**Proof** Suppose \( l^{(n)}_2 \) has been constructed inductively to give:

\[
l^{(n)}_2(F_2(X, Y)) = l^{(n)}_2(X) + l^{(n)}_2(Y) \quad (\equiv \text{denotes modulo terms of degree } \geq n)
\]

i.e. \( l^{(n)}_2(F_2 \left( l^{(n)}_2(X), l^{(n)}_2(Y) \right)) = X + Y \) \( \ldots \) \( \dagger \)

The l.h.s. of \( \dagger \) satisfies (1), (2), (3), (4) because \( F_2 \) does and they are easily seen to be properties preserved by power series transformations. Write l.h.s. of \( \dagger \) as \( G(X, Y) \):

Then \( G(X, Y) = \pi_n X + Y + \pi_n(X + Y) \quad (\pi \text{ homogeneous of degree } n) \)

\( \Gamma \) satisfies (i) & (iii) of 3.5.3 because \( G \) satisfies (1) & (3) \( \Gamma \) satisfies (ii) of 3.5.3 because \( G(X, Y) \) satisfies (2):-

i.e. \( G(X, G(Y, Z)) = G(G(X, Y), Z) \)
so \( X + (Y + Z + \Gamma(Y, Z)) + \Gamma(X, Y + Z + \Gamma(Y, Z)) = \Gamma(X + Y + \Gamma(X, Y)) + Z + \Gamma(X + Y + \Gamma(X, Y)). \)

so \( \Gamma(Y, Z) + \Gamma(X, Y + Z) \equiv_{\text{p}} \Gamma(X, Y) + \Gamma(X + Y, Z) \)

\( \Gamma \) satisfies (iv) of 3.5.3 because \( G(X, Y) \) satisfies (4):

\[
G(X, G(X, G(X, \ldots ))) = 0
\]

Now \( G(X, X) \equiv_{\text{p}} X + X + \Gamma(X, X) \)

so \( G(X, G(X, X)) \equiv_{\text{p}} (X + X + \Gamma(X, X)) + X + \Gamma(X, 2X + \Gamma(X, X)) \)

\[
= \Gamma(X + X) + \Gamma(X, 2X)
\]

\[
\ldots
\]

\[
0 = G(X, G(X, G(X, \ldots ))) \equiv_{\text{p}} pX + \Gamma(X, X) + \Gamma(X, 2X) + \ldots + \Gamma(X, (p-1)X)
\]

so \( \Gamma \) satisfies (iv).

Thus by lemma 3.5.3 \( \Gamma = a_n([X+Y]^n - X^n - Y^n) \) for some \( a_n \in A \).

Now put \( f(X) = X - a_nX^n \) and we have:\( (\text{since } G(X, Y) \equiv_{\text{p}} X + Y + \Gamma(X, Y) ):\)

\[
f(G(f^{-1}(X), f^{-1}(Y))) \equiv_{\text{p}} X + Y.
\]

So, putting \( l^{(a)}_2(X) = f(l^{(a)}_2(X)) \) we have \( l^{(a)}_2(F_2((l^{(a)}_2)^{-1}(X), (l^{(a)}_2)^{-1}(Y))) = X + Y \).

i.e. \( l^{(a)}_2(F_2(X, Y)) \equiv_{\text{p}} l^{(a)}_2(X) + l^{(a)}_2(Y) \) completing the induction.

The existence of \( l^{(a)}_2 \) follows by the completeness of \( A[[X]] \).

The \( a_n \) are given uniquely in the proof above (by the proof of 3.5.3) except when \( n = p^k \), in which case (again by the proof of 3.5.3) \( \Gamma \equiv 0 \), and we may then put \( x_{2i} = 0 \) for \( i = p^k - 1 \) and obtain a unique logarithm.
Corollary 3.5.5  Let A be an algebra over \( Z/p \), \( J(X,Y) \) a power series in \( X,Y \) over \( A \), \( F_2(X,Y) \) satisfying (1),(2),(3) & (4) of 3.5.4, then:

\[
\begin{align*}
(1) & \quad J(X,0) = 0 = J(0,X) \\
(11) & \quad J(X,Y) + J(F_2(X,Y),Z) = J(Y,Z) + J(X,F_2(Y,Z)) \\
(1ii) & \quad J(X,Y) = J(Y,X) \\
(1iv) & \quad J(X,X) + J(X,F_2(X,X)) + \ldots + J(X,F_2(X,F_2(X,X),\ldots X)) = 0
\end{align*}
\]

\( \iff \) \( J \) is of the form: \( f(F_2(X,Y)) - f(X) - f(Y) \) for some power series \( f(X) = a_i X^i \ldots \in A[[X]] \). The series is unique (up to multiplication by elements of \( A \)), if we require the coefficient of \( X^j \) to be zero for each \( j \).

**Proof**

(a) Functions of this form satisfy (**) :-

1. **h.s.** of (11) = \( f(F_2(X,Y)) - f(X) - f(Y) + f(F_2(F_2(X,Y),Z)) - f(F_2(X,Y)) - f(Z) = f(F_2(F_2(X,Y),Z)) - f(X) - f(Y) - f(Z) = r.h.s. of (11) by symmetry (\( F_2 \) is associative).

(b) If \( J \) satisfies (**) then it has the required form:-

Suppose \( f_{(n\cdot 0)} \) has been constructed inductively to give:

\[
J_{(n\cdot 0)} = f_{(n\cdot 0)}(F_2(X,Y)) - f_{(n\cdot 0)}(X) - f_{(n\cdot 0)}(Y)
\]

\( J_{(n-1)} \) satisfies (**) (by (a)) and \( J \) does by hypothesis.

Let \( J = \mu + \sum_{j=0}^{\infty} \Gamma_j X^j \) (\( \Gamma \) homogeneous of degree \( n \), \( \Gamma \) satisfies (1) & (iii) of (*) 3.5.3 because \( J \) & \( J_{(n-1)} \) satisfy (1) & (iii) of (**) (immediate)

\( \Gamma \) satisfies (ii) & (iv) of (**) 3.5.3 because \( J \) & \( J_{(n-1)} \) satisfy (ii) & (iv) of (**) (almost immediate - using \( F_2(X,Y) = X+Y \cdot \ldots \).
Thus \( \Gamma(\mathbf{X}, \mathbf{Y}) = a_n([\mathbf{X} + \mathbf{Y}]^n - \mathbf{X}^n - \mathbf{Y}^n) \) for some \( a_n \) by lemma 3.5.3.

Hence \( j - j_{n-1} = a_n((\mathbf{F}_2(\mathbf{X}, \mathbf{Y}))^n - \mathbf{X}^n - \mathbf{Y}^n) \), completing the induction step.

The uniqueness of the canonical form of \( f \) (with the coefficient of \( \mathbf{X}^n \) vanishing) follows in exactly the same way as the last part of Corollary 3.5.4.

Lemma 3.5.6 Let \( A \) be an algebra over \( \mathbb{Z}/p \), \( \Gamma(\mathbf{X}, \mathbf{Y}) \) a homogeneous polynomial of degree \( n \) over \( A \), then:

\[
\begin{align*}
\text{\( \Gamma \) satisfies (\dagger) if} & \\
(\text{i}) & \Gamma(\mathbf{X}, \mathbf{0}) = 0 \\
(\text{ii}) & \Gamma(\mathbf{X}, \mathbf{Y}) + \Gamma(\mathbf{X} + \mathbf{Y}, \mathbf{Z}) = \Gamma(\mathbf{X}, \mathbf{Y} + \mathbf{Z}) \\
(\text{iii}) & \Gamma(\mathbf{Y}, \mathbf{X}) + \Gamma(\mathbf{X} + \mathbf{Y}, \mathbf{Z}) = \Gamma(\mathbf{Y}, \mathbf{Z}) + \Gamma(\mathbf{Y} + \mathbf{Z}, \mathbf{X})
\end{align*}
\]

\( \iff \Gamma \) is of the form \( a(\mathbf{X}^n - [\mathbf{X} + \mathbf{Y}]^n) \) for some \( a \in A \).

\textbf{Proof} (c.f. 3.5.3)

\begin{enumerate}
\item [(a)] Functions of the form \( a(\mathbf{X}^n - [\mathbf{X} + \mathbf{Y}]^n) \) satisfy (\dagger) (by direct substitution).
\item [(b)] If \( \Gamma \) satisfies (\dagger) then it is of the form \( a(\mathbf{X}^n - [\mathbf{X} + \mathbf{Y}]^n) \):
\end{enumerate}

Suppose \( \Gamma = \sum_{i=0}^{n} a_i \mathbf{X}^i \mathbf{Y}^{n-i} \).

The term in \( \mathbf{X}^n \) in (ii) gives: \( a_n + a_n = a_n \Rightarrow a_n = 0 \).

Equating coefficients of \( \mathbf{X}^i \mathbf{Y}^{n-i} \) gives us:

From (i) \( a_i^i j (i+j) = a_i^i j (n-i) \) (i+j\#n)

From (iii) \( a_i^i j (i+j) = a_i^i j (n-i) \) (i+j\#n, i\#0)

Case (ii): \( n \) not divisible by \( p \)

Putting \( j = k-l, i = 1 \), we get \( a_k^1 = a_1^1 (n-1) (k-l) \)

Hence for \( k \) not divisible by \( p \): \( a_k^1 = \frac{1}{n} a_1^1 (n-1) \) (from (ii))

If \( k \) is divisible by \( p \), then \( n-k \) is not, and we have:

\( a_k^1 = \frac{1}{n(n-k)} a_1^1 \) (from (iii))
Putting $j=1$, $i=0$, gives: $a_i = na_o$.

So $\sum_{i=0}^{n} a_i X^i Y^{n-i} = a_n \left( X^n - [X+Y]^n \right)$ as required.

**Case 2: $n = p$**

Again, $a_i \begin{bmatrix} k \\ 1 \end{bmatrix} = a_i \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} (1 \leq k \leq n-1)$

So $a_k = \frac{1}{k(n-1)} a_i$.

$j=1$, $i=0$, gives $a_1 = pa_o = 0$.

So $\Gamma(X,Y) = a_o Y^p$ as required.

**Case 3: $n = pq$ ($q > 0$)**

$j=1$, $i=0$, gives $a_i = na_o = 0$

For any $k$ not divisible by $p$ we have:

$a_k = \frac{1}{k(n-1)} a_i = 0$

So $\Gamma'_e = \Gamma(X,Y) = \Gamma'(X^p, Y^q)$, where $\Gamma'$ is homogeneous of degree $q$. It is immediate that satisfies (i), (ii), (iii) and the result follows by induction, using Case 1 and Case 2.

**Corollary 3.5.7** Let $A$ be an algebra over $\mathbb{Z}/p$, $K(X,Y)$ a power series in $X,Y$ over $A$, and $F_2(X,Y)$ a power series satisfying (1), (2), (3), (4) of 3.5.4, then:

\[
\begin{cases} 
K(X,0) = 0 \\
K \text{ satisfies (\dag\dag)} \\
K(X,Y) + K(F_2(X,Y), Z)(1+K(X,Y)) = K(X,F_2(Y,Z)) \\
(1+K(Y,X))K(F_2(X,Y), Z) = K(Y,Z) + (1+K(Y,Z))K(F_2(Y,Z), X)
\end{cases}
\]

$K$ is of the form:

\[
\begin{align*}
f(X) &= l + y_2 x + y_3 x^2 + \cdots \\
f(F_2(X,Y)) &= l + y_2 F_2(X,Y) + y_3 F_2(X,Y)^2 + \cdots
\end{align*}
\]

for some $y_2 \in A$. Such $y_2$ are unique.
Proof (a) Functions of the form \( \frac{f(X)}{f(F(X,Y))} - 1 \) satisfy (\( \dagger \dagger \)):-

(i) is obvious.

(ii) l.h.s. = \( \frac{f(X)}{f(F_2(X,Y))} - 1 + \left( \frac{f(F_2(X,Y))}{f(F_2(F_1(X,Y),Z))} \right) \cdot \frac{f(X)}{f(F_2(X,Y))} \)

= \( \frac{f(X)}{f(F_2(F_2(X,Y),Z))} - 1 \) = r.h.s. (\( F_2 \) is associative)

(iii) l.h.s. = \( \frac{f(Y)}{f(F_2(Y,X))} - 1 + \left( \frac{f(F_2(Y,X))}{f(F_2(F_1(Y,X),Z))} \right) \cdot \frac{f(Y)}{f(F_2(Y,X))} \)

= \( \frac{f(Y)}{f(F_2(F_2(Y,X),Z))} - 1 \)

= \( \frac{f(Y)}{f(F_2(Y,Z))} + \frac{f(Y)}{f(F_2(F_2(Y,Z),X))} \cdot \frac{f(F_2(Y,Z))}{f(F_2(F_2(Y,Z),X))} \)

= r.h.s (\( F_2 \) is commutative & associative).

(b) If \( K \) satisfies (\( \dagger \dagger \)) then it has the required form:-

Suppose \( f_{(n,0)} \) has been constructed inductively to give:

\( K = \sum_{(n,0)} f_{(n,0)}(X) - 1 \)

\( K \) satisfies (\( \dagger \dagger \)) by (a) and \( K \) satisfies (\( \dagger \dagger \)) by hypothesis.

Let \( K = \sum_{(n,0)} f_{(n,0)}(X) - 1 \) (\( \Gamma \) homogeneous of degree \( n \)).

Then \( \Gamma \) satisfies (i), (ii), & (iii) of (\( \dagger \)) 3.5.6

because \( K \) & \( \Gamma \) satisfy (i), (ii) & (iii) of (\( \dagger \dagger \)).

\( F_2(X,Y) = X + Y + \ldots \)

Thus \( \Gamma = a(X^\infty - [X+Y]^\infty) \)

So \( K = \sum_{(n,0)} f_{(n,0)}(X) + aX^\infty - 1 = \sum_{(n,0)} f_{(n,0)}(X) + a(X+Y)^\infty - 1 \)

Putting \( f_{(n,0)}(X) = f_{(n,0)}(X) + aX^\infty \) completes the induction step. By the construction \( a \) is unique, and thus by completeness, in the limit \( f(X) \) is unique.
Proof of Theorem 3.5.1

(1): The associativity rule ((2) of 3.4) for \( F \) gives (on the \( F_4 \) term):

\[
F_4\left(\left(\left(F_2(\beta_1, \beta_2), (\min_3)\right), (\max_3)\right)\right) = F_4\left(\left(F_2(\beta_1, (\min_3)), (\max_3)\right)\right)
\]

Written out at length this is:

\[
F_4\left(\left(\beta_1, (\min_3)\right), (\max_3)\right) + \alpha_3 \cdot F_4\left(\left(\beta_1, (\max_3)\right), (\min_3)\right) + \alpha_2 \cdot F_4\left(\left(\min_3, (\min_3)\right), (\max_3)\right)
\]

Picking out from \( A \) the term in \( \alpha, \alpha_2 \):

\[
H_2(\beta_2, \beta_3) + H_2(\beta_2, \beta_3) \cdot \left[ H_2(F_2(\beta_2, \beta_3), \beta_3) \right] = \alpha_3 \cdot F_4\left(\left(\beta_1, (\min_3)\right), (\max_3)\right)
\]

Now \( \alpha_3 = -\alpha_2 \), so the commutativity rule ((3) of 3.4) for \( F \)
gives: \( H_2(\beta_2, \beta_3) = -H_2(\beta_2, \beta_3) \) and in particular \( H_2(\beta_2, \beta_3) = 0 \).

Thus, putting \( \beta_2 = \beta_3 \) in \( A_{12} \), we obtain:

\[
(1 + H_2(\beta_2, \beta_3)) \cdot H_2(\beta_2, F_2(\beta_2, \beta_3)) = 0
\]

Suppose the lowest term in \( H_2 \) has degree \( n \), i.e.:

\[
H_2(\beta_2, \beta_3) = \sum_{r_3, r_3} a_3(\beta_2^{r_3} - \beta_3^{r_3})
\]

\( H_\delta \) has degree \( > 0 \) (since \( F \) satisfies (1) of 3.4) so:

\[
H_2(\beta_2, F_2(\beta_2, \beta_3)) \equiv 0
\]

Hence:

\[
H_2(\beta_2, \beta_2 + \beta_3) \equiv 0.
\]

Take the least \( r_3 \) with \( a_3 \equiv 0 \), say \( r_3^* \), and divide by \( \beta_2^{r_3} \); then the term in \( \beta_3^{r_3} \) gives:

\[
a_3 \beta_2^{r_3} = 0. \quad \text{i.e. } a_3 = 0.
\]

Thus all \( a_i = 0 \) so \( H_2(\beta_2, \beta_3) \equiv 0. \)
(ii): Finding a logarithm is equivalent to solving:

\[ l_1(F_1(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2})) = l_1(\frac{\alpha_1}{\beta_1}) + l_1(\frac{\alpha_2}{\beta_2}) \quad \text{(I)} \]

\[ l_2(F_2(\beta_1, \beta_2)) = l_2(\beta_1) + l_2(\beta_2) \quad \text{(II)} \]

(1), (2), (3) & (4) for \( F \) ensure \( F_2 \) satisfies the conditions of Corollary 3.5.4 (Lazard's Theorem) and hence there is a unique solution to (II) of the required form, for \( l_2 \).

Writing out (I) at length:

\[ F_1(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}) = l_1 + l_2 + l_3 + \cdots + \sigma_1 \beta_1 \alpha_1 \beta_1 + \sigma_2 \beta_2 \alpha_2 \beta_2 + \cdots = \alpha(1 + \frac{\alpha_1}{\beta_1} \alpha_2 \beta_2 + \cdots + \sigma_1 \beta_1 \alpha_1 \beta_1 + \cdots) \]

Solving (I') is equivalent to solving two equations (one for the term independent of \( \alpha_1 \) & \( \alpha_2 \) and one for the term in \( \alpha_1 \); that in \( \alpha_1 \) is the same as for \( \alpha_1 \))

As usual, put \( F_1(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}) = \alpha_1 + \alpha_2 H_2(\beta_1, \beta_2) + \alpha_1 H_2(\beta_1, \beta_2) + H_1(\beta_1, \beta_2) \) in (I')

\[ (H_2 = 0) \]

Then the coefficients of \( \alpha_1 \) in (I') give us to solve:

\[ [1 + H_2(\beta_1, \beta_2)][1 + \beta_1 \beta_2 + \cdots] = 1 + \beta_1 \beta_2 + \cdots \quad \text{(III)} \]

The coefficients independent of \( \alpha_1 \) & \( \alpha_2 \) in (I') give us to solve:

\[ H_2(\beta_1, \beta_2)[1 + \beta_2 \beta_2 + \cdots + \sigma_1 \beta_1 \beta_2 + \sigma_2 \beta_2 \beta_2 + \cdots] = \sigma_1 \beta_1 \beta_2 + \sigma_2 \beta_2 \beta_2 + \cdots \quad \text{(IV)} \]

The proof now proceeds by showing that the formal group law rules for \( F \) provide sufficient conditions to give a unique solution of (III) & (IV) for \( y_2, \sigma_1, \sigma_2 \) of the required form. We assemble the necessary data:

The term of \( A \) independent of \( \alpha_1 \) & \( \alpha_2 \) gives:

\[ H_1(\beta_1, \beta_2) + H_2(F_2(\beta_1, \beta_2), \beta_2) + H_2(\beta_1, \beta_2) + H_2(F_2(\beta_1, \beta_2), \beta_1) = \]

\[ = H_1(\beta_1, \beta_2) + H_2(F_2(\beta_1, \beta_2), \beta_2) + H_2(\beta_1, \beta_2) + H_2(F_2(\beta_1, \beta_2), \beta_1) \quad \text{(A1)} \]
The term of \( A \) in \( \alpha_1 \) gives:

\[
H_\phi(\beta, \beta) + H_\phi(F_2(\beta, \beta), \beta_3)(1 + H_\phi(\beta, \beta)) = H_\phi(\beta, F_2(\beta, \beta)) \] ..........................(A2)

(The term of \( A \) in \( \alpha_3 \) is the same, by symmetry.)

The term of \( A \) in \( \alpha_2 \) gives:

\[
H_\phi(\beta, \beta)(1 + H_\phi(\beta, \beta)) \cdot H_\phi(F_2(\beta, \beta), \beta_3) = H_\phi(F_2(\beta, \beta), \beta_3)(1 + H_\phi(\beta, \beta)) H_\phi(F(\beta, \beta), \beta_3) \] ..........................(A3)

Also \( F_1((\beta, \beta), 0) = \infty \) (by condition (1) of 3.4)

So \( H_\phi(\beta, \beta)(1 + H_\phi(\beta, \beta)) = 0 \) ..........................(B1)

& \( H_\phi(\beta_3, 0) = 0 \) ..........................(B2).

(A2),(A3),(B1) give that \( H_\phi(X, Y) \) satisfies the conditions (i), (ii), & (iii) of Corollary 3.5.7. Thus \( H_\phi(\beta, \beta) \) is uniquely expressible as:

\[
\frac{1 + \psi_\beta + \psi_\beta^2 + \cdots}{1 + \psi_\beta^2(\beta, \beta) + \psi_\beta^3(\beta, \beta) + \cdots} = -1
\]

i.e. we have a unique solution to (III).

It remains to solve (IV) for \( \gamma_{x_{i+1}} \).

Writing \( \gamma_\beta + \gamma_\beta^2 + \cdots \) as \( h(\beta) \), (IV) is:

\[
H_\phi(\beta, \beta)[1 + \psi_\beta^2(\beta, \beta) + \psi_\beta^3(\beta, \beta) + \cdots] = h(\beta) + h(\beta) - h(F_2(\beta, \beta)) \] ..........................(IV')

Call the l.h.s. of (IV') \( J(\beta, \beta) \).

Now 3.3.5 gives us a unique solution of (IV') for \( h \) if and only if \( J(\beta, \beta) \) satisfies (**) of 3.5.5, so this is all that it remains to check to complete the proof of 3.5.1.

(B2) gives that \( J(\alpha, \beta) \) satisfies (i) of (**) .

\[
F_1((\alpha, \alpha), (\beta, \beta)) = F_1((\beta, \beta), (\beta, \beta)) \] (by condition (3) of 3.4)

so that \( J(\beta, \beta) \) satisfies (ii) of (**).

To show \( J(\beta, \beta) \) satisfies (**)(ii):

Substituting \( H_\phi(\beta, \beta) = \frac{1 + \psi_\beta + \cdots}{1 + \psi_\beta^2(\beta, \beta) + \cdots} = -1 \) in (Al) we have:

\[
l.h.s. = \frac{1}{1 + \psi_\beta^2(\beta, \beta) + \cdots} [J(\beta, \beta) + J(F_2(\beta, \beta), \beta_3)]
\]
Similarly \( r.h.s = \frac{1}{1+\chi(\beta_1, \beta_2, F(\beta_1, \beta_2))} \left[ J(\beta_1, \beta_2) + J(\beta_1, F(\beta_1, \beta_2)) + \cdots \right] \)

As \( F_1 \) is associative, the denominators are equal, and putting \( 1.h.s \) of \((\text{Al}) = r.h.s \) of \((\text{Al})\) gives us that \( J(\beta, \beta) \) satisfies \((**)(11)\).

To show that \( J(\beta, \beta) \) satisfies \((**)(iv)\)

\[
\left[ \frac{[n]}{F}(\gamma) \right]_{F_1} \equiv \left[ \frac{n}{F}(\gamma) \right]_{F_1} \equiv F(\gamma, F(\gamma, F(\gamma, \cdots ))) \quad (\gamma = (\frac{\alpha}{\beta}))
\]

(so \( \left[ \frac{n}{F}(\gamma) \right]_{F_1} = \left[ \frac{n}{F}(\beta) \right] \))

\( \left[ \frac{n}{F}(\gamma) \right]_{F_1} = F(\gamma, [n-1]_{F_1}(\gamma)) \)

\( \left[ \frac{n}{F}(\gamma) \right]_{F_1} = \alpha + [n-1]_{F_1}(\gamma) \)

\( \left[ \frac{n}{F}(\gamma) \right]_{F_1} = \alpha H_0(\beta, [n-1]_{F_1}(\beta)) + [n-1]_{F_1}(\gamma) H_0(\beta, [n-1]_{F_1}(\beta)) \)

Let \( G_\alpha(\beta) \) be the term of \( \left[ \frac{n}{F}(\gamma) \right]_{F_1} \) not involving \( \alpha \).

Then: \( G_\alpha(\beta) = G_{n-1}(\beta) + G_{n-1}(\beta) H_0(\beta, [n-1]_{F_1}(\beta)) \)

Substituting \( H_0(\beta, \beta) = \frac{1+y_2(\beta, \beta) + \cdots + 1}{1+y_2(\beta, \beta) + \cdots} \)

Then \( G_\alpha(\beta) = G_{n-1}(\beta) \left( \frac{1+y_2([n-1]_{F_1}(\beta)) + \cdots + H(\beta, [n-1]_{F_1}(\beta))}{1+y_2([n]_{F_1}(\beta)) + \cdots} \right) \)

Thus \( G_\alpha(\beta) = \frac{1}{1+y_2([n]_{F_1}(\beta)) + \cdots} \left[ J(\beta, [n-1]_{F_1}(\beta)) + \cdots + H(\beta, [n-1]_{F_1}(\beta)) + \cdots \right] \)

So \( G_\alpha(\beta) = \frac{1}{1+y_2([p]_{F_1}(\beta)) + \cdots} \left[ J(\beta, [p-1]_{F_1}(\beta)) + J(\beta, [p-2]_{F_1}(\beta)) + \cdots \right] \)

But \((4)\) of \( 3.4 \) gives: \( [p]_{F_1}(\gamma) = 0 \), so \( G_\alpha(\beta) = 0 \).

Thus \( J \) satisfies \((iv)\) of \((**),\) of \( 3.5.5 \), completing the proof of \( 3.5.1 \).
Proof of Corollary 3.5.2

It is sufficient to check the equations

\[ l_i \left( \mathcal{F}_i \left( \gamma_1, \lambda_1 \right) \right) = l_i \left( \gamma_1 \right) + l_i \left( \lambda_1 \right) \quad (i = 1, 2) \]

do indeed define a unique \(Z/p\)-formal group law \(\mathcal{F}_A = \left( \mathcal{F}_i \right)\).

(For then, given any \(Z/p\)-formal group \((A, \mathcal{F})\) we have a canonical ring homomorphism \(\phi : A_\mathcal{F} \rightarrow A\) under which the coefficients \(x_i, y_i, \alpha_i, \beta_i\) of \(\left( \begin{array}{c} l_i \\ l_i \end{array} \right)\) map to the corresponding coefficients of the canonical logarithm for \(\mathcal{F}\), and this map \(\phi\) is the unique ring map sending \(\mathcal{F}_A\) to \(\mathcal{F}\) such that \(x_i, y_i\) poly. and \(\alpha_i, \beta_i\) ext. give the largest ring allowed by the commutativity conditions)

We need to show first that \(\left( \begin{array}{c} l_i \\ l_i \end{array} \right)\) has a unique inverse

\[ l^{-1} = \left( \begin{array}{c} l^{-1}_i \\ l^{-1}_i \end{array} \right) \]

such that \(l^{-1}_i(l_i) = \text{id} \quad i.e. \quad \text{such that}

\[ \left( \begin{array}{c} l^{-1}_i(l_i) \\ \alpha_i \end{array} \right) = \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right) \quad \text{...... (i)} \]

\[ \left( \begin{array}{c} l^{-1}_i \\ l_i \end{array} \right) = \left( \begin{array}{c} \beta_i \\ \alpha_i \end{array} \right) \quad \text{...... (ii)} \]

Consider (ii): \(l_i \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right) = 0\) and \(l_i \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right) = \alpha + \ldots\) so \(l^{-1}_i \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right)\) is independent of \(\alpha\). Thus \(l^{-1}_i \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right)\) is uniquely defined and is just \(l^{-1}_i\) (since \(l_i(X) = X + \ldots, l_i\) has a unique inverse series \(l^{-1}_i\)).

Consider (i): \(\text{let } g \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right) = \alpha + a_i \beta + b_i \alpha \beta + a_3 \beta^2 + b_4 \alpha \beta^2 + \ldots \)

\[ = \alpha(1 + b_i \beta + b_4 \alpha \beta + \ldots) + a_i \beta + a_3 \beta^2 + \ldots \]

Then \(g \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right) = \left( \begin{array}{c} \alpha + \sigma_i \beta + y_i \alpha \beta + a_3 \beta^2 + \ldots \end{array} \right) \left( \begin{array}{c} 1 + b_1 \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right) + b_4 \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right)^2 + \ldots \end{array} \right) + \ldots \)

\[ + a_i \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right) + a_3 \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right)^2 + \ldots \]

Thus \(g \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right) = \alpha \left( \begin{array}{c} 1 + y_i \beta + y_i \alpha \beta^2 + \ldots \end{array} \right) \left( \begin{array}{c} 1 + b_1 \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right) + b_4 \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right)^2 + \ldots \end{array} \right) = 1 \)

\[ \text{...... (iii)} \]

\[ \& \left( \sigma_i \beta + y_i \alpha \beta^2 + \ldots \right) \left( 1 + b_1 \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right) + \ldots \right) + a_i \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right) = 0 \]

\[ \text{...... (iv)} \]
Since $l_{2}(\beta) = \beta + \ldots$ we can use (iii) to define $b_{2i}$ inductively. 
(iv) to define $a_{2i+1}$ inductively.

These are unique, and so we have a unique inverse $l^{1}$ to $l$, i.e. $F_{A}^{*}(\bar{F}_{1})$ is uniquely defined by $F_{A}((\tilde{\gamma}_{1}, \tilde{\gamma}_{2}, \tilde{\gamma}_{3})) = l^{-1}(l(\tilde{\gamma}_{1}) + l(\tilde{\gamma}_{2}) + l(\tilde{\gamma}_{3}))$

It only remains to check that $F_{A}$ satisfies (1), (2), (3), (4) & (5) of 3.4 and so is a $Z/p$-formal group law. This is straightforward, e.g. for (2):

$$F_{A}(F_{A}(\tilde{\gamma}_{1}, \tilde{\gamma}_{2}, \tilde{\gamma}_{3})) = l^{-1}(l(1(\tilde{\gamma}_{1}^{*}) + l(\tilde{\gamma}_{2}^{*}) + l(\tilde{\gamma}_{3}^{*})))$$

$$= l^{-1}(l(\tilde{\gamma}_{1}^{*}) + l(\tilde{\gamma}_{2}^{*}) + l(\tilde{\gamma}_{3}^{*}))$$

$$= F_{A}(\tilde{\gamma}_{1}, F_{A}(\tilde{\gamma}_{2}, \tilde{\gamma}_{3}))$$

The other conditions are just as easy.

Remark 3.5.8

Consider the cohomology theory $\tilde{H}^{*}(\cdot; Z/p) \otimes A_{V} = h^{*}(\cdot)$. If $l$ is the canonical logarithm of $F_{A}$ on $A_{V}$, and

$$l^{-1}(\alpha) = (\alpha + \alpha a_{1} + \alpha b_{2} + \alpha^{2} a_{2} + \ldots)$$

$$l^{-1}(\beta) = (\beta + \beta a_{1} + \beta^{2} a_{2} + \ldots)$$

put

$$\left(\frac{\alpha_{k}}{\beta_{k}}\right) = \left(\frac{\alpha_{1} + \alpha_{2} + \alpha_{3} + \ldots}{\beta_{1} + \beta_{2} + \beta_{3} + \ldots}\right)$$

This then gives an "algebraic" realisation of the universal $Z/p$-formal group law as a cohomology theory, since under

$$m: BZ/p \times BZ/p \to BZ/p$$

we have:

$$m^{*}: (\alpha_{1}) \mapsto (\alpha_{1} + \alpha_{2}) = l_{1}^{*}(\alpha_{1})$$

$$m^{*}: (\beta_{1}) \mapsto (\beta_{1} + \beta_{2}) = l_{1}^{*}(\beta_{1})$$

so

$$m^{*}: (\alpha_{1}) \mapsto l_{1}^{*}(1(\alpha_{1}^{*}) + l(\alpha_{2}^{*})) = F_{A}(1(\alpha_{1}^{*}) + l(\alpha_{2}^{*}))$$

However, what we would like is a geometric theory $V^{*}(\cdot)$ where the $\alpha_{V}$ & $A_{V}$ are naturally occuring "Euler classes" (as with $e_{V}$ in the $Z/2$-case). This will be developed in the following sections.