Chapter 3
The logistic map, period-doubling and universal constants

We consider the discrete time dynamical system known as the logistic map

\[ x_{n+1} = \mu x_n (1 - x_n) \]


We digress to consider the corresponding differential equation:

\[ \dot{x} = \alpha x - \beta x^2 \]

The equation for uncontrolled population growth is \( \dot{x} = \alpha x \), which has solution \( x = x_0 e^{\alpha t} \). The term \( ' - \beta x^2' \) is introduced as to counteract this growth when the population \( x \) becomes large (for example, there is not enough food to go round once the population becomes too big).

This model has very simple dynamics (a repelling fixed point at \( x = 0 \) and an attracting fixed point at \( x = \alpha / \beta \)) so it was a surprise to find that the corresponding discrete time system has much more complicated behaviour.

To ‘discretise’ \( \dot{x} = \alpha x - \beta x^2 \) we put \( \dot{x} = \Delta x / \Delta t \), to get:

\[ \Delta x = \Delta t (\alpha x - \beta x^2) \]

Setting \( \Delta x = x_{n+1} - x_n \) we get:

\[ x_{n+1} = (1 + \alpha \Delta t) x_n - \beta \Delta t x_n^2 \]

A change of variable of the form \( y_n = k x_n \) now puts this into the form

\[ y_{n+1} = \mu y_n (1 - y_n) \]

(It is easily checked that we must take \( \mu = 1 + \alpha \Delta t \) and \( k = \beta \Delta t / \mu \) to obtain this.)

Note that the discrete time system is a good approximation to the continuous times system when \( \Delta t \) is small. However, as we shall see, its behaviour becomes more complicated when \( \Delta t \) (or equivalently \( \mu \)) is larger.
We first observe that the graph of the logistic map $f_\mu(x) = \mu x (1 - x)$ is symmetric about the line $x = 1/2$ (since $f_\mu(x) = f_\mu(1 - x)$) and that the function has a maximum at $x = 1/2$. This maximum value is $f_\mu(1/2) = \mu/4$. Thus for $\mu \in [0, 4]$ the function $f_\mu$ maps the closed interval $[0, 1]$ into itself. We shall restrict our attention to these values of $\mu$.

**Fixed points**

The fixed points of $f_\mu$ are given by:

$$\mu x (1 - x) = x$$

Thus they are

$$x = 0 \quad \text{and} \quad x = 1 - \frac{1}{\mu}$$

Next we compute the derivative of $f_\mu$ at the fixed points:

$$f'_\mu(x) = \mu(1 - 2x) \quad \text{so} \quad f'_\mu(0) = \mu \quad \text{and} \quad f'_\mu(1 - 1/\mu) = 2 - \mu$$

Thus $x = 0$ is an **attractor** for $0 \leq \mu < 1$ and a **repeller** for $\mu > 1$.

And $x = 1 - 1/\mu$ is an **attractor** for $1 < \mu < 3$ and a repeller for $\mu > 3$.

**Comment**

For $1 \leq \mu < 3$ it can be shown that all initial points $x_0 \in (0, 1)$ lie in the basin of attraction on $1 - 1/\mu$, so there can be no periodic points of prime period $> 1$ for these values of $\mu$.

**Period 2 cycles**

At $\mu = 3$ we have $f'_\mu(1 - 1/\mu) = -1$ and so $(f^2_\mu)'(1 - 1/\mu) = (-1)(-1) = +1$.

When $\mu$ is just above 3, the slope of the graph of $f^2_\mu$ at the fixed point becomes greater than one and this graph intersects the line $y = x$ at two new points either side of the fixed point. The new fixed points of $f^2_\mu$ are not fixed points of $f_\mu$ so they must form a new cycle of $f_\mu$ of period 2. Moreover the slope of $f^2_\mu$ is $< 1$ at each of the new points, so this period 2 cycle is an attractor. It can be shown that while this 2-cycle remains an attractor, every point of $(0, 1)$, except the repelling fixed point and its pre-images, is in the basin of attraction of the 2-cycle. Thus there are no other periodic cycles except for the 2-cycle and the two fixed points $(0$ and $1 - 1/\mu)$.

As $\mu$ increases further we reach a value where the period 2 cycle changes type, from attracting to repelling, and a new period 4 cycle is born. This process repeats, to give a **period-doubling cascade**.
When does the period 2 cycle change type? Let the orbit be \( \{p_1, p_2\} \). We first have to compute this orbit. We recall that \( p_1 \) and \( p_2 \) are the solutions of \( f_{\mu}^2(x) = x \) other than \( x = 0 \) and \( x = 1 - 1/\mu \). Now

\[
f_{\mu}^2(x) = x \Rightarrow \mu(\mu x(1 - x))(1 - \mu x(1 - x)) - x = 0
\]

\[
\Rightarrow x(\mu + 1 - \mu)(\mu^2 x^2 - (\mu^2 + \mu)x + (1 + \mu)) = 0
\]

So \( p_1 \) and \( p_2 \) are the solutions of:

\[
(**) \quad x^2 - \left(\frac{\mu + 1}{\mu}\right)x + \frac{\mu + 1}{\mu^2} = 0
\]

So, in particular, real \( p_1 \) and \( p_2 \) exist if and only if

\[
\left(\frac{\mu + 1}{\mu}\right)^2 \geq \frac{4(\mu + 1)}{\mu^2}
\]

\[
\Leftrightarrow \mu + 1 \geq 4 \Leftrightarrow \mu \geq 3
\]

Thus the period 2 cycle exists for all values of \( \mu \geq 3 \).

But to find when it is an attractor we need to compute the value of \( f_{\mu}^{'\mu}(p_1)f_{\mu}^{'\mu}(p_2) \) and determine for what values of \( \mu \) its modulus is < 1.

\[
f_{\mu}^{'\mu}(p_1)f_{\mu}^{'\mu}(p_2) = (\mu - 2\mu p_1)(\mu - 2\mu p_2)
\]

\[
= \mu^2(1 - 2(p_1 + p_2) + 4p_1p_2)
\]

which, using \((*)\),

\[
= \mu^2(1 - 2\left(\frac{\mu + 1}{\mu}\right) + 4\left(\frac{\mu + 1}{\mu^2}\right))
\]

\[
= -\mu^2 + 2\mu + 4
\]

But \(-\mu^2 + 2\mu + 4 = 1\) when \( \mu = 3 \), and for \( \mu > 3 \) it decreases monotonically. It reaches the value 0 when \( \mu = 1 + \sqrt{5} \) and the value \(-1\) when \( \mu = 1 + \sqrt{6} \).

Thus the period 2 cycle is an attractor for \( 3 < \mu < 1 + \sqrt{6} \), and becomes a repeller for \( \mu > 1 + \sqrt{6} \).

The period-doubling repeats, to produce, at the \( n \)th doubling, an attracting cycle of period \( 2^n \). Let \( \mu_k \) be the value of \( \mu \) where the \( k \)th period-doubling occurs. Thus

\[
\mu_0 = 1
\]

\[
\mu_1 = 3
\]

\[
\mu_2 = 1 + \sqrt{6} = 3.449
\]
This sequence accumulates at a limit:
\[ \mu_{\infty} = 3.61547 \ldots \]

Moreover, writing
\[ d_k = \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} \]

it was observed experimentally by Feigenbaum and Cvitanovic (and others) around 1975 that the sequence \( \{d_k\} \) has a limit
\[ d_{\infty} = 4.669202 \ldots \]

now often called the Feigenbaum ratio. It was also observed that one gets the same ‘universal value’ of \( d_{\infty} \) for all analytic families of ‘quadratic-like maps’ (one humped maps with non-zero second derivative at the peak).

The period-doubling route to chaos has been observed in many different contexts (chemical reactions, electronic circuits, dripping taps - at least in theory,...) and the same universal constant \( d_{\infty} = 4.669202 \ldots \) always appears.

A ‘renormalization theory’ explaining the existence of this constant was developed during the 1980s, and Feigenbaum’s conjectures were proved using computer-assisted methods. At the end of the 1980’s Dennis Sullivan produced a full pure mathematical proof, using ideas from complex analysis and hyperbolic geometry. See the 1993 book *One-dimensional dynamics* of van Strien and de Melo for this proof. Other proofs were developed by McMullen and by Lyubich in the early 1990’s.

The idea of renormalization is that at \( \mu = \mu_{\infty} \) (the period-doubling limit) the central region of the graph of \( f_\mu^2 \) looks almost exactly like the graph of \( f_\mu \) but rescaled and turned upside down. Similarly we get smaller and smaller regions of \( f_\mu^4, f_\mu^8, \) etc on which the graph gets closer and closer to a ‘universal’ shape. Define
\[ \mathcal{R} f_\mu(x) = L_\mu \circ f_\mu^2 \circ L_\mu^{-1}(x) \]

where \( L_\mu \) is a linear map (of the form \( x \rightarrow ax + b \)) which rescales the real line and moves the origin. Then repeatedly applying \( \mathcal{R} \) to \( f_{\mu_{\infty}} \) sends it closer and closer to a function \( F \) satisfying the ‘functional equation’
\[ F(x) = L \circ F^2 \circ L^{-1}(x) \]

One can show there exist a unique symmetric quadratic-like \( F \) and linear function \( L \) which satisfy this equation. Then starting from any family of quadratic-like maps and picking the map in this family which is at the period-doubling limit, one can show that by repeatedly applying \( \mathcal{R} \) to this map one gets closer and closer to \( F \). This process shows that near to the period-doubling limit the same pictures and
scaling ratios are always obtained, and this in turn can be used to proved the existence of the universal
Feigenbaum ratio $d_\infty$ (we shall not attempt to explain the details here: an outline of the strategy can be
found in the book by P.A.Glendinning, *Stability, Instability and Chaos*).

**The bifurcation diagram for the logistic family of maps**

The bifurcation diagram is drawn using a computer program like the following:

```plaintext
for $\mu := 0$ to 4 step 0.01
    $x := 0.01$; (any random value in (0,1) will do)
for $i := 1$ to 10000 step 1
    $x := \mu \ast x \ast (1 - x)$;
    if $i > 1000$ then plot ($\mu, x$);
next i
next $\mu$
```

Note that to allow the orbit to settle down on an attractor (if there is one) we compute the first 1000
points of the orbit before plotting anything. We plot a large number of points of each orbit (9000) because
for some values of $\mu$ the orbit is quite chaotic.

From such a plot, we see that new periodic orbits are born as $\mu$ increases. It can be shown that none die
(they just switch to repelling)- see Milnor and Thurston’s paper in Springer Lecture Notes in Mathematics
volume 1342 (1989) pages 465-563. By the time the period 3 attracting orbit is born there are repelling
periodic orbits of all periods $n \neq 2$ by Sarkovskii’s Theorem, but we do not see these on the plot.

The birth of the period 2 attractor is an example of a period-doubling (or pitchfork) bifurcation. The
‘handle’ of the pitchfork is the $(\mu, x)$ graph of the attracting fixed point $x = 1 - 1/\mu$ and this curve
continues as the ‘middle prong’ when the fixed point becomes a repeller. The other two prongs are made
up of the graph of the period 2 attracting cycle $x^2 - ((\mu + 1)/\mu)x + (\mu + 1)/\mu^2 = 0$.

By contrast the period 3 attractor is born via a tangent (or saddle) bifurcation. Here, at the value
$\mu = 3.829$, the graph of $f_\mu^3$ touches the line $y = x$ tangentially at three points (it still crosses the line
$y = x$ at the point $x = 1 - 1/\mu$). As $\mu$ passes this value what happens is that a pair of complex conjugate
period 3 orbits collide and become a real period 3 orbit, which then separates into a pair of period 3
orbits, one attracting and one repelling. For values of $\mu$ just below 3.829 the system behaves for long
periods as if it had a period 3 attractor, but has bursts of chaotic behaviour in between these long periods.
This phenomenon is called intermittency.
In the next chapter we shall examine the fully developed chaos exhibited by the logistic map at the parameter value $\mu = 4$. 