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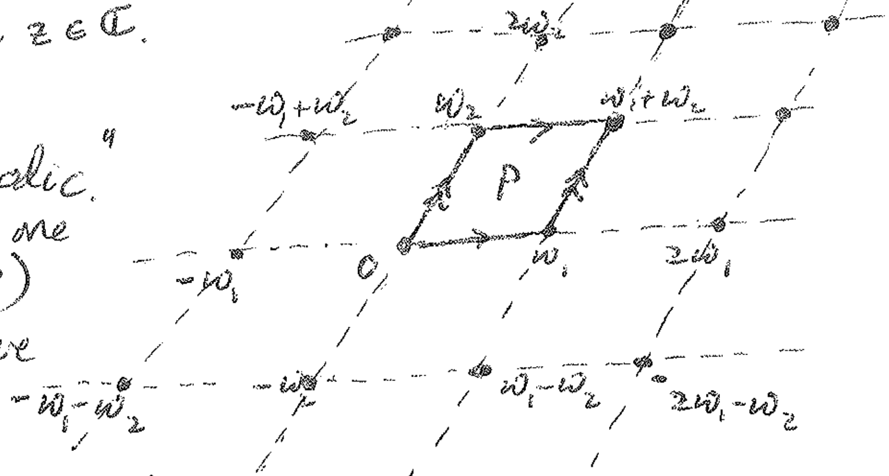
MTH6111 Complex Analysis
Friday 25th March 2011

Notes on elliptic functions, the Schwarz Reflection Principle, and the proof of Picard's Little Theorem

(These notes are additional to the typed lecture notes, where there are no pictures.)

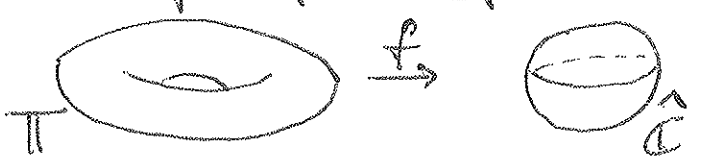
Elliptic Functions (§24 of lecture notes)

A meromorphic function f on $\mathbb{C}$ is said to be *elliptic* with respect to the lattice $\Omega = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\} \subset \mathbb{C}$ if $f(z+\omega) = f(z)$ $\forall \omega \in \Omega$ and $z \in \mathbb{C}$.



Such an f is "doubly-periodic." Once you know its values on one parallelogram (for example P) you know its values everywhere on $\mathbb{C}$.

We can regard f as a function from a torus $\mathbb{T}$ (P with opposite pairs of edges identified) to the Riemann sphere $\hat{\mathbb{C}}$. The *order* of f is the number of points of $\mathbb{T}$ mapping to a "typical" point of $\hat{\mathbb{C}}$ (it is the analogue of the *degree* of a rational map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$).



Idea of proof of 24.2

(i) Every elliptic function of order 0 is constant

Order 0 $\Rightarrow$ we can assume f has no poles, but then $f|_P$ is bounded (since P is closed and bounded) so f is bounded on $\mathbb{C}$, so constant (by Liouville)

(ii) There are no elliptic functions of order 1

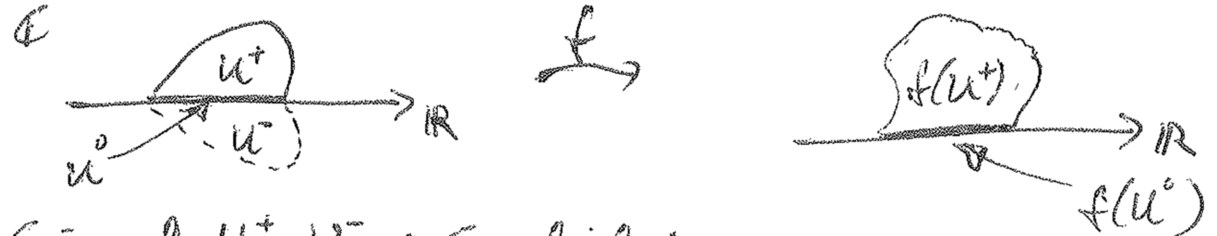
Order 1 $\Rightarrow$ single pole in P, but this contradicts the residue theorem (integrate f around the boundary of P and observe that opposite sides cancel)

2 There are elliptic functions of all orders $N \geq 2$:-

For $N \geq 3$ take $F_N(z) = \sum_{w \in \Omega} \frac{1}{(z-w)^N}$

For $N=2$ this expression does not converge, but we may replace it by $\wp(z) = \frac{1}{z^2} + \sum'_{w \in \Omega} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$, the Weierstrass $\wp$-function

Schwarz Reflection Principle (§23 of lecture notes)



Given $f : U^+ \cup U^0 \to \mathbb{C}$ which is

- holomorphic on U^+
- continuous on $U^+ \cup U^0$
- has $f(U^0) \subset \mathbb{R}$

we can extend f to a holomorphic function $f : U^+ \cup U^0 \cup U^- \to \mathbb{C}$ by defining $f(z)$ for any $z \in U^-$ to be $f(z) = \overline{f(\bar{z})}$ (where $\bar{z}$ = complex conjugate of z).

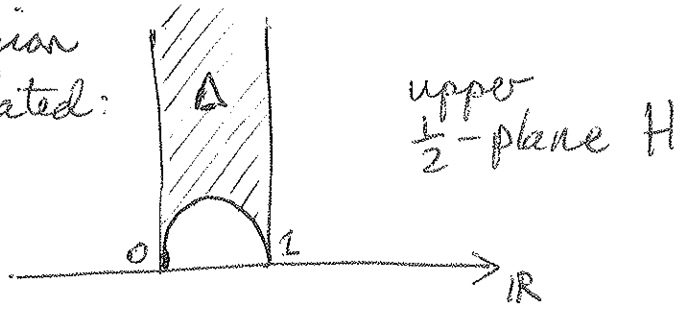
Picard's (Little) Theorem (§25 of lecture notes)

If $f : \mathbb{C} \to \mathbb{C}$ is entire and there at least two points of $\mathbb{C}$ not in the image of f, then f is constant.

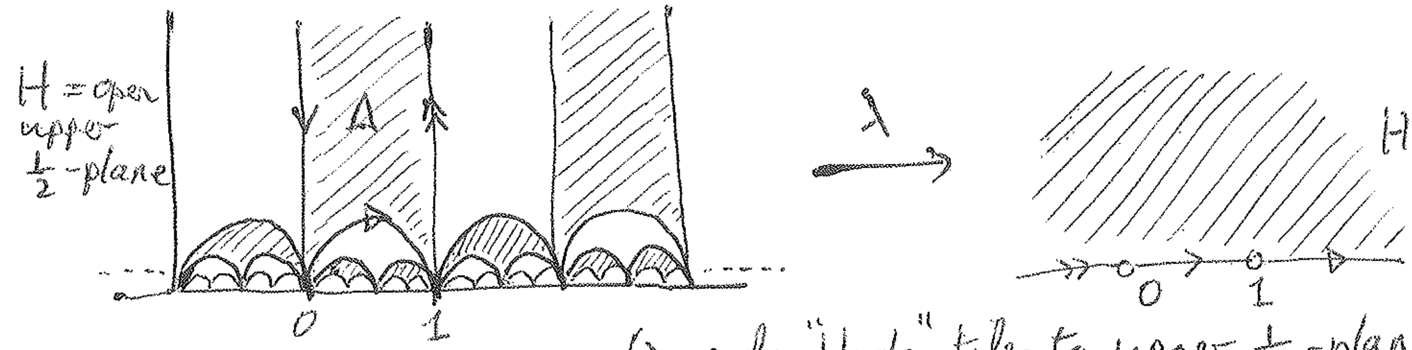
Idea of proof

Suppose (without loss of generality) the 2 points of $\mathbb{C}$ not in the image of f are 0 and 1.

Let Δ be the (open) region illustrated:



③ By the Riemann Mapping Theorem there is a conformal bijection λ sending Δ to the complex upper $\frac{1}{2}$-plane H, sending $\infty \to 0$, $0 \to 1$ and $1 \to \infty$.
By the Schwarz reflection principle we can extend λ to "tiles" obtained by "reflecting" Δ in its 3 sides. We can repeat the process on the "free" sides of these new tiles etc, until in the limit we have extended λ to a conformal map $H \overset{\lambda}{\twoheadrightarrow} \mathbb{C} - \{0, 1\}$:



(λ sends "black" tiles to upper $\frac{1}{2}$-plane, "white" tiles to lower $\frac{1}{2}$-plane and its image misses 0 & 1)

One can even write an explicit formula for λ (see notes).

Let $\nu : H \to \Delta$ denote the branch of λ^{-1} sending H to Δ.

Now given any entire $f : \mathbb{C} \to \mathbb{C} \setminus \{0, 1\}$, choose any z_0 with $f(z_0) \in H$, and analytically continue νf from a neighbourhood of z_0 to a holomorphic function $g : \mathbb{C} \to H$ (there is an extension, and it is unique, as $\mathbb{C}$ is simply-connected). But we know $\exists$ Möbius $\phi : H \to D$ (open unit disc). Now $g \circ \phi$ is bounded, hence constant by Liouville. So g is constant. So f is constant.