Mating quadratic maps with the modular group II

Shaun Bullett and Luna Lomonaco

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Abstract

In 1994 S. Bullett and C. Penrose introduced the one complex parameter family of $(2:2)$ holomorphic correspondences $\mathcal{F}_a$:

$$\left(\frac{aw-1}{w-1}\right)^2 + \left(\frac{aw-1}{w-1}\right)\left(\frac{az+1}{z+1}\right) + \left(\frac{az+1}{z+1}\right)^2 = 3$$

and proved that for every value of $a \in [4,7] \subset \mathbb{R}$ the correspondence $\mathcal{F}_a$ is a mating between a quadratic polynomial $Q_c(z) = z^2 + c,\ c \in \mathbb{R}$ and the modular group $\Gamma = PSL(2,\mathbb{Z})$. They conjectured that this is the case for every member of the family $\mathcal{F}_a$ which has a in the connectedness locus.

We prove here that every member of the family $\mathcal{F}_a$ which has a in the connectedness locus is a mating between the modular group and an element of the parabolic quadratic family $Per_1(1)$.

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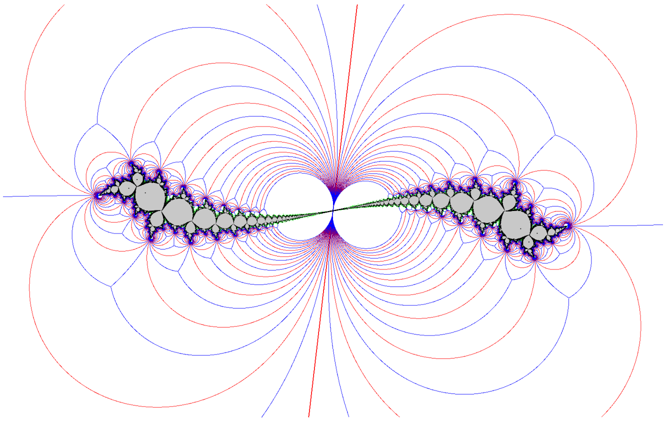

Figure 1: Limit set for $\mathcal{F}_a$, $a = 4.565 + 0.420i$.

1 Introduction

The analogies between the iteration of holomorphic maps and the action of Kleinian groups were first enumerated by Sullivan in the mid 1980s. His landmark paper [S], where he proved the conjecture of Fatou that there are no wandering domains for a rational map on the Riemann sphere, includes the first version of what it is now called Sullivan's dictionary, in which definitions, theorems and conjectures in the world of holomorphic maps are related to analogous definitions, theorems and conjectures in the world of Kleinian groups. Sullivan draws attention to deep parallels between the Fatou set F_f and Julia set J_f of a holomorphic map f on $\widehat{\mathbb{C}}$, and the ordinary set $\Omega(G)$ and limit set $\Lambda(G)$ respectively of a finitely generated Kleinian group G acting on $\widehat{\mathbb{C}}$, and his proof of the no wandering domains theorem for rational maps is inspired by the method used to prove Ahlfors' finiteness theorem in the world of Kleinian groups.

Both rational maps and finitely generated Kleinian groups can be regarded as particular cases of correspondences. An n-to-m holomorphic correspondence on $\widehat{\mathbb{C}}$ is a multi-valued map $\mathcal{F} : z \to w$ defined by a polynomial relation $P(z, w) = 0$. A rational map $f(z) = p(z)/q(z)$ becomes an n-to-1 correspondence defined by $P(z, w) = 0$, where $P(z, w) = wq(z) - p(z)$, and any finitely generated Kleinian group G with generators

$$\gamma_j(z) = \frac{a_j z + b_j}{c_j z + d_j}$$

can be regarded as an $(n:n)$ correspondence by taking

$$P(z,w) = \prod_{j=1}^{n} \left(w(c_j z + d_j) - (a_j z + b_j) \right).$$

In particular, since

$$\alpha(z) = z+1 \quad \text{and} \quad \beta(z) = \frac{z}{z+1}$$

generate the modular group $\Gamma = PSL(2,\mathbb{Z})$, the orbits of Γ on $\widehat{\mathbb{C}}$ are the orbits of the $(2:2)$ correspondence defined by

$$(w-(z+1))(w(z+1)-z) = 0.$$

In the early 1990s the first author and C. Penrose observed behaviour such as that illustrated in Figure 1, in a particular family of $(2:2)$ correspondences, namely the one parameter family $\mathcal{F}_a$ defined by

$$\left(\frac{aw-1}{w-1}\right)^2 + \left(\frac{aw-1}{w-1}\right)\left(\frac{az+1}{z+1}\right) + \left(\frac{az+1}{z+1}\right)^2 = 3.$$

Computer plots appeared to show two copies (denoted in this article by $\Lambda_{a,-}$ and $\Lambda_{a,+}$) of the filled Julia set of a quadratic polynomial, together with an action of the modular group on the complement (denoted Ω_a here), prompting the question as to whether in the world of holomorphic correspondences there might exist 'matings' between quadratic polynomials and the modular group. Bullett and Penrose [BP] constructed an abstract combinatorial mating between any member of the quadratic family (with connected filled Julia set) and the modular group (see Section 1.1). Holomorphic correspondences realising these combinatorial matings are holomorphic realisations of Minkowski's *question mark function* [Min], a homeomorphism from the unit interval to the positive real line which sends a real number expressed in binary to a real number with a corresponding continued fraction expression. On the binary expression side of the mating is the Douady-Hubbard coding of rays for quadratic polynomials, which is key to combinatorial descriptions of Julia sets and renormalisation theory. On the continued fraction side, the action of the modular group is related to the generation of Farey sequences of rationals, and thence to the Riemann Hypothesis (we thank Charles Tresser for drawing our attention to the work of Franel [F] and Landau [Lan], showing that the Riemann Hypothesis is equivalent to certain conditions concerning the uniformity of distribution of such sequences).

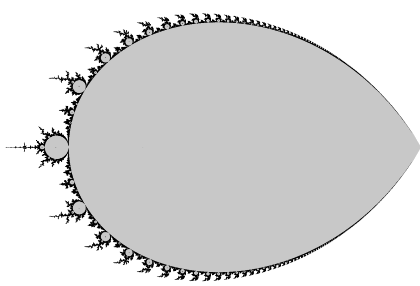

Figure 2: M_Γ

The main result of [BP] is that for a in the real interval $[4, 7]$ the holomorphic correspondence $\mathcal{F}_a$ is indeed a mating between a (real) quadratic polynomial and the modular group. More generally, Bullett and Penrose conjectured that each $\mathcal{F}_a$ for which the parameter a is in the connectedness locus for the family is a mating between a quadratic polynomial and the modular group. Their conjecture has since been proved for a large class of values of the parameter a, by applying Haissinsky's technique of 'pinching' to polynomial-like maps (see [BHai]). But the technique is not applicable for *all* values of a in the connectedness locus.

Whatever the value of a, the branch of $\mathcal{F}_a$ which fixes $z = 0$ is parabolic, with multiplier at the parabolic fixed point equal to 1 (see Proposition 3.6). This fact makes the use of polynomial-like mappings tricky and finally inefficient, and suggests that the family $\mathcal{F}_a$ might have significant differences from the family of quadratic polynomials. On the other hand, it also suggests that the optimum description of the correspondences $\mathcal{F}_a$ might be as matings between the modular group and members of some family of *parabolic* quadratic maps. This is indeed the case, the family of maps being

$$P_A : z \to z + \frac{1}{z} + A, \quad A \in \mathbb{C},$$

which we recall are the quadratic rational maps with a parabolic fixed point of multiplier 1 at infinity and critical points at ± 1 (in Milnor's notation, the maps in $Per_1(1)$).

Definition 1.1. We say that $\mathcal{F}_a$ is a mating between the rational quadratic map $P_A : z \to z + 1/z + A$ and the modular group $\Gamma = PSL(2, \mathbb{Z})$ if

(i) the 2-to-1 branch of $\mathcal{F}_a$ for which $\Lambda_{a,-}$ is invariant, is hybrid equivalent to P_A on $\Lambda_{a,-}$, and

(ii) when restricted to a $(2:2)$ correspondence from Ω_a to itself, $\mathcal{F}_a$ is conformally conjugate to the pair of Möbius transformations $\{\alpha, \beta\}$ from the complex upper half plane $\mathbb{H}$ to itself.

Formal definitions of the sets $\Lambda_{a,-}$, $\Lambda_{a,+}$ and Ω_a are given in Section 3. In the same Section we also define the *Klein combination locus* $\mathcal{K} \subset \mathbb{C}$ and the *connectedness locus* $\mathcal{C}_\Gamma \subset \mathcal{K}$ of the family of correspondences $\mathcal{F}_a$. Given these concepts, we are in a position to state the main result of this paper:

Main Theorem. *For every $a \in \mathcal{C}_\Gamma$ the correspondence $\mathcal{F}_a$ is a mating between some rational map $P_A : z \to z + 1/z + A$ and Γ.*

The layout of this paper is as follows. In Section 2 we assemble facts concerning *Fatou coordinates* and *parabolic-like mappings* that will be needed later. In Section 3 we investigate some dynamical properties of the family $\mathcal{F}_a$ and in Section 4 we prove:

Theorem A. *For every $a \in \mathcal{C}_\Gamma$, when restricted to a $(2:2)$ correspondence from Ω_a to itself, $\mathcal{F}_a$ is conformally conjugate to the pair of Möbius transformations $\{\alpha, \beta\}$ from the complex upper half plane $\mathbb{H}$ to itself.*

In Section 5 we prove that every $\mathcal{F}_a$ with a in the *Klein combination locus* $\mathcal{K}$ can be surgically modified to become a single-valued *parabolic-like* map in the sense of [L] on a neighbourhood of the backward limit set $\Lambda_{a,-}$. Since this parabolic-like map can then be *straightened* ([L]) into a rational map of the form $P_A : z \to z + 1/z + A$, we obtain the following:

Theorem B. *For every parameter value $a \in \mathcal{K}$, after a surgery supported outside the limit set, the branch of $\mathcal{F}_a$ fixing $\Lambda_{a,-}$ restricts to a parabolic-like mapping, and therefore on $\Lambda_{a,-}$ is hybrid equivalent to a member of the family $Per_1(1)$ of quadratic rational maps.*

Note that since the Julia set of a rational map is the closure of the set of repelling periodic points, and quasiconformal maps preserve the nature of periodic points, the theorem implies the following:

Corollary 1.2. *For each $a \in \mathcal{K}$, the boundary of $\Lambda_{a,-}$ is the closure of the set of repelling periodic points of the branch of $\mathcal{F}_a$ fixing $\Lambda_{a,-}$.*

The Main Theorem is consequence of Theorems A and B.

Remark 1.3. *As we shall see, the closed disc $\mathcal{D} = \{a : |a-4| \leq 3\}$ is contained in the Klein combination locus $\mathcal{K}$ (apart from the point $a = 1$, where the correspondence $\mathcal{F}_a$ is undefined). Let M_Γ denote $\mathcal{C}_\Gamma \cap \mathcal{D}$. This set M_Γ (Figure 2) was first plotted in [BP]; it is conjectured to be homeomorphic to the classical Mandelbrot set. We conjecture that M_Γ is the whole of $\mathcal{C}_\Gamma$. However the overall structure of the Klein combination locus $\mathcal{K}$ is as yet poorly understood and we cannot exclude the possibilty the $\Lambda_{a,-}$ is connected for some value of $a \in \mathcal{K} \setminus \mathcal{D}$. What we can say is that the only point of M_Γ on the boundary of $\mathcal{D}$ is $a = 7$: this will be proved in the forthcoming article [BL], as a consequence of a new inequality of Yoccoz type.*

Acknowledgement. The authors are very grateful to Adam Epstein for first suggesting to them in 2011 that parabolic-like mappings might be applied to the family of correspondences $\mathcal{F}_a$, and to Adam Epstein, Carsten Petersen and Pascale Roesch for very helpful discussions at various stages of the work reported on here. This research has been partially supported by EU Marie-Curie IRSES Brazilian-European partnership in Dynamical Systems (FP7-PEOPLE-2012-IRSES 318999 BREUDS) and the Fundação de amparo a pesquisa do estado de São Paulo (Fapesp, process 2013/20480-7).

2 Preliminaries

This section is devoted to a summary of results we will use during the article.

2.1 Petals and Fatou coordinates

A holomorphic map $g(z) = z + bz^2 + \ldots$, with $b \neq 0$, defined in a neighbourhood of the origin, has a parabolic fixed point at the origin with multiplicity 1. A complex number $\mathbf{v}$ points in the repelling direction if $b\mathbf{v}$ is real and positive, and a complex number $\mathbf{w}$ points in the attracting direction if $b\mathbf{w}$ is real and negative. An open set in a neigbourhood of the origin is called an attracting petal if it is mapped into itself and if each orbit eventually absorbed by it converges to the origin from the attracting direction $\mathbf{v}$. Similarly, a repelling petal is an open set contained in its image and with orbits escaping from the origin in the repelling direction $\mathbf{w}$.

There exists a well-established body of knowledge concerning attracting and repelling petals at parabolic fixed points of holomorphic functions g, and *Fatou coordinates* on these petals. We shall make use of petals with the properties listed in the following Proposition.

Proposition 2.1. *For every holomorphic function g as above, and every angle $0 < \theta < \pi$, inside every neighbourhood of 0 there exists a repelling petal U_θ^+ containing an open sector of angle 2θ centered at the origin and symmetric with respect to the repelling direction. Each of these petals is equipped with a conformal homeomorphism Φ^+ (known as a Fatou coordinate) from U_θ^+ to a subset V_θ^+ of the complex plane consisting of all points $w = u + iv$ to the left of some curve which has asymptotes $u = -|v|\cot(\theta) - c$, with c large so that $|w|$ is large for all $w \in V_\theta^+$ (see [M2]), with the following properties:*

(i) Φ^+ is a composition $\psi^{-1}\phi$ where $\phi(z) = 1/(-bz)$ and ψ (defined on V_θ^+) is asymptotic to the identity, in the sense that $\lim_{|w|\to\infty} \psi(w)/w = 1$ for all $w \in V_\theta^+$;

(ii) Φ^+ conjugates g^{-1} on U_θ^+ to $w \to w - 1$ on V_θ^+.

Proof. This is an immediate consequence of Chapter 10 in [M], or Chapter 6.5 in [Be]. For the estimate of the asymptotes of the repelling petal in the w coordinate, see the proof of Theorem 7.2 in [M2] □

An *attracting petal* U_θ^- is a repelling petal for g^{-1}, and has a Fatou coordinate conjugating g to $w \to w + 1$ on the corresponding domain V_θ^+ in $\mathbb{C}$. We observe that U_θ^+ and U_θ^- are foliated by *invariant curves* (invariant under g^{-1} and g respectively), corresponding to the respective foliations of V_θ^+ and V_θ^- by horizontal lines. When $U_\theta^+ \cap U_\theta^-$ is non-empty (which is always the case when $\theta > \pi/2$) the two foliations on the intersection will usually be different: nevertheless for both these foliations on the intersection, leaves which correspond to horizontal lines in the w-plane sufficiently far above or below the real axis, extend to invariant (under both g and g^{-1}) topological circles, *horocycles*, in the z-plane.

2.2 $Per_1(1)$ and parabolic-like maps

Consider the family of quadratic rational maps having a parabolic fixed point of multiplier 1 at ∞. Normalizing by fixing the critical points at ± 1, this family is

$$Per_1(1) = \{P_A(z) = z + 1/z + A \mid A \in \mathbb{C}\}$$

For a map in $Per_1(1)$, denoting by Λ the parabolic basin of attraction of infinity, we can define the filled Julia set of P_A to be $K_A = \widehat{\mathbb{C}} \setminus \Lambda$ (the map $P_0(z) = z + 1/z$ is the unique map in the family $Per_1(1)$ with two parabolic attracting petals, and we set $K_0 = \overline{\mathbb{H}_l}$).

Parabolic-like maps are objects that locally behave as members of the family $Per_1(1)$, i.e. objects which extend the notion of polynomial-like maps to maps with a parabolic external class. More specifically:

Definition 2.2. A parabolic-like map is a 4-tuple (f, U', U, γ) where

- U', U are open subsets of $\mathbb{C}$, with U', U and $U \cup U'$ isomorphic to a disc, and U' not contained in U,
- $f : U' \to U$ is a proper holomorphic map of degree d with a parabolic fixed point at $z = z_0$ of multiplier 1,
- $\gamma : [-1, 1] \to \overline{U}, \gamma(0) = z_0$ is an arc, forward invariant under f, C^1 on $[-1, 0]$ and on $[0, 1]$, and such that

$$f(\gamma(t)) = \gamma(dt), \;\; \forall -\frac{1}{d} \leq t \leq \frac{1}{d},$$

$$\gamma([\frac{1}{d}, 1) \cup (-1, -\frac{1}{d}]) \subseteq U \setminus U', \;\; \gamma(\pm 1) \in \partial U.$$

  It resides in repelling petal(s) of z_0 and it divides U', U into Ω', Δ' and Ω, Δ respectively, such that $\Omega' \subset\subset U$ (and $\Omega' \subset \Omega$), $f : \Delta' \to \Delta$ is an isomorphism and Δ' contains at least one attracting fixed petal of z_0. We call the arc γ a *dividing arc*.

The filled Julia set of a parabolic-like map (f, U', U, γ) is the set of points that never escape $\Omega' \cup \{z_0\}$, this is $K_f = \{z \in U' \mid \forall n \geq 0, \;\; f^n(z) \in U' \setminus \Delta'\}$, and the Julia set is defined as $J_f := \partial K_f$ (see [L]). By the Straightening Theorem for parabolic-like maps, any degree 2 parabolic-like map is hybrid equivalent to a member of the family $Per_1(1)$, a unique such member if the filled Julia set is connected.

3 Dynamics of $\mathcal{F}_a$

We consider the family of $(2 : 2)$ holomorphic correspondences on the Riemann sphere which have the form $\mathcal{F}_a : z \to w$, where

$$\left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3$$

for a parameter $a \in \mathbb{C}$, $a \neq 1$. The reason for studying this particular family is the following lemma (the content of which is in [BP], repeated here to establish notation) together with Proposition 1.4 of [BHai], which states that every mating between a quadratic map and the modular group which supports a *compatible involution* (see [BHai]) is conformally conjugate to a member of this family.

Lemma 3.1. $\mathcal{F}_a = J_a \circ Cov_0^Q$, *where*

$$J_a(Z) = \frac{(a+1)Z - 2a}{2Z - (a+1)}$$

is the involution which has fixed points 1 and a, and $Cov_0^Q : z \to w$ *is the deleted covering correspondence of the rational map* $Q(z) = z^3 - 3z$.

Proof. Consider the map $Q(Z) = Z^3 - 3Z$. It has a double critical point at infinity and simple critical points at ± 1, and up to pre- and post-composition by Möbius transformations, every degree 3 rational map with exactly 3 distinct critical points is equivalent to $Q(Z)$.

Let $Cov^Q : Z \to W$ be the $(3:3)$ covering correspondence of Q, which is the correspondence exchanging the preimages of Q, or in other words acting on the fibres of Q. This is the correspondence defined by

$$Q(Z) = Q(W),$$

or more explicitly by

$$Z^3 - 3Z = W^3 - 3W.$$

Let $Cov_0^Q : z \to w$ be the $(2:2)$ correspondence defined by

$$\frac{Q(Z) - Q(W)}{Z - W} = 0,$$

that is,

$$Z^2 + ZW + W^2 = 3.$$

This is called the *deleted covering correspondence* of Q, since its graph is obtained from that of Cov^Q by deleting the graph of the identity.

Post-composing this last correspondence by the involution $W \to J_a(W)$ we obtain the $(2:2)$ correspondence defined by the polynomial

$$Z^2 + Z(J_a(W)) + (J_a(W))^2 = 3.$$

This is the correspondence

$$Z^2 + Z\left(\frac{(a+1)W - 2a}{2W - (a+1)}\right) + \left(\frac{(a+1)W - 2a}{2W - (a+1)}\right)^2 = 3,$$

which is, via the change of coordinates

$$Z = \frac{az+1}{z+1},$$

the correspondence $\mathcal{F}_a$. $\square$

Note that in the coordinate z, the involution J_a becomes $z \leftrightarrow -z$. The choice of whether to work in the coordinate Z or the coordinate z depends on whether it is more convenient to have a simple expression for Cov_0^Q or J_a. We denote by P the common fixed point of Cov_0^Q and J_a (P is the point $Z = 1$ or $z = 0$ in our two coordinate systems).

By a *fundamental domain* for Cov_0^Q we shall mean a maximal open set which is disjoint from its image under Cov_0^Q. (In this article fundamental domains will always be open sets.)

Definition 3.2. The *Klein combination locus* $\mathcal{K}$ for the family of correspondences $\mathcal{F}_a$ is the set of parameter values a for which there exist simply-connected fundamental domains Δ_{Cov} and Δ_{J_a} for Cov_0^Q and J_a respectively, bounded by Jordan curves, such that

$$\Delta_{Cov} \cup \Delta_{J_a} = \hat{\mathbb{C}} \setminus \{P\}.$$

We call such a pair of fundamental domains $(\Delta_{Cov}, \Delta_{J_a})$ a *Klein combination pair.*

Definition 3.3. For a in $\mathcal{D} = \{a : |a-4| \leq 3\}$, the *standard pair of fundamental domains* is that given by taking Δ_{Cov} to be the region of the Z-plane $\mathbb{C}$ to the right of $Cov_0^Q((-\infty, -2])$, and Δ_{J_a} to be complement of the closed round disc D_a which has centre on the real axis and boundary circle through the points 1 and a.

Proposition 3.4. *For all $a \in \mathcal{D}$ (apart from the parameter value $a = 1$ where the correspondence is undefined), the standard pair of fundamental domains is a Klein combination pair. Hence $\mathcal{D} \setminus \{1\} \subset \mathcal{K}$.*

Proof. The real line interval $L = [-\infty, +2]$ has inverse image $Q^{-1}(L)$ the line interval L itself, together with a curve L' which crosses L orthogonally at $Z = 1$ and runs off towards ∞ in directions approaching angles $\pm\pi/3$ to the positive real axis (Fig. 3). This line L' is the image of $[-\infty, -2]$ under Cov_0^Q: an elementary computation shows that

$$L' = \{\left(1 + \frac{t}{2}\right) \pm i\sqrt{3\left(t + \left(\frac{t}{2}\right)^2\right)} : t \in [0, \infty]\}.$$

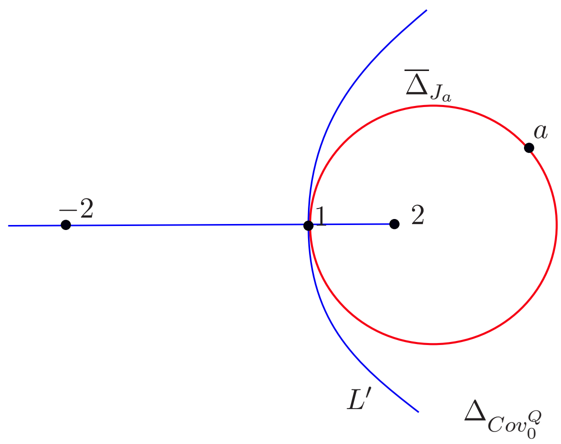

Figure 3: Standard fundamental domains for $\mathcal{F}_a$.

Now the component of $\mathbb{C} \setminus L'$ which lies to the right of L' is a fundamental domain for Cov_0^Q, that is to say it is a maximal open set which is disjoint from its image under Cov_0^Q (see also Example 1.2 in [B]). But this component is our *standard* fundamental domain for Cov^Q (Definition 3.3.)

The standard Δ_{J_a} is self-evidently a fundamental domain for the involution J_a, so it only remains to verify that for $a \in \mathcal{D} \setminus \{1\}$, the domains Δ_{Cov} and Δ_{J_a} satisfy the Klein combination condition. However a straightforward computation shows that L' meets the circle which has centre $Z = 4$ and radius 3 at the single point $Z = 1$. It follows that $D_a \setminus \{1\} \subset \Delta_{Cov}$ for all $a \in \mathcal{D} \setminus \{1\}$, and so $\Delta_{Cov} \cup \Delta_{J_a} \supseteq \hat{\mathbb{C}} \setminus \{1\}$ for every such value of a. $\square$

Proposition 3.5. *For every $a \in \mathcal{K}$ and Klein combination pair $(\Delta_{Cov}, \Delta_{J_a})$, the correspondence $\mathcal{F}_a$ has the following properties (here we denote by D_a the complement of Δ_{J_a} in $\hat{\mathbb{C}}$: so D_a is a closed topological disc with boundary a Jordan curve):*

- *$\mathcal{F}_a(D_a) \subset D_a$ and $\partial\mathcal{F}_a(D_a) \cap \partial D_a = \{P\}$;*
- *the restricted correspondence $\mathcal{F}_a| : D_a \to \mathcal{F}_a(D_a)$ is a 1-to-2 correspondence;*
- *the restricted correspondence $\mathcal{F}_a| : \mathcal{F}_a^{-1}(\overline{\Delta}_{J_a}) \to \overline{\Delta}_{J_a}$ is a 2-to-1 map, conjugate to $\mathcal{F}_a^{-1}| : \mathcal{F}_a(D_a) \to D_a$.*

Proof. Since $\Delta_{Cov} \cup \Delta_{J_a} = \hat{\mathbb{C}} \setminus \{P\}$ and $D_a = \Delta_{J_a}^c$, we have that $D_a \subset \Delta_{Cov} \cup \{P\}$. Hence $Cov_0^Q(\Delta_{Cov}) \subset \Delta_{J_a}$, and so $\mathcal{F}_a(\Delta_{Cov}) \subset \Delta_{Cov}$. Noting that

$$Cov_0^Q : \Delta_{Cov} \to Cov_0^Q(\Delta_{Cov})$$

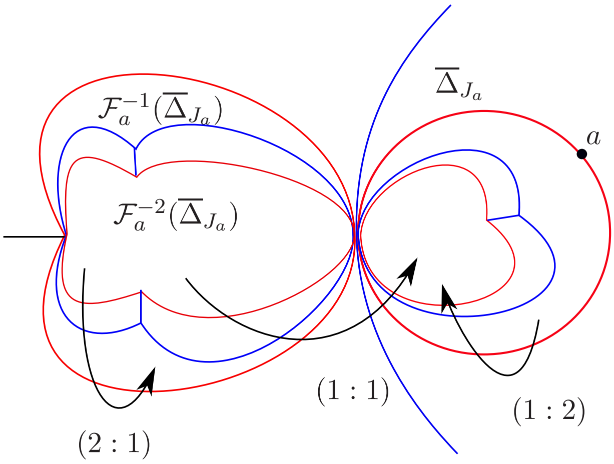

Figure 4: Images and preimages of $\overline{\Delta}_{J_a}$.

is 1-to-2, we deduce that

$$\mathcal{F}_{a|\Delta_{Cov}} : \Delta_{Cov} \to \mathcal{F}_a(\Delta_{Cov})$$

is 1-to-2. In particular, since $D_a \setminus \{P\} \subset \Delta_{Cov}$, we see that $\mathcal{F}_a(D_a) \subset D_a$, $\mathcal{F}_a(D_a) \cap D_a = \{P\}$ and thus

$$\mathcal{F}_a| : D_a \to \mathcal{F}_a(D_a)$$

is 1-to-2 (see Figure 4).

Similarly, since $(\mathcal{F}_a)^{-1}(\Delta_{J_a}) \subset Cov_0^Q(\Delta_{Cov})$ and

$$Cov_0^Q : Cov_0^Q(\Delta_{Cov}) \to \Delta_{Cov}$$

is 2-to-1, we have that

$$\mathcal{F}_{a|(\mathcal{F}_a)^{-1}(\overline{\Delta}_{J_a})} : (\mathcal{F}_a)^{-1}(\overline{\Delta}_{J_a}) \to \overline{\Delta}_{J_a}$$

is 2-to-1 and it is conjugate by J_a to

$$\mathcal{F}^{-1}_{a|\mathcal{F}_a(D_a)} : \mathcal{F}_a(D_a) \to D_a$$

(the closure $\overline{\Delta}_{J_a}$ includes the point P).

$\square$

We next examine the behaviour of $\mathcal{F}_a$ around P $(Z = 1)$.

Proposition 3.6. *Let $\zeta = Z - 1$. When $a \neq 7$ the power series expansion of the branch of $\mathcal{F}_a$ which fixes $\zeta = 0$ has the form:*

$$\zeta \to \zeta + \frac{a-7}{3(a-1)}\zeta^2 + \dots$$

and so the Leau-Fatou flower at the fixed point has a single attracting petal. When $a = 7$ the expansion has the form:

$$\zeta \to \zeta + \frac{1}{27}\zeta^4 + \dots$$

and so the flower at the fixed point has three attracting petals.

Proof. By Lemma 3.1, $\mathcal{F}_a = J_a \circ Cov_0^Q$, where J_a is the involution which has fixed points 1 and a:

$$J_a(Z) = \frac{(a+1)Z - 2a}{2Z - (a+1)}$$

and $Cov_0^Q : Z \to W$ where $Z^2 + ZW + W^2 = 3$. Therefore the branch of Cov_0^Q fixing $Z = 1$ is $Z \to W$ where

$$W = \frac{-Z + (12 - 3Z^2)^{1/2}}{2}.$$

Changing coordinates to ζ, ω where $Z = \zeta + 1$ and $W = \omega + 1$, so that the fixed point is at $\zeta = 0$, this branch of Cov_0^Q becomes:

$$\omega = -\frac{\zeta}{2} + \frac{3}{2}\left(\left(1 - \frac{2\zeta}{3} - \frac{\zeta^2}{3}\right)^{1/2} - 1\right) = -\zeta - \frac{\zeta^2}{3} - \frac{\zeta^3}{9} - \frac{2\zeta^4}{27} - \dots$$

In these coordinates the involution J_a is:

$$\zeta \to -\zeta\left(\frac{1}{1 - \frac{2\zeta}{a-1}}\right) = -\zeta - \frac{2\zeta^2}{a-1} - \frac{4\zeta^3}{(a-1)^2} - \frac{8\zeta^4}{(a-1)^3} - \dots$$

Composing the two power series and collecting up terms we deduce that the branch of $\mathcal{F}_a = J_a \circ Cov_0^Q$ which fixes $\zeta = 0$ sends ζ to:

$$\zeta + \frac{a-7}{3(a-1)}\zeta^2 + \left(\frac{a-7}{3(a-1)}\right)^2\zeta^3 + \left(\frac{2}{27} - \frac{2}{3(a-1)} + \frac{4}{(a-1)^2} - \frac{8}{(a-1)^3}\right)\zeta^4 + \dots$$

completing the proof. ∎

For $a \neq 7$ there is a unique *repelling direction* at the parabolic fixed point. From Proposition 3.6, in the ζ coordinate this is the direction

$$\zeta = \frac{\bar{a} - 7}{\bar{a} - 1}.$$

For $a = 7$, there are three repelling directions: $\zeta = 0, e^{2\pi i/3}, e^{4\pi i/3}$.

Definition 3.7. Let P be the parabolic fixed point of our correspondence $\mathcal{F}_a$, $a \neq 7$. We call the line defined by the repelling direction the *parabolic axis* at P, and we say that a differentiable curve ℓ passing through P is *transverse to the parabolic axis* if ℓ crosses this axis at a non-zero angle. (For $a = 7$ we adopt the convention that the 'parabolic axis' is the real axis, in both the Z-coordinate and the z-coordinate.)

Corollary 3.8. *For $a \neq 7$, given any smooth curve ℓ passing through P transversely to the parabolic axis, there is a repelling petal U_θ^+ and Fatou coordinate Φ^+ on U_θ^+ such that $\Phi^+(\ell)$ (in the $w = u+iv$ plane) intersects every horizontal leaf $v = c$ in $V_\theta^+ = \Phi^+(U_\theta^+)$ which corresponds to a sufficiently large value of $|c|$.*

Proof. The line ℓ meets the repelling direction at P at some angle $0 < \alpha < \pi$. Choose θ with $\alpha < \theta < \pi$. By Proposition 2.1, as we travel along ℓ towards P from either side, the final part of our journey is contained in U_θ^+. The result follows, since $\Phi^+ : U_\theta^+ \to V_\theta^+$ sends a line meeting the repelling direction at P at angle α to a curve the points $w(t)$ of which have $\lim_{t\to\infty} |w(t)| = \infty$ and $\lim_{t\to\infty} \arg(w(t)) = \pi - \alpha$. $\square$

Proposition 3.9. *For $a \in \mathcal{K}$, we may always choose a Klein combination pair $(\Delta_{Cov}, \Delta_{J_a})$ of fundamental domains which have boundaries which are smooth at P and transverse to the parabolic axis.*

Proof. By definition the Jordan curves bounding Δ_{Cov} and Δ_{J_a} meet only at P. By making small perturbations to these curves if need be, we can ensure they are both smooth, apart from an angle of $2\pi/3$ on $\partial\Delta_{Cov}$ at the double critical point ($Z = \infty$) of Q. At P the smooth curves $\partial\Delta_{Cov}$ and $\partial\Delta_J$ are tangent to one another (since the Klein combination condition excludes the possibility that they cross). For $a \in int(\mathcal{D})$, that is $|a - 4| < 3$, the boundaries of the standard pair $(\Delta_{Cov}, \Delta_{J_a})$ at their intersection P ($Z = 1$) are parallel to the imaginary axis in the Z-plane, and as a lies inside the circle in the Z-plane which has diameter the real interval $[1, 7]$, we know that

$$arg\left(\frac{\bar{a} - 7}{\bar{a} - 1}\right) \neq \pm\frac{\pi}{2}$$

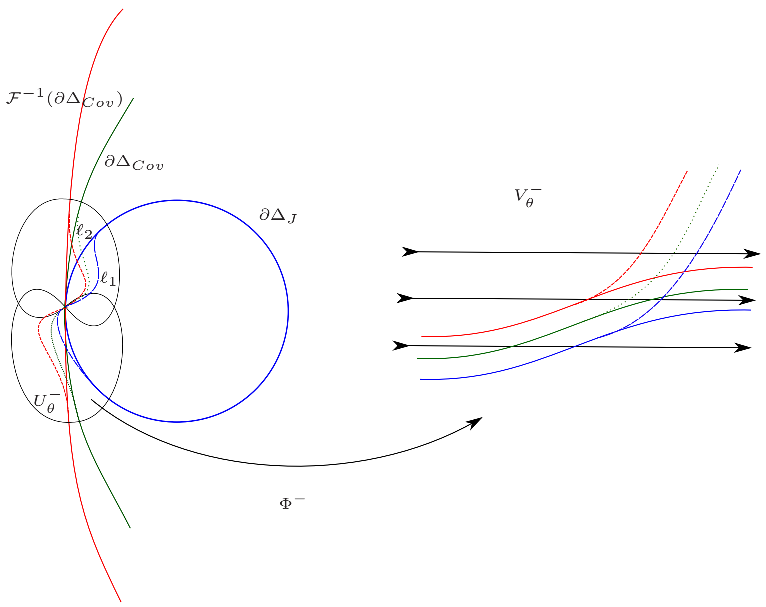

Figure 5: Changing domains

so the parabolic axis is tranverse to the imaginary axis and we are done. When $a = 7$, by our convention the parabolic axis is the real axis, which is transverse to the imaginary axis, so again we are done.

However for $a \in \partial\mathcal{D} \setminus \{7\}$ the boundaries of the standard pair are *tangent* to the parabolic axis, and so small horocycles at P are tangent to $\partial\Delta_{J_a}$ there. We shall see that in this situation, by making a small modification to the boundaries of the standard pair near P, we can construct a new Klein combination pair which have boundaries transverse to the parabolic axis. More generally, suppose that for some $a \in \mathcal{K}$ we have Klein combination domains Δ_J and Δ_{Cov} whose boundaries approach P tangentially to the parabolic axis at P. Choose an angle $0 < \theta < \pi/2$ and attracting and repelling petals $U_\theta^\pm$ which are sufficiently small that they do not intersect. Using the fact that the invariant foliations on these petals give us a complete picture of the dynamics of $\mathcal{F}_a$ on them, we can modify the part of $\partial\Delta_J$ which lies in the repelling petal by replacing a small segment by a curve ℓ_1 which approaches P transversely to the parabolic axis and meets $Cov_0^Q(J(\ell_1))(= \mathcal{F}_a^{-1}(\ell_1))$ only at the point P (figure 5). Next we modify $\partial\Delta_{Cov}$ on the same petal, replacing a segment with a curve ℓ_2 lying between $\mathcal{F}_a^{-1}(\ell_1)$ and ℓ_1. Finally on the attracting petal we replace a segment of $\partial\Delta_J$ by $J(\ell_1)$ and a segment of $\partial\Delta_{Cov}$ by $Cov_0^Q(\ell_2)$. Since Cov_0^Q acts on a neighbourhood of P as an involution with fixed point P, rotating one side of figure 5 to the other, we see that $\ell_1 \cup J(\ell_1)$ meets $\ell_2 \cup Cov_0^Q(\ell_2)$ only at P, and so we can use these as boundaries of modified fundamental domains which still satisfy the Klein combination condition. □

Definition 3.10. For $a \in \mathcal{K}$, with $(\partial\Delta_{Cov}, \partial\Delta_J)$ chosen with boundaries transverse to the parabolic axis at P, the forward limit set of $\mathcal{F}_a$ is defined to be

$$\Lambda_{a,+} = \bigcap_{n=1}^{\infty} \mathcal{F}_a^n(\Delta_{Cov}),$$

the backward limit set is defined to be

$$\Lambda_{a,-} = \bigcap_{n=1}^{\infty} \mathcal{F}_a^{-n}(J_a(\Delta_{Cov})) = J_a(\Lambda_+)$$

and the limit set is defined to be $\Lambda_a = \Lambda_{a,+} \cup \Lambda_{a,-}$, noting that by Proposition 3.5 we have $\Lambda_{a,+} \cap \Lambda_{a,-} = \{P\}$. The regular set Ω_a is defined to be $\hat{\mathbb{C}} \setminus \Lambda_a$.

Note that, by Proposition 3.5, the involution J_a conjugates $\mathcal{F}_a$ on $\Lambda_{a,-}$ to $\mathcal{F}_a^{-1}$ on $\Lambda_{a,+}$ (see also the fifth of the 'Comments on Theorem 2' in [B]).

Remark 3.11. *The partition of $\hat{\mathbb{C}}$ into Λ and Ω is independent of the choice of Klein combination domains, provided these domains have boundaries transverse to the parabolic axis at P. For what can go wrong if we do not make this requirement, see Remark 4.1 following the proof of Theorem 1 below.*

By the definition and Proposition 3.5, the sets Λ_a and Ω_a are completely invariant under $\mathcal{F}_a$, and $\mathcal{F}_a$ on $\Lambda_{a,-}$ is conjugate to the correspondence $\mathcal{F}_a^{-1}$ on $\Lambda_{a,+}$.

Definition 3.12. The connectedness locus for the family $\mathcal{F}_a$ is the subset $\mathcal{C}_\Gamma$ of $\mathcal{K}$ for which $\Lambda_{a,-}$, and hence also $\Lambda_{a,+}$ and Λ_a, is connected.

Since $\Delta_{Cov} \cup \Delta_{J_a} = \hat{\mathbb{C}} \setminus \{P\}$, the proof of Theorem 2 in [B], which is a version for correspondences of the Klein Combination Theorem [K, Mas] (sometimes informally known as the 'Ping-Pong Theorem'), shows that $\mathcal{F}_a$ acts on Ω_a properly discontinuously (see the 4th point of Theorem 2 in [B]) and faithfully (since it acts freely on the set Ω'_a obtained from Ω_a by removing the grand orbit of fixed points of J and Cov_0^Q), with fundamental domain

$$\Delta = \Delta_{Cov} \cap \Delta_J.$$

(The theorem in [B] is stated for correspondences $\mathcal{F} = Cov^P * Cov^Q$, where P, Q are rational maps and Cov^P and Cov^Q are the covering correspondences. Writing $J(z) = -z$ and $P(z) = z^2$ we have $J = Cov_0^P$ and thus our $\mathcal{F}_a$ has the form $Cov_0^P \circ Cov_0^Q$. Note that if $Cov^P * Cov^Q$ acts freely on Ω'_a, then $Cov_0^P \circ Cov_0^Q$ acts faithfully on Ω_a, where Cov_0^P, Cov_0^Q are the deleted covering correspondences of P and Q respectively.)

4 Proof of Theorem A

Recall that the Möbius transformations

$$\alpha : z \to z + 1, \qquad \beta : z \to \frac{z}{z+1}$$

generate the action of $PSL(2, \mathbb{Z})$ on the open upper half-plane $\mathbb{H}$.

Proof of Theorem A. We start by observing that by Definition 3.10 and Proposition 3.5, for every $a \in \mathcal{C}_\Gamma$ the regular set $\Omega = \Omega_a$ is open and simply connected, and therefore there exists a Riemann map $\phi : \Omega \to \mathbb{H}$. We will now prove that:

1. there exist Möbius transformations σ of order 2 and ρ of order 3, both in $PSL_2(\mathbb{R})$, such that ϕ conjugates $\mathcal{F}_a$ to $\{\sigma\rho, \sigma\rho^{-1}\}$;
2. the free product $\langle\sigma\rangle * \langle\rho\rangle$ is a faithful and discrete representation of $C_2 * C_3$ in $PSL_2(\mathbb{R})$;
3. this representation is conjugate to $PSL_2(\mathbb{Z})$.

Step 1. Note that on a neighborhood of ∞ the Böttcher map conjugates the map $Q(Z) = Z^3 - 3Z$ to the map $Z \to Z^3$. It follows that on a neighbourhood of $Z = \infty$, the covering correspondence of Q is conjugate to that of $Z \to Z^3$ via a homeomorphism $\hat{\varphi}$, say. This can be extended to a conjugacy $\hat{\varphi}$ on the whole of Ω, since the only critical point of Q on Ω is the double critical point at $Z = \infty$. Thus Cov^Q on Ω is conjugate via $\hat{\varphi}$ to $\{I, \hat{\rho}, \hat{\rho}^2\}$ on some simply-connected open set $\Omega' \subset \hat{\mathbb{C}}$, where $\hat{\rho}(z) = e^{2\pi i/3}$, and so Cov_0^Q is conjugate to $\{\hat{\rho}, \hat{\rho}^2\}$. If $R : \Omega' \to \mathbb{H}$ is a Riemann map, $\phi : R \circ \hat{\varphi} : \Omega \to \mathbb{H}$ is a Riemann map conjugating the action of Cov_0^Q on Ω to the action of an order 3 rotation ρ on $\mathbb{H}$. On the other hand, since $J_{a|\Omega}$ is an involution, $J_{a|\Omega}$ is conjugate by ϕ to some involution σ on $\mathbb{H}$. Therefore $\mathcal{F}_a = J_a \circ Cov_0^Q$ is conjugate by ϕ on Ω to $\{\sigma\rho, \sigma\rho^{-1}\}$.

Step 2. By the correspondence ping-pong theorem ([B]) we have that $\mathcal{F}_a$ acts on Ω faithfully and properly discontinuously. Since ϕ is a homeomorphism, $\langle\sigma\rangle * \langle\rho\rangle$ also acts faithfully and properly discontinuously on $\mathbb{H}$. Therefore (since σ is an involution and ρ is an order 3 rotation) $\langle\sigma\rangle * \langle\rho\rangle$ is a faithful and discrete representation of $C_2 * C_3$ in $PSL_2(\mathbb{R})$.

Step 3. To complete the proof we must prove that the representation of $C_2 * C_3$ on $\mathbb{H}$ is the standard representation as $PSL(2,\mathbb{Z}) \subset PSL(2,\mathbb{R})$. For every discrete representation of $C_2 * C_3$ the orbifold $\mathbb{H}/(\langle\sigma\rangle * \langle\rho\rangle)$ is conformally isomorphic to a sphere with a $2\pi/3$-cone point, a π-cone point, and either a single boundary component or a puncture point (a cusp is conformally equivalent to a neigbourhood of a puncture point). The representation is conjugate to $PSL_2(\mathbb{Z})$ if and only if the orbifold $\mathbb{H}/(\langle\sigma\rangle * \langle\rho\rangle)$ has a puncture point. Since ϕ is an isomorphism, $\mathbb{H}/(\langle\sigma\rangle * \langle\rho\rangle)$ is conformally equivalent to $\Omega/\langle\mathcal{F}_a\rangle$. By Proposition 3.6 the point P ($Z = 1$) is a parabolic fixed point of $\mathcal{F}_a$. Let $(\Delta_{J_a}, \Delta_{Cov})$ be a Klein combination pair with boundaries transverse to the parabolic axis (such a pair exists by Proposition 3.9). By Proposition 2.1 there exists a repelling petal U_θ^+ containing all points of the line $\partial\Delta_{J_a}$ which lie sufficiently close to P, and by Corollary 3.8 the image of this line under the Fatou coordinate Φ^+ meets all lines $v = c$ in the w-plane (where $w = u + iv$) which have $|c|$ sufficiently large. Writing W for the intersection between $\Delta_{J_a} \setminus \mathcal{F}_a^{-1}(\Delta_{J_a})$ and the petal, we deduce that for $|c|$ sufficiently large, $\Phi^+(W)$ intersects the horizontal line $v = c$. So $W \setminus \{P\}$, after quotienting by the boundary identification induced by $\mathcal{F}_a^{-1}$, is conformally bijective to a pair of neighbourhoods of the ends of $V_\theta^+/\langle w \to w-1\rangle$, that is to a pair of punctured discs. Hence $\Omega/\langle\mathcal{F}_a\rangle$ has a pair of puncture points (one either side of the parabolic axis) corresponding to P. $\square$

Remark 4.1. *If we were to choose Δ_J and Δ_{Cov} with boundaries approaching $Z = 1$ tangentially to the parabolic axis, then the image under Φ^+ of points of $\partial\Delta_J$ sufficiently close to P might lie below some level $v = c$, in which case $(W \setminus \{P\})/\mathcal{F}_a$ would be an annulus rather than a punctured disc and we would find that the new set Ω would differ from that in the case of a transverse intersection: a horodisc at $Z = 1$, together with the grand orbit of this horodisc, would be excised from the Ω of the transverse case. The representation of $C_2 * C_3$ on $\mathbb{H}$ would no longer be that of $PSL(2,\mathbb{Z})$, but $\Lambda(\mathcal{F}_a)$ would also be changed, by the addition of a countable union of discs, attached at the points of the grand orbit of $Z = 1$. In Definition 3.10 we required Δ_J and Δ_{Cov} to have boundaries transverse to the parabolic axis, in order that the partition of $\hat{\mathbb{C}}$ into Ω and Λ be uniquely defined.*

5 Proof of Theorem B

5.1 Useful properties of $\mathcal{F}_a$

For the proof of Theorem B we shall need to convert the branch of $\mathcal{F}_a$ which fixes $\Lambda_{a,-}$ into a parabolic-like map by quasiconformal surgery. The next two results set the scene. Proposition 5.1 ensures that the branch of $\mathcal{F}_a$ which fixes $\Lambda_{a,-}$ is locally holomorphic everywhere but on a neigbourhood of S_a (the preimage of the parabolic fixed point). Proposition 5.2 ensures we have a sector at S_a which can support the surgery that will turn this branch into a parabolic-like map.

Proposition 5.1. *For every $a \in \mathcal{K}$, the restricted correspondence $\mathcal{F}_a| : \mathcal{F}_a^{-1}(\Delta_{J_a}) \to \Delta_{J_a}$ extends locally to a holomorphic 1-to-1 map in a neighbourhood of every $z \in \overline{\mathcal{F}_a^{-1}(\Delta_{J_a})}$, except at the preimage S_a of the parabolic fixed point P, neighbourhoods of which are mapped $(1:2)$ to neighbourhoods of P by a correspondence conjugate to $\zeta \to \pm\sqrt{\zeta}$.*

Proof. By the ping-pong theorem for correspondences ([B]), Ω_a is tiled by bijective copies of $\Delta_{Cov} \cap \Delta_{J_a}$, so in the interior of $\mathcal{F}_a^{-1}(\Delta_{J_a})$ the branch fixing Λ_- is a holomorphic 1-to-1 map. Given any $Z \in \partial(\mathcal{F}_a^{-1}(\Delta_{J_a}))$ which does not map to P ($Z = 1$) we may deform the boundary of Δ_{J_a} so that Z is inside Δ_{J_a}. The local 1-to-1 property now follows from the fact that the branches of $\mathcal{F}_a^{-1}$ are bijective on the interiors of tiles. Finally, a neighbourhood of S_a ($Z = -2$) is mapped 1-to-2 by $\mathcal{F}_a$ to a neighbourhood of P ($Z = 1$), since $\mathcal{F}_a = J_a \circ Cov_0^Q$, and $Cov_0^Q : Z \to W$ is the $2:2$ correspondence with $W = (-Z \pm \sqrt{12 - 3Z^2})/2$. The local conjugacy here to $\zeta \to \pm\sqrt{\zeta}$ also follows. □

Proposition 5.2. *For every $a \in \mathcal{K}$ and Klein combination pair $(\Delta_{J_a}, \Delta_{Cov})$ for $\mathcal{F}_a$, with boundaries transverse to the parabolic axis at the fixed point P, there exist a closed topological disc $V_a \subset \hat{\mathbb{C}}$ and angles $\theta_1 = \theta_{a,1} > 0$ and $\theta_2 = \theta_{a,2} > 0$, with $\theta_1 + \theta_2 < \pi$, with the following properties:*

1. *$\Lambda_{a,-} \subset V_a$ and $\Lambda_{a,-} \cap \partial V_a = \{P\}$;*
2. *the boundary ∂V_a of V_a is smooth away from the parabolic fixed point P, where it meets $\partial\Delta_{J_a}$ at angles θ_1 and θ_2 (so at P the boundary ∂V_a has a 'cone' of angle $\hat{\theta} = \pi - (\theta_1 + \theta_2)$);.*
3. *$V'_a = \mathcal{F}_a^{-1}(V_a) \subset V_a$, and $\partial V'_a \cap \partial V_a = \{P\}$;*
4. *the boundary $\partial V'_a$ of V'_a is smooth everywhere but at P, where it forms a cone of angle $\hat{\theta}$, and at the preimage S_a of P, where it forms a cone of angle $2\hat{\theta}$;*

5. *inside every neighbourhood of P there exist $\mathcal{F}_a$-invariant arcs $\gamma_i : [0,1] \to \bar{V}_a$, $i = 1,2$, emanating from P on the two sides of the parabolic axis, each C^1 and satisfying $\gamma_i(t)[1/2,1) \subset V_a \setminus V_a'$.*

Proof. By Proposition 3.9, for every $a \in \mathcal{K}$ we can choose a Klein combination pair $(\Delta_{Cov}, \Delta_{J_a})$ of fundamental domains which have boundaries which are smooth at P and transverse to the parabolic direction. By Proposition 3.5,

$$\mathcal{F}_a^{-1}(\Delta_{J_a}) = Cov_0^Q \circ J_a(\Delta_{J_a}) = Cov_0^Q(D_a) \subset \Delta_{J_a}.$$

We shall construct V_a by making a small change to the boundary of Δ_{J_a} in a neighbourhood of P, so that while V_a is not a fundamental domain for J_a it retains the property that $\mathcal{F}_a^{-1}(V_a) \subset V_a$ and gains the other properties listed.

Suppose firstly that $a \neq 7$, so we are in the 'single petal' case. Let ℓ denote $\partial\Delta_{J_a}$ and let the angles at P between ℓ and the parabolic axis be α_1 and $\alpha_2 = \pi - \alpha_1$. Choose θ such that $\max(\alpha_1, \alpha_2) < \theta < \pi$ (such an angle θ exists since we started from a Klein combination pair $(\Delta_{Cov}, \Delta_{J_a})$ which have boundaries which are smooth at P and transverse to the parabolic direction). Let U_θ^+ be a repelling petal containing an open sector of angle 2θ centered at P given by Proposition 2.1, and $\Phi^+ : U_\theta^+ \to V_\theta^+$ be a repelling Fatou coordinate (where V_θ is the subset of $\mathbb{C}$ consisting to all the $w = u + iv$ to the left of a curve which has asymptotes $u = -|v|\cot(\theta) - c$, with c large). As ℓ is C^1 at P, and so is $F_a^{-1}(\ell)$, with the same tangent at P, we know that for every point $R \in \ell$ sufficiently close to P the open straight line segment (in whatever coordinate we are working in) from R to $\mathcal{F}_a^{-1}(R)$ lies in $\Delta_{J_a} \setminus \mathcal{F}_a^{-1}(\Delta_{J_a})$. Thus we can foliate the intersection W between $\Delta_{J_a} \setminus \mathcal{F}^{-1}(\Delta_{J_a})$ and a suitable neighbourhood of P by straight line segments. The set W has two components, which we denote W_1 and W_2, one each side of the parabolic axis at P. Write D_i $(i = 1,2)$ for $\bigcup_{n=0}^{\infty} \mathcal{F}_a^{-n}(W_i)$. The sets D_i are foliated by piecewise-linear leaves, each of which is invariant under $\mathcal{F}_a^{-1}$ and crosses each line $\mathcal{F}_a^{-n}(\ell)$ exactly once. In $\Phi^+(D_i) \subset V_\theta$ they become leaves invariant under $w \to w - 1$. Consider a set of these leaves which are integer distances apart at the points where they meet $\Phi^+(\ell)$. Together with the lines $(\Phi^+(\mathcal{F}_a^{-n}(\ell)))_{n\geq 0}$ they create a 'skew grid' in each of the $\Phi^+(D_i)$, $i = 1,2$.

Choose $0 < \theta_1 < \alpha_1$ and $0 < \theta_2 < \alpha_2$. Using the skew grids as coordinate systems, we can now construct in each $\Phi^+(D_i)$, $i = 1,2$, a smooth curve m_i which at one end joins $\Phi^+(\ell)$ smoothly, at the other is asymptotic to a line at angle θ_i to the horizontal as w tends to infinity, and in between crosses each leaf of the foliation exactly once, and each line $\Phi^+(\mathcal{F}_a^{-n}(\ell))$ exactly once. Note that the lines $(\Phi^+)^{-1}(m_i)$ lie outside $\Lambda_{a,-}$ since every point of $(\Phi^+)^{-1}(D_i)$ eventually leaves Δ_{J_a} under some iterate of the branch of $\mathcal{F}_a$ which fixes P. We now define V_a to be the domain bounded by $\ell = \partial\Delta_{J_a}$ as modified by $(\Phi^+)^{-1}(m_1)$ and $(\Phi^+)^{-1}(m_2)$ in a neighbourhood of P, and define V'_a to be $\mathcal{F}_a^{-1}(V_a)$. The first three properties stated in the Proposition are immediate, and the 4th property follows from Proposition 5.1. Finally, for the 5th property we note that each leaf of the foliation satisfies all the requirements except that it is piecewise-linear and not (in general) C^1. We rectify this by replacing a straight line path in W_i between some $R \in \ell$ and $\mathcal{F}_a^{-1}(R)$ by a C^1 curve n_i in W_i which meets ℓ at R and $\mathcal{F}_a^{-1}(\ell)$ at $\mathcal{F}_a^{-1}(R)$ at angles which sum to π, and set γ_i to be $\bigcup_{n=0}^{\infty} \mathcal{F}_a^{-n}(n_i)$, parametrized appropriately.

In the case $a = 7$ we have to modify the argument above to allow for the fact that we have three attracting petals and three repelling petals. We omit details, but remark that the key difference is that whereas for $a \neq 7$ we can choose θ_1 and θ_2 such that $\hat{\theta} = \pi - (\theta_1 + \theta_2)$ is arbitrarily small, for $a = 7$, with the standard domains, by taking Fatou coordinates on appropriate overlapping attracting and repelling petals, one can show that θ_1 and θ_2 must satisfy $\hat{\theta} > 2\pi/3 > 0$ but that they can be chosen with $\hat{\theta}$ arbitrarily close to $2\pi/3$.

$\square$

5.2 Proof of Theorem B

For every $a \in \mathcal{K}$, and therefore in particular for every $a \in \mathcal{C}_\Gamma$, the correspondence $\mathcal{F}_a$, restricted to a neighbourhood of $\Lambda_{a,-}$, satisfies all the conditions necessary for it to be a *parabolic-like map* in the sense of [L], except one: on a neighbourhood of the point $S_a = \mathcal{F}_a^{-1}(P) \setminus \{P\}$ it is not a single-valued map, as such a neighbourhood is mapped one-to-two onto a neighbourhood of P. However by redefining $\mathcal{F}_a$ on a 'sector' at P lying outside $\Lambda_{a,-}$, and adjusting the complex structure on this sector and its inverse images, we shall now modify a restriction of the branch of $\mathcal{F}_a$ fixing $\Lambda_{a,-}$, to yield a parabolic-like map $\widetilde{\mathcal{F}}$.

To simplify notation let us omit the dependence on the parameter a. By Proposition 5.2, at the parabolic fixed point P the boundary $\partial V'$ of V' forms a cone of angle $\hat{\theta} = \pi - (\theta_1 + \theta_2)$, and at the preimage S of P it forms a cone of angle $2\hat{\theta}$. Possibly by reducing θ_1, θ_2 and $\hat{\theta}$ we can choose $\epsilon > 0$ small enough so that the round disc $D_2 = \mathbb{D}(S, \epsilon)$ intersects V' in a sector of angle

$2\hat{\theta}$, so that $D_1 = \mathcal{F}(D_2)$ intersects V' in a sector of angle $\hat{\theta}$, and moreover $\partial V \cap \gamma_i \not\subset \overline{D}_1$ (where V is the set, γ_1, γ_2 the invariant arcs, and $\hat{\theta}$ the angle given by Proposition 5.2). Hence denoting by $\hat{T}_2$ the sector $(\pi - 2\theta_2, S, \pi + 2\theta_1)$ and by $\hat{T}_1$ the sector $(3\pi/2 - \theta_2, P, \pi/2 + \theta_1)$, both $\hat{T}_1$ and $\hat{T}_2$ are outside V', and in particular $\hat{T}_2 \in V \setminus V'$. Set $\phi : D_2 \to \mathbb{D}$, $\phi(z) = (z - S)/\epsilon)$, and let $\psi : D_1 \to \mathbb{D}$ be the Riemann map sending P to 0. Then $\phi \circ \mathcal{F}^{-1} \circ \psi^{-1}$ is a degree 2 proper and holomorphic map from the unit disc into itself, with a unique fixed point at $z = 0$, and so pre- or post-composing with a rotation we can assume it to be the map $P_0(z) = z^2$.

We are now going to modify $\mathcal{F}$ on $\hat{T}_2$ by quasiconformal surgery. Lift to logarithm coordinates, and define the quasiconformal map

$$G : \{x + iy \mid x < 0,\ y \in [\pi, 3\pi]\} \to \{x + iy \mid x < 0,\ y \in [0, 2\pi]\}$$

as follows:

$$G(z) = \begin{cases} 2z - 2\pi i & \text{on } \{x + iy \mid x < 0,\ y \in [\pi, 3\pi/2 - \theta_2]\} \\ \text{quasiconformal interpolation} & \text{on } \{x + iy \mid x < 0,\ y \in [3\pi/2 - \theta_2, 5\pi/2 + \theta_1]\} \\ 2z - 4\pi i & \text{on } \{x + iy \mid x < 0,\ y \in [5\pi/2 + \theta_1, 3\pi]\} \end{cases}$$

Then the map $f = \phi^{-1} \circ exp \circ G \circ log \circ \psi : D_1 \to D_2$ is also quasiconformal. Define $U = V \cup D_1$, $U' = \mathcal{F}^{-1}(U)$, and the map $F : U' \to U$ to be:

$$F = \begin{cases} f^{-1} & \text{on } D_2 \\ \mathcal{F} & \text{on } U' \setminus D_2 \end{cases}$$

The map $F : U' \to U$ is continuous, because it coincides with $\mathcal{F}$ everywhere but on $\hat{T}_2$, and along the boundaries of $\hat{T}_2$ inside D_2 it is continuous by construction. So the map F is quasiregular, proper of degree 2, and holomorphic everywhere but on the sector $\hat{T}_2$.

Setting $\tilde{\mu} = (f^{-1})^*(\mu_0)$, and spreading $\tilde{\mu}$ by the dynamics of F, we obtain on U the Beltrami form:

$$\overline{\mu} = \begin{cases} \widetilde{\mu} & \text{on } \hat{T} \\ (F^n)^*\widetilde{\mu} & \text{on } F^{-n}(\hat{T}) \\ \mu_0 & \text{on } U \setminus F^{-n}(\hat{T}) \end{cases}$$

Since the sector $\hat{T}_2$ lies outside $F^{-1}(V)$, it follows that $F^{-i}(\hat{T}_2)$ lies outside $F^{-i-1}(V)$, and therefore the preimages of the sectors $F^{-i}(\hat{T}_2)$ where we change the structure do not intersect each others. Hence the Beltrami form $\overline{\mu}$ is F-invariant, and by the Measurable Mapping Theorem there exists a quasiconformal map $\varphi : U \to \mathbb{D}$ such that $\varphi^*\mu_0 = \overline{\mu}$. Let us define

$$\widetilde{\mathcal{F}} := \varphi \circ F \circ \varphi^{-1} : \mathcal{V}' = \varphi(U') \to \mathcal{V} = \varphi(U),$$

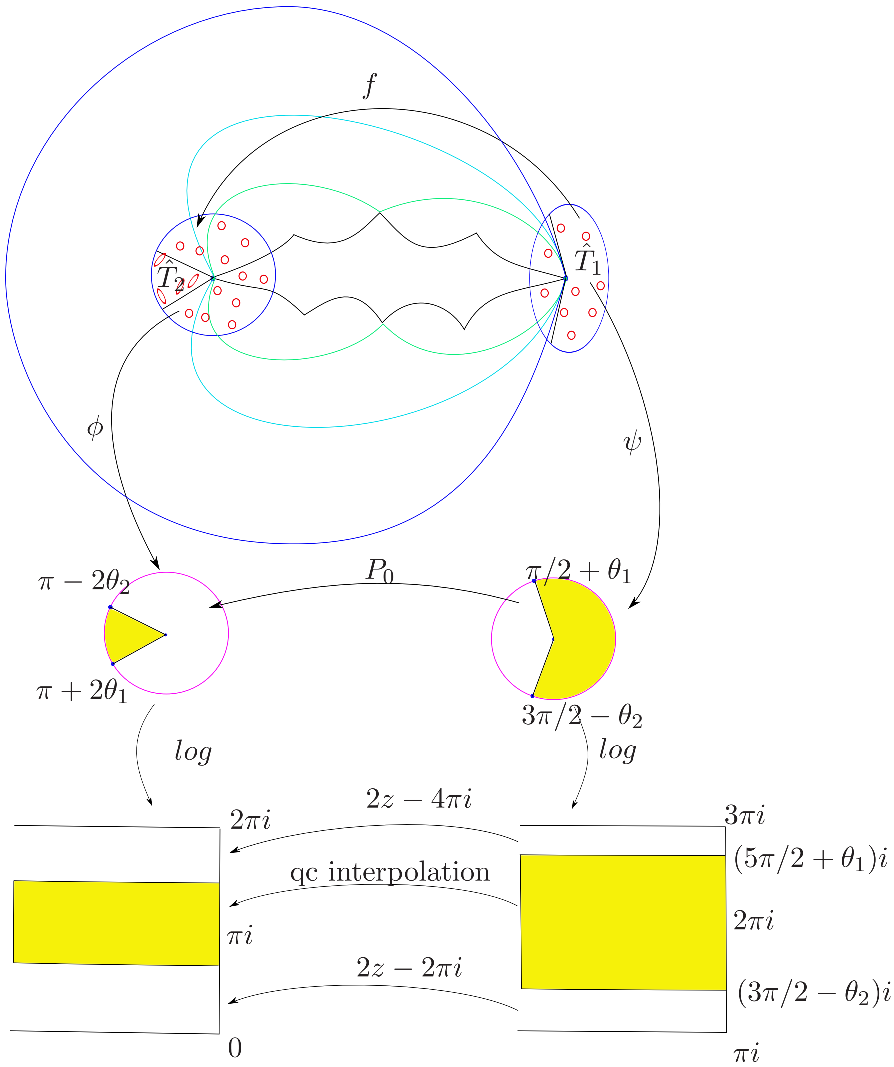

Figure 6: surgery construction

and set $\gamma_+ = \varphi(\gamma_1) \cap \overline{\mathcal{V}}$ and $\gamma_- = \varphi(\gamma_2) \cap \overline{\mathcal{V}}$ (where γ_1 and γ_2 are the invariant arcs given by Proposition 5.2). Then $\gamma = \gamma_+ \cup \gamma_-$ is a dividing arc in the sense of Definition 2.2, and $(\widetilde{\mathcal{F}}, \mathcal{V}', \mathcal{V}, \gamma)$ is a degree 2 parabolic-like map, with filled Julia set $K = \varphi(\Lambda_-)$. The map $\widetilde{\mathcal{F}}$ is quasiconformally conjugate to $\mathcal{F}$ everywhere but on the sector $\hat{T}_2$ and its image, which do not intersect the filled Julia set K. Moreover, this quasiconformal conjugacy is holomorphic everywhere but on the preimages of $\hat{T}_2$ (which do not intersect the filled Julia set K). Therefore $\widetilde{\mathcal{F}}$ is hybrid conjugate to $\mathcal{F}$ on K. By the Straightening Theorem for parabolic-like maps (see [L]), this implies that $\mathcal{F}$ is hybrid conjugate to a member of the family $Per_1(1)$ on Λ_-.

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School of Mathematical Sciences
Queen Mary University of London, London E1 4NS, UK
s.r.bullett@qmul.ac.uk

Instituto de Matemática e Estatistica da Universidade de São Paulo
Rua do Matão, 1010 - CEP 05508-090 - São Paulo - SP, Brasil
lluna@ime.usp.br