Chapter 4
Topological conjugacy and symbolic dynamics

Conjugacy is about when two maps have precisely ‘the same’ dynamical behaviour. Equivalently it is about ‘changing the co-ordinate system’ we use to describe a map.

Let \(f : X \to X\) and \(g : Y \to Y\) be continuous maps. Think of \(X\) and \(Y\) as subsets of \(\mathbb{R}^n\) for some \(n\), but they can be any metric spaces or indeed topological spaces.

Definition
A map \(h : X \to Y\) is a topological conjugacy from \(f : X \to X\) to \(g : Y \to Y\) if:

(i) \(h\) is a homeomorphism from \(X\) to \(Y\) (in other words it \(h\) is a bijection and both \(h\) and \(h^{-1}\) are continuous), and

(ii) \(hf = gh\) (as maps), or equivalently \(g = hfh^{-1}\), or \(f = h^{-1}gh\).

Lemma
(a) \(f : X \to X\) is conjugate to itself.
(b) If \(h\) is a conjugacy from \(f : X \to X\) to \(g : Y \to Y\), then \(h^{-1}\) is a conjugacy from \(g : Y \to Y\) to \(f : X \to X\).
(c) If \(h\) is a conjugacy from \(f : X \to X\) to \(g : Y \to Y\) and \(k\) is a conjugacy from \(g : Y \to Y\) to \(j : Z \to Z\), then \(kh\) is a conjugacy from \(f : X \to X\) to \(j : Z \to Z\).

Proof
(a) \(I_X f = f I_X (= f)\) (where \(I_X\) denotes the identity map \(X \to X\)).

(b) \(h\) a homeomorphism implies \(h^{-1}\) a homeomorphism; and \(hf = gh\) implies \(fh^{-1} = h^{-1}g\).

(c) \(h\) and \(k\) homeomorphisms implies \(kh\) is a homeomorphism; also \(hf = gh\) and \(kg = jk\) imply that \(khf = kgh = jkh\). □

This lemma tells us that conjugacy is an equivalence relation.

Lemma If \(h\) is a conjugacy from \(f : X \to X\) to \(g : Y \to Y\) then \(h\) is also a conjugacy from \(f^n\) to \(g^n\) for every positive integer.
Proof

\[ hf = gh \Rightarrow hf h^{-1} = g \Rightarrow (hf h^{-1})(hf h^{-1}) \cdots (hf h^{-1}) = g^n \Rightarrow hf^n h^{-1} = g^n \Rightarrow hf^n = g^n h. \]

Lemma A conjugacy \( h \) from \( f : X \to X \) to \( g : Y \to Y \) sends orbits of \( f \) to orbits of \( g \), and periodic orbits to periodic orbits of the same period.

Proof

Let \( x_0 \in X \) and let \( y_0 = h(x_0) \). By definition,

\[ O^+(x_0) = \{ f^n(x_0) : n > 0 \} \]

so

\[ h(O^+(x_0)) = \{ hf^n(x_0) : n > 0 \} = \{ g^n h(x_0) : n > 0 \} = \{ g^n y_0 : n > 0 \} = O^+(y_0). \]

Let \( x_0 \) be a fixed point of \( f \) (so \( f(x_0) = x_0 \)) and let \( y_0 = h(x_0) \).

\[ g(y_0) = gh(x_0) = hf(x_0) = h(x_0) = y_0. \]

Thus

\[ h(Fix(f)) \subseteq Fix(g) \]

Applying the same reasoning to \( h^{-1} \) in place of \( h \), we have

\[ h^{-1}(Fix(g)) \subseteq Fix(f) \]

Hence \( h \) is a bijection from \( Fix(f) \) to \( Fix(g) \).

Finally, suppose \( x_0 \) is a point of prime period \( n \) for \( f \), and let \( y_0 = h(x_0) \). Then

\[ f^k(x_0) = x_0 \quad \text{and for} \quad m < k \quad f^m(x_0) \neq x_0 \]

So

\[ g^k(y_0) = g^k h(x_0) = hf^k(x_0) = h(x_0) = y_0 \]

and for \( m < k \)

\[ g^m(y_0) = g^m h(x_0) = hf^m(x_0) \neq h(x_0) \]

(As \( h \) is a bijection, \( f^m(x_0) \neq x_0 \Rightarrow hf^m(x_0) \neq h(x_0) \).) Thus \( g^m(x_0) \neq y_0 \). Thus a conjugacy from \( f \) to \( g \) carries every dynamical feature \( f \) across to an identical feature of \( g \). In fact if the conjugacy is
diffeomorphism we can say even more more - the numbers we get from \( g \) as the multipliers of orbits are the same numbers as those we get from \( f \) as the multipliers of orbits:

**Lemma** If \( h \) is a diffeomorphism then the multiplier \( g'(y_0) \) of each fixed point of \( g \) is equal to the multiplier \( f'(x_0) \) of the corresponding fixed point of \( f \).

**Proof**

Let \( y_0 = h(x_0) \). By the chain rule,

\[
g'(y_0) = (hf^{-1})'(y_0) = h'(x_0)f'(x_0)(h^{-1})'(y_0)
\]

But \((h^{-1})'(y_0) = (h'(x_0))^{-1}\) (since \( y_0 = h(x_0) \)).

□

It often happens that given a map we can find a conjugacy which puts it into a more convenient form.

**Example 1**

Let \( \mu > 0 \). The map \( f(x) = \mu x(1 - x) \) is conjugate to a map \( g(x) = x^2 + c \) for a suitable value of \( c \).

**Proof**

We try to find a conjugacy of the form \( h(x) = ax + \beta \).

\[
hf = gh \Leftrightarrow \alpha(\mu x(1 - x)) + \beta = (\alpha x + \beta)^2 + c \quad \forall x
\]

\[
\Leftrightarrow -\mu \alpha x^2 + \mu \alpha x + \beta = \alpha^2 x^2 + 2\alpha \beta x + \beta^2 + c
\]

\[
\Leftrightarrow -\mu = \alpha \quad \text{and} \quad \mu = 2\beta \quad \text{and} \quad c + \beta^2 = \beta
\]

Thus \( f(x) = \mu x(1 - x) \) is conjugate to \( g(y) = y^2 + c \) when \( c = \mu/2 - (\mu/2)^2 \), the conjugacy being \( y = -\mu x + \mu/2 \).

□

**Example 2**

The map \( x \to x^2 + c \) has exactly one period 3 orbit if (and only if) \( c = -7/4 \). Thus the logistic map has a single period 3 orbit when \( \mu = 1 + 2\sqrt{2} \). (This tells us the value of \( \mu \) where the period 3 orbit is ‘born’ is \( 2 + 2\sqrt{2} = 3.829... \))

**Proof**

We shall prove the ‘if’ statement, but not the ‘only if’ statement.

\[
x \to x^2 + c \to (x^2 + c)^2 + c = x^4 + 2cx^2 + c^2 + c \to (x^4 + 2cx^2 + c^2 + c)^2 + c
\]
\[ x^8 + 4cx^6 + (6c^2 + 2c)x^4 + 4c(c^2 + c)x^2 + c^4 + 2c^3 + c^2 + c \]

Subtracting \( x \) from this expression and dividing by \( x^2 - x + c \) gives us the following equation with solution the points of prime period 3:

\[
x^6 + x^5 + (3c + 1)x^4 + (2c + 1)x^3 + (3c^2 + 3c + 1)x^2 + (c^2 + 2c + 1)x + (c^3 + 2c^2 + c_1) = 0
\]

In general this has 6 solutions, and in general we may expect them to be complex. However when \( c = -7/4 \) it is easily checked that the left hand side is a perfect square, and the equation becomes:

\[
(x^3 - x^2/2 - 9x/4 - 1/8)^2 = 0
\]

which has exactly 3 roots, all of them real. The value \( c = -7/4 \) corresponds to \( \mu = 1 + 2\sqrt{2} \), by Example 1, so this is the value of \( \mu \) at which the period 3 orbit is born. It can be shown that for all \( c < -7/4 \) (or equivalently for all \( \mu > 1 + 2\sqrt{2} \)) there are two distinct orbits on the real line which have period 3. □

**Example 3**

The tent map \( T \), defined by \( T(x) = 2x \) for \( 0 \leq x \leq 1/2 \) and by \( T(x) = 2(1 - x) \) for \( 1/2 \leq x \leq 1 \), is conjugate to the logistic map \( f_4(x) = 4x(1 - x) \), via the conjugacy \( h(x) = \sin^2(\pi x/2) \).

**Proof**

The map \( h \) is a homeomorphism from \([0,1]\) to \([0,1]\), since \( h(0) = 0, h(1) = 1 \), and \( h \) is continuous and monotone increasing on \([0,1]\) (as \( h'(x) = (\pi/2) \sin(\pi x) > 0 \forall x \in (0,1) \)). Indeed \( h \) is a diffeomorphism from \((0,1)\) to \((0,1)\) (though its inverse fails to be differentiable at the two end points 0 and 1).

For all \( x \in [0,1] \) we have:

\[
f_4h(x) = 4 \sin^2 \frac{\pi x}{2} \left(1 - \sin^2 \frac{\pi x}{2}\right) = \sin^2 \pi x
\]

For \( 0 \leq x \leq 1/2 \) we have:

\[
hT(x) = h(2x) = \sin^2 \pi x
\]

and for \( 1/2 \leq x \leq 1 \) we have:

\[
hT(x) = h(2 - 2x) = \left( \sin \frac{\pi(2 - 2x)}{2} \right)^2 = (\sin(\pi - \pi x))^2 = \sin^2 \pi x
\]

Thus \( hT(x) = f_4h(x) \) for all \( x \in [0,1] \). □

**Note.** This is special to \( \mu = 4 \). Other ‘tent maps’ of different heights are not conjugate to other \( f_\mu \). For example in the family of ‘tent maps’ of increasing heights, the whole period-doubling sequence takes place at a single parameter value (the value where the slope of the tent is 1).
Conjugacy carries across other properties in addition to the ones we have already proved. For example it is easily proved that if $h : X \to Y$ is a homeomorphism from $X$ to $Y$ and $S$ is a dense subset of $X$ then $h(S)$ is a dense subset of $Y$. So any conjugacy carries a dense set of orbits to a dense set of orbits.

As an example, consider the periodic orbits of the tent map $T$. By induction

$$T^n(x) = 0 \quad \text{for} \quad x = \frac{2k}{2^n} \quad 0 \leq k \leq 2^{n-1}$$
$$T^n(x) = 1 \quad \text{for} \quad x = \frac{2k - 1}{2^n} \quad 1 \leq k \leq 2^{n-1}$$

and $T^n$ is linear between these points of its graph. It follows from the graph of $T^n$ that $T^n$ has $2^n$ fixed points, and hence that $T$ has $2^n$ points of period dividing $n$. Moreover there is a periodic point of $T$ in each interval of the form $[k2^{-n}, (k + 1)2^{-n}]$. Thus the periodic points of $T$ are dense in $[0, 1]$, and hence the periodic points of $f_4$ are dense in $[0, 1]$. Moreover the periodic points of $T$ are all repellers (since the multiplier of a prime period $n$ point of $T$ has modulus $2^n$) and hence the same is true for the periodic points of $f_4$. Finally, we can compute all these periodic points for $f_4$ by computing them for $T$ (where the computations are straightforward) and then applying the conjugacy $h$.

Symbolic Dynamics

One way to describe orbits of the tent map (and hence also orbits of the logistic map $f_4(x) = 4x(1 - x)$) is by their ‘left-right’ itineraries: to each $x_0 \in [0, 1]$ we assign a sequence of L’s and R’s by writing ‘L’ if the $n$th point of the orbit of $x_0$ is in $[0, 1/2]$ and ‘R’ if it is in $[1/2, 1]$. This raises two questions:

(i) For each sequence of L’s and R’s does there necessarily exist a point $x_0$ with this as its itinerary?

(ii) Can two different initial points have the same itinerary?

In fact for the tent map (or $f_4$) the answer to (i) is ‘yes’ and the answer to (ii) is ‘no’ (provided we exclude orbits which contain the ‘ambiguous’ point 1/2). As a consequence we can reduce the study of $T$ (or $f_4$) to a study of all possible sequences of L’s and R’s. We shall investigate a slightly simpler map (the doubling map) this way, but the methods and proofs for $T$ are very similar.

The doubling map

$$D(x) = 2x \mod 1$$

for $x \in [0, 1)$. (Equivalently $z \to z^2$ on the unit circle in $\mathbb{C}$.)

Recall that, like the tent map, this map too has a dense set of periodic points: any rational of the form $p/(2^n - 1)$ is periodic, of period dividing $n$. 

5
The best coordinate system for investigating the doubling map is the real numbers expressed in base 2.

Write \( x \in [0,1) \) in binary form:

\[
x = \frac{b_1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \ldots \quad b_j = 0 \text{ or } 1
\]

We have an obvious map

\[
h : S = \{(b_1, b_2, \ldots) : b_j = 0 \text{ or } 1\} \to [0,1)
\]

We define the shift map \( \sigma : S \to S \) by

\[
\sigma(b_1, b_2, b_3, \ldots) = (b_2, b_3, \ldots)
\]

Then

\[
h\sigma = Dh
\]

However although \( h \) is surjective it is not injective (for example \( h(0,1,1,\ldots) = 0 + 1/4 + 1/8 + 1/16 + \ldots = 1/2 \) and \( h(1,0,0,\ldots) = 1/2 \)). Thus the map \( h \) is what is known as a semi-conjugacy rather than a conjugacy.

To each point \( x \in [0,1] \) we assign a sequence of 0’s and 1’s by writing down a binary representation of \( x \).

**Examples**

\[
1/3 = .010101\ldots \quad \text{which we denote } \overline{01}
\]

is periodic, of prime period 2.

\[
1/7 = .001001\ldots = \overline{001}
\]

is periodic, of prime period 3.

**Remark** Recall that \( r \in \mathbb{R} \) is rational if and only if the binary representation of \( r \) eventually repeats, in blocks, that is to say if and only if the corresponding binary sequence is preperiodic.

Clearly there are non-periodic orbits of \( \sigma \), for example the orbit of

\[
01001000100001\ldots
\]

but do there exist points of \( [0,1) \) whose orbits under \( D \) are dense in \( [0,1) \)?

Construct a binary sequence by writing down all possible blocks of length 1, then all possible blocks of length 2, then all possible blocks of length 3 etc.: 

\[
0, 1, \{0,0\}, \{0,1\}, \{1,0\}, \{1,1\}, \{0,0,0\}, \ldots
\]
Let $s_0$ be this sequence and let $x_0 = h(s_0)$, the real number which has $s_0$ as its binary representation.

**Proposition** The orbit of $x_0$ is dense in $[0,1)$.

**Proof**

Let $y_0$ be any point in $[0,1)$. Let the base 2 representation of $y_0$ be:

$$y_0 = 0.b_1 b_2 b_3 \ldots$$

Given any $\epsilon > 0$ let $n$ be such that $2^{-n} < \epsilon$. We shall show that there exists an orbit point $x_N = D^N(x_0)$ such that $|y - x_N| \leq 2^{-n}$. By the construction of $x_0$ there exists an $N$ such that the binary sequence for $x_N$ starts with $b_1, b_2, \ldots, b_n$. Thus

$$|y - x_N| \leq 2^{-n}$$

as required. □

**Remark**

There is a theorem (due to G.H. Hardy) that if an infinite sequence of 0’s and 1’s is chosen at random, then with probability 1 it will contain all possible finite sequences of 0’s and 1’s, and hence the orbit under $D$ of the corresponding point of $[0,1)$ will be dense in $[0,1)$. Another application of this theorem is that if you sit a monkey down at a typewriter then, with probability 1, he (or she) will eventually type out the complete works of William Shakespeare (though the solar system might have come to an end before that actually happens ....).