A holomorphic correspondence at the boundary of the Klein combination locus

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Abstract

We investigate an explicit holomorphic correspondence on the Riemann sphere with striking dynamical behaviour: the limit set is a fractal resembling the one-skeleton of a tetrahedron and on each component of the complement of this set the correspondence behaves like a Fuchsian group.

1 Introduction

A one (complex) parameter family of holomorphic correspondences $\mathcal{F}_a$ containing matings between the modular group and quadratic polynomials was discovered by the first author and Christopher Penrose nearly twenty years ago [3], and further investigated in [2] and [1]. Two naturally defined subsets of interest in the parameter space are:

(1) The *connectivity locus* $\mathcal{M}$.
(2) The *Klein combination locus* $\mathcal{C} \supset \mathcal{M}$.

Both loci will be given precise mathematical definitions below, in Section 2. But we mention now that $\mathcal{M}$ is conjectured to be the set of values of a such that $\mathcal{F}_a$ is a mating between $PSL_2(\mathbb{Z})$ and $q_c : z \to z^2 + c$ for some $c \in M$, the Mandelbrot set, and that $\mathcal{C}$ is the interior of the set of values of a for which the action of $\mathcal{F}_a$ is 'faithful and discrete', in a sense not yet formalised. The set $\mathcal{M}$ is conjectured to be homeomorphic to the Mandelbrot set and, and the set $\mathcal{C}$ is conjectured to be homeomorphic to a topological disc.

The correspondences $\mathcal{F}_a$ for $a \in \mathcal{C} \setminus \mathcal{M}$ may be thought as 'matings between $PSL_2(\mathbb{Z})$ and maps q_c for which the Julia set is a Cantor set'. A classification of these correspondences will be presented in [4]. Any two such $\mathcal{F}_a$ are quasiconformally conjugate, provided each has no *critical relations*. The exceptional values of a, for which $\mathcal{F}_a$ has some critical relation, form a countable set of isolated points in $\mathcal{C} \setminus \mathcal{M}$. Our interest in the current article is in the behaviour of $\mathcal{F}_a$ as a approaches the outer boundary of $\mathcal{C}$ (avoiding the exceptional values)

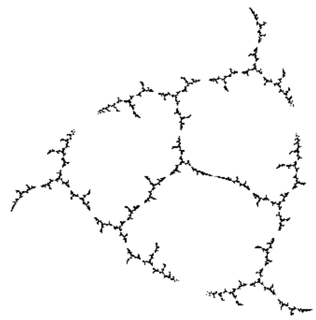

Figure 1: The limit set of the correspondence $\mathcal{F}_{a_{1/3}}$. The 'gaps' are an illusion arising from the presence of parabolic points.

and what happens when a reaches that boundary. A good analogy is the behaviour of a one (complex) parameter family of Kleinian groups, representations $G \to PSL_2(\mathbb{C})$ of a finitely generated abstract group G indexed by a parameter a, as a approaches the boundary of the slice in which the representations are discrete and faithful. In this paper we shall focus on one particular boundary point of $\mathcal{C}$, which we call the *Penrose point*, and investigate the dynamical behaviour (Figure 1) of the correspondence which has this as its parameter value, in particular showing that the complement of the limit set has four components and that for each component the subcorrespondence which stabilises it is conjugate to a Fuchsian group. More general results and conjectures concerning correspondences in the family $\mathcal{F}_a$, and the structure of $\mathcal{C}$ and its boundary, will be presented elsewhere.

2 The family of matings, the connectivity locus $\mathcal{M}$, and the Klein combination locus $\mathcal{C}$

In this section we briefly introduce the family of holomorphic correspondences $\mathcal{F}_a$ in which we are interested. For more details the interested reader should consult the orginal paper [3].

The first ingredient in the definition of $\mathcal{F}_a$ is the notion of a covering correspondence. For a given rational map $q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ the associated *covering correspondence* Cov^q is defined by the relation

$$(z, w) \in Cov^q \Leftrightarrow q(z) = q(w).$$

For each $z \in \hat{\mathbb{C}}$, by letting $Cov^q(z) = \{w : (z, w) \in Cov^q\}$ we can consider the correspondence to be a multifunction. Doing so allows us to consider correspondences from a dynamical perspective. Taking this point of view it is often

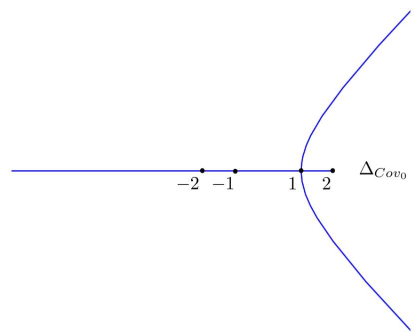

Figure 2: Fundamental domains for Cov_0: the curves are the preimage of $[-\infty, 2]$ under $Q(z) = z^3 - 3z$.

convenient to consider the associated *deleted covering correspondence* Cov_0^q defined by

$$(z, w) \in Cov_0^q \Leftrightarrow \frac{q(z) - q(w)}{z - w} = 0.$$

Noting that all rational maps are branched covering maps of the sphere to itself we make use of branch cuts to understand how Cov^q acts as a transformation of the Riemann sphere. For the purposes of the current article we make the following definition of a *fundamental domain* for this action.

Definition 1 *A fundamental domain for the action of Cov^q is any component of $\hat{\mathbb{C}} \setminus q^{-1}(\ell)$, where ℓ is any piecewise smooth simple arc which starts and ends at two of the critical values of q and passes through all the others.*

Up to conjugacy there are only two possibilities for covering correspondences of cubic *polynomials*. This is because in addition to the critical point ∞, the polynomial either has a double critical point (which we may take to be at 0) or two simple critical points. We restrict attention to the second case and to the particular cubic polynomial $Q(z) = z^3 - 3z$. The critical points of Q are $-1, 1$ and ∞, and the corresponding critical values are $2, -2$ and ∞. The preimages of 2 are -1 and 2, the preimages of -2 are 1 and -2, and the only preimage of ∞ is ∞. Consider the preimage under Q of the real interval $[-\infty, 2]$ (see Figure 2). We note that $\hat{\mathbb{C}} \setminus Q^{-1}([-\infty, 2])$ consists of three components and that Cov_0^Q maps each of these components homeomorphically onto each of the others. Thus any choice of component gives us a fundamental domain for the action Cov^Q on $\hat{\mathbb{C}}$. We choose as our fundamental domain $\Delta_{Cov} = \Delta_{Cov_0}$ the right-hand component in Figure 2, with boundary the curves running from 1 to ∞, at asymptotic angles $\pm\pi/3$, together with the cut from 1 to 2. The correspondence Cov_0^Q maps the cut $[1, 2]$ one-to-two onto the interval $[-2, 1]$, sending 1 to -2 and to 1, and sending 2 to the single point, -1. The point

-2 also has a unique image under Cov_0^Q, namely 1, but all other points of $\hat{\mathbb{C}}$ (except ∞) have two distinct images.

The $(2;2)$ correspondences $\mathcal{F}_a$ with which we shall be concerned are defined for all $a \in \mathbb{C}$ with $a \neq 1$. We set $\mathcal{F}_a$ to be the composition $J \circ Cov_0^Q$ where $J = J_a$ is the (unique) involution of $\hat{\mathbb{C}}$ which has fixed points $+1$ and a. Any round circle C passing through $+1$ and a is mapped to itself by J and divides $\hat{\mathbb{C}} \setminus C$ into two components, either of which can be taken as a fundamental domain Δ_J for J. (This is a fundamental domain in the sense of Definition 1 if we view J as Cov_0^q where q is the projection from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}/J$.) A variant of the Klein Combination Theorem tells us that if

1. $\Delta_{Cov} \cap \Delta_J \neq \emptyset$, and
2. $\overline{\Delta}_{Cov} \cup \overline{\Delta}_J = \hat{\mathbb{C}}$

then $\Delta = \Delta_{Cov} \cap \Delta_J$ is a fundamental domain (in an appropriate sense) for the action of the free product $\{Id, J\} * Cov^Q$ acting on the union, Ω, of the images of Δ under iterated applications of J and Cov^Q together with those points which lie on the boundary of a *finite* number of images of Δ, and that this is a proper discontinuous action. The grand orbits of the action of the free product are precisely the grand orbits of the composite $(2:2)$ correspondence $\mathcal{F}_a = J \circ Cov_0^Q$ (see [3]).

Remark The sense in which we are using the term 'fundamental domain' here is that the images of the (open) tile Δ have closures which together cover Ω and that these closures meet only at boundaries.

We can satisfy the two Klein combination conditions stated above by, for example, taking a in the real interval $(1, 7]$. For such values of a, the circle C having centre on the real axis, and passing through 1 and a, meets the boundary of Δ_{Cov} at the single point 1, and so if we take as Δ_J the component of $\hat{\mathbb{C}} \setminus C$ containing ∞ then Δ_{Cov} as chosen above and Δ_J satisfy the conditions. Now let D_J denote the closed disc complementary to Δ_J. This disc has image $D_1 = \mathcal{F}_a(D_J)$ contained in D_J, and $\mathcal{F}_a^{-1} : D_1 \to D_J$ is a *pinched quadratic-like map* with pinch point 1 and critical point 2. Let $\Lambda_+ = \bigcap_{n=1}^{\infty} (\mathcal{F}_a)^n(D_J)$, $\Lambda_- = J(\Lambda_+)$ and $\Lambda = \Lambda_+ \cup \Lambda_-$. Then Λ and Ω partition $\hat{\mathbb{C}}$, and for certain values of a (see [3, 2, 1]) $\mathcal{F}_a$ is a mating in the sense of the definition below.

Represent the action of the modular group $PSL_2(\mathbb{Z})$ on the complex upper half-plane by the *modular correspondence*,

$$(z, w) \in F_{Mod} \Leftrightarrow ((\tau_1(z) - w)(\tau_2(z) - w) = 0.$$

where

$$\tau_1(z) = z + 1 \qquad \text{and} \qquad \tau_2(z) = \frac{z}{z+1}.$$

Definition 2 *Let $q_c : z \to z^2 + c$ be a quadratic polynomial with connected filled Julia set K_c. We say that a $(2:2)$ holomorphic correspondence F is a mating between q_c and the modular group $PSL_2(\mathbb{Z})$ if:*

1. *There exists a region Ω invariant under the action of F and a conformal homeomorphism h from Ω to the upper-half plane conjugating the action of F to F_{Mod}.*

2. *The complement of Ω in $\hat{\mathbb{C}}$ is the union of two sets Λ_+ and Λ_-, with $\Lambda_+ \cap \Lambda_-$ consisting of a single point, and with two quasiconformal homeomorphisms $h_+ : \Lambda_+ \to K_c$ and $h_- : \Lambda_- \to K_c$, conformal on interiors, which respectively conjugate the action of F^{-1} restricted to Λ_+, and that of F restricted to Λ_-, to the action of q_c on K_c.*

When considering real values of a it is convenient to choose fundamental domains Δ_{Cov} and Δ_J which are symmetric about the real axis, but in general in order for the action of $\mathcal{F}_a$ on Ω to be a faithful action of the free product of the cyclic group $\{Id, J\}$ of order two with the correspondence Cov of order three, all we need is the existence of some pair of fundamental domains Δ_{Cov}, Δ_J satisfying appropriate Klein combination conditions. Noting that both boundaries must contain the point 1, since 1 is both a critical point of Q and a fixed point of J, we make the following definition.

Definition 3 *The Klein combination locus for the family $\mathcal{F}_a$ is the set $\mathcal{C}$ of values of $a \in \hat{\mathbb{C}}$ such that there exist fundamental domains Δ_{Cov}, Δ_J for Cov^Q and J with:*

1. $\partial\Delta_{Cov} \cap \partial\Delta_J = \{1\}$*, and*
2. $\overline{\Delta}_{Cov} \cup \overline{\Delta}_J = \hat{\mathbb{C}}$

If $a \in \mathcal{C}$, then for Λ_+ to be connected (and hence $\Lambda = \Lambda_+ \cup \Lambda_-$ to be connected) all we need is the additional property that the critical point 2 of the pinched quadratic-like map $(\mathcal{F}_a)^{-1} : D_1 \to D$ should lie in $(\mathcal{F}_a)^n(D_J)$ for all $n > 0$.

Definition 4 *The connectivity locus for the family $\mathcal{F}_a$ is the subset $\mathcal{M} \subset \mathcal{C}$ of values of a for which there is a pair of fundamental domains Δ_{Cov}, Δ_J satisfying the conditions in Definition 3, and with the additional property that $2 \in \bigcap_{n=1}^{\infty} (\mathcal{F}_a)^n(D_J)$ (where $D_J = \hat{\mathbb{C}} \setminus \Delta_J$).*

A computer plot of the connectivity locus was presented in the orginal paper on this family of correspondences [3]. It is conjectured that the family $\mathcal{F}_a$ contains a mating of q_c with $PSL_2(\mathbb{Z})$ for every value of $c \in M$, the Mandelbrot Set, and that $\mathcal{M}$ is homeomorphic to M. The first conjecture was proved for a large subset of values of $c \in M$ in [1]. Our interest in the current article is in $\mathcal{C} \setminus \mathcal{M}$, and in particular a specific point on the boundary of $\mathcal{C}$. The way that we have defined $\mathcal{C}$ ensures that $\mathcal{C} \setminus \mathcal{M}$ is an open subset of the parameter space, and thus that the boundary points are outside $\mathcal{C}$.

Remark There is experimental evidence which suggests that for all a outside the closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ the periodic points of $\mathcal{F}_a$ are dense in $\hat{\mathbb{C}}$. It is tempting to hope that by analogy with the study of deformations of Kleinian groups there might be some definition of 'faithful and discrete' for a correspondence action which would allow us to characterise $\overline{\mathcal{C}}$ as the set of all values of $a \in \hat{\mathbb{C}}$ for which $\mathcal{F}_a$ satisfies this yet to be formulated property.

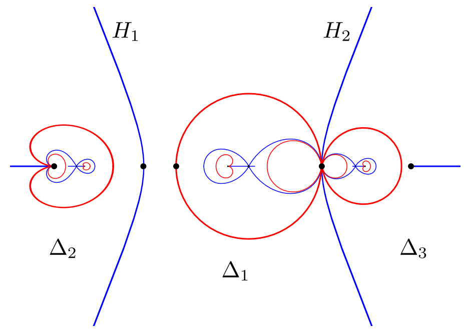

Figure 3: The fundamental domain Δ_1 and its first few images under combinations of Cov and J. (The points marked by black dots on the real axis are: $-2, -1, a_*, 1$ and 2.)

3 Dynamics of the correspondence $\mathcal{F}_a$ when a lies in $\mathcal{C} \setminus \mathcal{M}$

When $a \in \mathcal{C} \setminus \mathcal{M}$ we still have the 'Klein combination' set-up for Δ_{Cov} and Δ_J, and we still have a partition of the Riemann sphere into invariant regions Ω and $\Lambda = \Lambda_+ \cup \Lambda_-$, but the filled Julia set $\Lambda_+ = \bigcap_{n=1}^{\infty} (\mathcal{F}_a)^n(D_J)$ of the associated 'pinched quadratic-like map' is no longer connected. Indeed it is totally disconnected. Thus Ω is now the complement in $\hat{\mathbb{C}}$ of a Cantor set, and the action of $\mathcal{F}_a$ on Ω can no longer be conjugate to that of the modular group on the complex upper half-plane. Nevertheless the action of $\mathcal{F}_a$ on Ω, remains 'discontinuous', in the sense that the space of grand orbits on Ω has the structure of an orbifold. As will be shown in [4] and a future article, apart from a countable set of isolated parameter values where the singular points 2 and -2 of Cov_Q lie on the same grand orbit of $\mathcal{F}_a$, all the correspondences $\mathcal{F}_a$ with $a \in \mathcal{C} \setminus \mathcal{M}$ lie in a single quasi-conformal conjugacy class, and can be obtained from one another by deformations of the complex structure on the orbifold $\mathcal{O} = \Omega/\langle \mathcal{F}_a \rangle$. In the final section of this paper we investigate an example where one can follow a ray in deformation space all the way to the boundary.

We begin by describing the dynamics of a *generic* correspondence, that is to say an $\mathcal{F}_a$ with $a \in \mathcal{C} \setminus \mathcal{M}$ which does not have any critical coincidences. For example any value of a in the real interval $(-1, +1)$ will do. We pick some a in this interval as the 'base point' of our parameter space and denote it by a_*. We let H_1 denote the image of $[2, \infty]$ under Cov_0 and let H_2 be the image of $[-\infty, 2]$. The region bounded by H_1 and H_2 is a fundamental domain for Cov, and the Klein Combination Theorem tells us that the region Δ_1 bounded by H_1, H_2, and the circle through $a(= a_*)$ and 1 centred on the real axis, is a fundamental domain for the action of $J \circ Cov_0$ on the union of its images. This fundamental domain and its two images Δ_2 and Δ_3 under Cov_0 are shown in Figure 3. Write

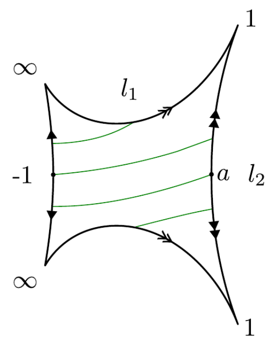

Figure 4: Δ_1 marked with its boundary identifications and the geodesic $\gamma_{1/3}$.

Δ for the union of $\Delta_1, \Delta_2, \Delta_3$ together with the parts $[-\infty, -2)$, $[2, \infty]$, H_1 and $H_2 \setminus \{1\}$ of their boundaries which meet only finitely many images of Δ_1. The complement of Δ consists of the closed discs D_1, D_2, and D_3. Now $J(\Delta)$ is contained in D_1 and $Cov_0 \circ J(\Delta)$ consists of two components, one contained in D_2 and the other in D_3. Continued iteration of J and Cov_0 gives us an infinite collection of images of Δ meeting along pairwise common edges, with the complement of their union Ω (including the common edges) a Cantor set contained within the real line.

The orbifold $\mathcal{O}$, the grand orbit space of Ω under the correspondence, is the quotient of Δ_1 under the boundary identifications indicated in Figure 4. It is a sphere with four cone points of types $\pi, \pi, 2\pi/3$ and π/∞ (the last one being a puncture point). One way to deform the complex structure on $\mathcal{O}$ is to contract a geodesic, for example a geodesic arc from the cone point -1 to the cone point a. Choices for such an arc correspond to rationals p/q with q odd, as follows. If we 'de-identify' the point 1 on the boundary of Δ_1 then we have a (hyperbolic) rectangle. We label two of the sides l_1 and l_2 as shown in the diagram, and we say that a geodesic from -1 to a has *slope* p/q, and denote it by $\gamma_{p/q}$, if it intersects l_1 and l_2 in p points and q points respectively: the example $\gamma_{1/3}$ is illustrated in Figure 4. The geodesic $\gamma_{p/q}$ lifts to a *lamination* $\Gamma_{p/q}$ on Ω: we simply mark the same 'pattern' on each 'tile'.

The global lamination corresponding to $\gamma_{1/3}$ is illustrated in Figure 5. It is tempting to try to deform $\mathcal{F}_{a_*}$ to a correspondence $\mathcal{F}_a$ with a on the boundary of $\mathcal{C}$, by contracting the leaves of $\Gamma_{p/q}$ to points, but there are technical difficulties. Even once one has excluded the possibility of a topological obstruction arising from a pair of leaves with the same end points, it is a daunting technical task to construct isotopies shrinking unions of arcs to points in such a way that the resulting correspondence is holomorphic (see [1]). We shall discuss this approach to the structure of the boundary of $\mathcal{C}$ elsewhere. Here we content ourselves with showing explicitly that the $\gamma_{1/3}$ pinch point can be reached, and describing

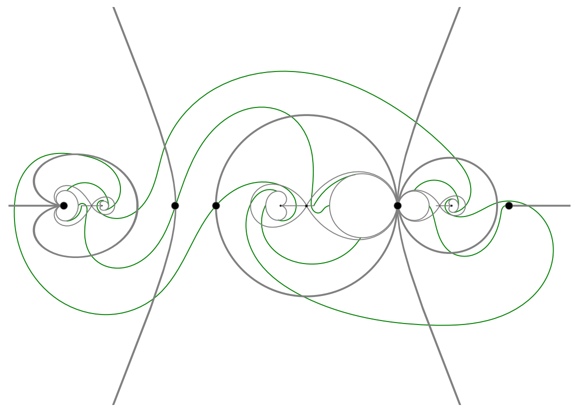

Figure 5: The global lamination $\Gamma_{1/3}$.

the behaviour of the correspondence there. The way that we shall do this is by identifying a unique candidate for the parameter value where the pinched dynamics could occur.

4 An example on the boundary: Penrose point

It is apparent from Figure 5 that the leaf of $\Gamma_{1/3}$ which passes through -1 also passes through one of the two points $J \circ Cov(a)$. Thus if we can pinch the geodesic $\gamma_{1/3}$ on the orbifold $\mathcal{O}$, then this will happen at a value of a where $-1 \in J \circ Cov(a)$. But then we will also have $a \in Cov(J(-1))$ and so, since a is fixed by J, we will now have both $-1 \in \mathcal{F}_a(a)$ and $a \in \mathcal{F}_a(J(-1))$. Since $J(-1) \in \mathcal{F}_a(-1)$ for any value of a, we see that the three points $a, -1, J(-1)$ will become a 3-cycle.

We next note that $J(-1) \in Cov(a)$ if and only if

$$a^2 + aJ(-1) + J(-1)^2 = 3.$$

But

$$J(z) = \frac{(a+1)z - 2a}{2z - (a+1)}$$

so

$$J(-1) = \frac{1+3a}{3+a}.$$

Thus

$$(a^2-3)(3+a)^2 + a(1+3a)(3+a) + (1+3a)^2 = 0,$$

which simplifies to

$$a^4 + 9a^3 + 25a^2 - 9a - 26 = 0,$$

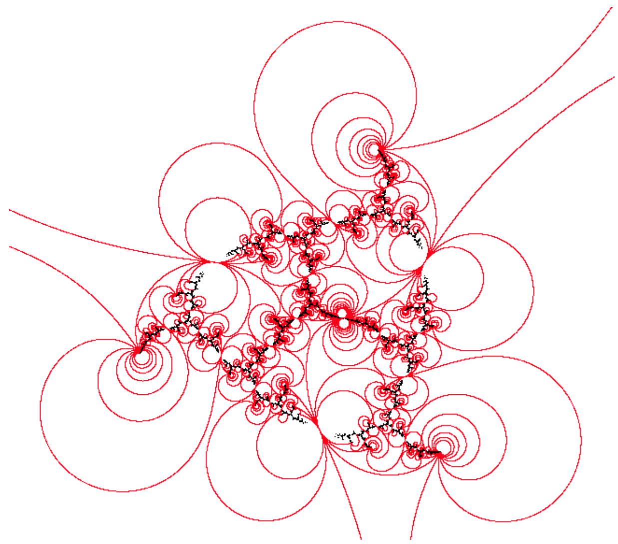

Figure 6: The correspondence $\mathcal{F}_{a_{1/3}}$. The complement Ω of the limit set is tiled by 'ideal hexagons' (Theorem 1).

and thence to

$$(a^2 + 9a + 26)(a^2 - 1).$$

We deduce that

$$a = -\frac{9}{2} + \frac{\sqrt{23}}{2}i$$

or its complex conjugate, and we denote the value which has positive imaginary part by $a_{1/3}$. (The value with negative imaginary part will correspond to pinching $\gamma_{-1/3}$.)

Figure 6 illustrates the dynamics of $\mathcal{F}_a$ for $a = a_{1/3}$. Shown in red are the images of the circular arc which runs from -1 through $+1$ to $J(-1)$. Shown in black is the grand orbit of the point $+1$ under all branches, forward and back, of the iterated correspondence $\mathcal{F}_a$. An equivalent description of the black set is that it is the set of all images of the point $+1$ under finite 'words' made up of the symbols J and Cov.

Remark *We have assigned $a_{1/3}$ the name 'Penrose point' in honour of Chris Penrose who originally found this example and plotted the limit set in the early 1990's (unpublished). Our re-discovery of it, and the analysis of its Fatou and limit sets reported on here, provided a key impetus for the formulation of conjectures (which we shall present elsewhere) describing the structure of the boundary of the Klein combination locus $\mathcal{C}$. 'Penrose point' lies at the tip of a promontory on the 'outer shoreline' of $\mathcal{C}$.*

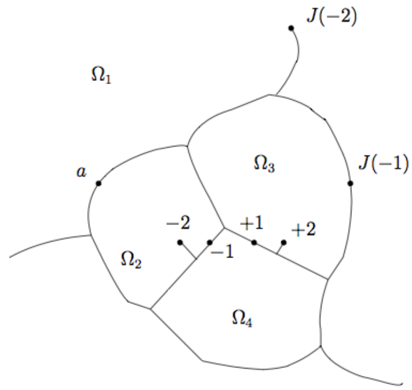

Figure 7: The components of the Fatou set and the positions of key points (schematic picture).

Theorem 1 *The correspondence $\mathcal{F}_{a_{1/3}}$ is obtained from $\mathcal{F}_{a_*}$ by pinching the geodesic $\gamma_{1/3}$ on the orbifold $\Omega(\mathcal{F}_{a_*})/\langle\mathcal{F}_{a_*}\rangle$. Moreover:*

1. *The ordinary set $\Omega = \Omega(\mathcal{F}_{a_{1/3}})$ of $\mathcal{F}_{a_{1/3}}$ is tessellated into ideal hexagons by the images of the circular arc from -1 through $+1$ to $J(-1)$. The action of $\mathcal{F}_{a_{1/3}}$ on Ω is a faithful action of the free product of the cyclic group $\{Id, J\}$ with the $(3:3)$ correspondence Cov (where Cov acts on the hexagon containing ∞ as a pair of rotations fixing ∞).*

2. *Ω has four connected components Ω_k, $k = 1,2,3,4$, each conformally homeomorphic to the (open) upper half-plane.*

3. *For each of $k = 1,2,3,4$ the action of the set of branches of iterates of the correspondence $\mathcal{F}_{a_{1/3}}$ which stabilise Ω_k is conformally conjugate to the action of a free product of groups $C_3 * C_\infty$ on the upper half-plane, where C_3 is generated by an elliptic Möbius transformation of order three and C_∞ is generated by a parabolic Möbius transformation.*

Proof The positions of the key points we shall need in the proof are indicated in Figure 7. The circular arc referred to in the statement of the theorem is made up of an arc L from -1 to $+1$, and its image $J(L)$ from $+1$ to $J(-1)$. Consider the images of $L \cup J(L)$ under the identity, Cov, and $J \circ Cov$. These are shown in the left hand plot of Figure 8. We assert that $L \cup J(L)$, together with its image under Cov running from -1 to $a = a_{1/3}$ (through a cusp at -2), and its image under $J \circ Cov$ running from a to $J(-1)$ (through a cusp at $J(-2)$), form a piecewise smooth simple closed curve, invariant under J. Following the same notation

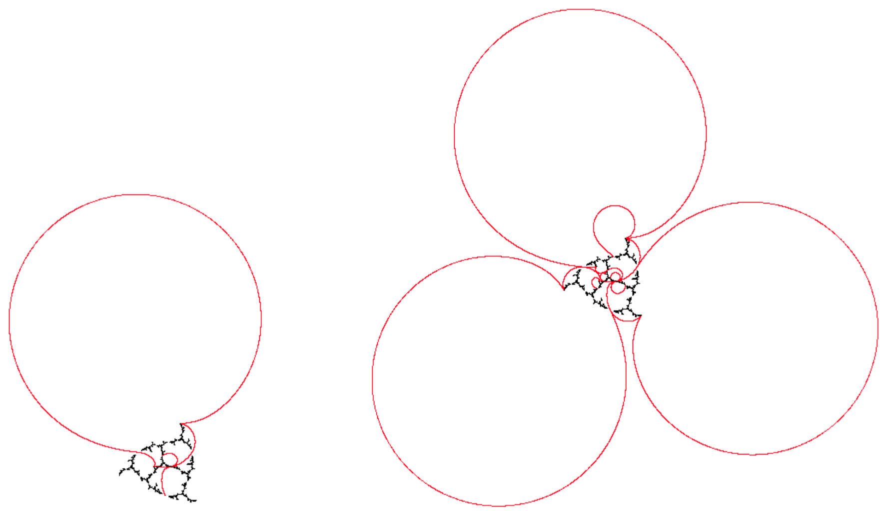

Figure 8: Left: the images of $L \cup J(L)$ under Cov and $JCov$ (these form a simple closed curve with cusps at -2 and $J(-2)$, together with two spurs emanating from $+1$). Right: the images of this set under a further application of Cov.

as in earlier sections of this paper, denote the component of the complement of this curve containing ∞ by Δ_J and the closure of the other component by D_J. We further assert that D_J and its images under Cov (plotted on the right in Figure 8) have disjoint interiors, that they are pairwise contiguous along L, $J(L)$ and the branch of $Cov(J(L))$ emanating from $+1$, and that the union of D_J with its two images is therefore also again a topological disc bounded by a piecewise smooth curve.

These assertions can be proved as follows (we omit details). Firstly, local analysis around the points of the period three cycle (which is parabolic), and around the (also parabolic) fixed point $+1$ and its image -2 under Cov, can be used to verify that the intersections of the arcs with neighbourhoods of the end points of L and their images are arranged as shown. Away from these neighbourhoods the arcs and their images are a definite distance apart and numerical estimation can be used complete the proof of the assertions. The same methods, of local analysis around the period three cycle and numerical estimation away from it, can be used to prove that the cycle can be 'unpinched' by a suitable small perturbation of the parameter value $a_{1/3}$, and thus that $\mathcal{F}_{a_{1/3}}$ is the correspondence obtained from $\mathcal{F}_{a_*}$ by pinching $\gamma_{1/3}$.

We deduce from the assertions above that D_J can be extended, as follows, to become a fundamental domain for the action of Cov on $\hat{\mathbb{C}}$. Join a to ∞ by any smooth curve M which is disjoint from its images under Cov and from $\partial\Delta_J$. Then the simple closed curve made up of M, the branch of $Cov(M)$ running from ∞ to $J(-1)$, and segments of the boundary of D_J running fom $J(-1)$ to -1 and -1 to a, will together bound a fundamental domain Δ_{Cov} for Cov. Now observe that the complement Δ_J of D_J is a fundamental domain for J, and that

$$\overline{\Delta}_{Cov} \cup \overline{\Delta}_J = \hat{\mathbb{C}}.$$

This is precisely the condition we need to apply the Klein Combination Theorem (in informal language, the 'ping-pong principle'). The Klein Combination Theorem has a simple statement in the case that the two fundamental domains concerned have disjoint boundaries, or when they meet at a single point (as in Section 2) but one has to take care when the boundaries meet along arcs, as in our situation here. The intersection $\Delta = \Delta_J \cap \Delta_{Cov}$ is the interior of a triangle which has two vertices of angle zero and one vertex of angle $2\pi/3$ (the vertex at ∞). Let $\mathcal{H}$ denote the ideal hexagon formed by the union of Δ with its two images under Cov, together with the points which lie on the boundaries of two of these trianges, and ∞ (which lies on the boundary of all three). Thus $\mathcal{H}$ is the external region (containing ∞) in the plot on the right in Figure 8. The ping-pong principle tells us at once that $\{Id, J\} * Cov$ acts freely on the union of images of $int(\mathcal{H})$, but we can do better than this and include the edges of $\mathcal{H}$ (though not its vertices). To see this, consider the following four ideal hexagons, which are disjoint apart from certain of their vertices: $\mathcal{H}_1 = \mathcal{H}$, $\mathcal{H}_2 = J(\mathcal{H})$, and the two images of $\mathcal{H}_2$ under Cov, which we denote $\mathcal{H}_3$ and $\mathcal{H}_4$. We note that any application of J or a branch of Cov to any of these $\mathcal{H}_i$ takes it either to another of the $\mathcal{H}_i$ or to a hexagon that has an edge in common with one of them. Setting Ω to be the union of the images of $\mathcal{H}$ together with its edges (but not its vertices) we deduce that $F_{a_{1/3}}$ has a proper discontinuous action on Ω, that this is a faithful action of the free product of $\{Id, J\}$ with Cov^Q, and that the 'centres' of the hexagons (the images of the point $\infty \in \mathcal{H}$), each have stabiliser a conjugate of the correspondence Cov. This gives us Statement 1 of the Theorem. Furthermore, the components of Ω are built up inductively from the four $\mathcal{H}_i$ by adjoining ideal hexagons along edges. Thus Ω is the disjoint union of four components, each containing one of the $\mathcal{H}_i$, and each homeomorphic to a disc (Statement 2).

The way that the four components of Ω are mapped to one another by Cov and J is determined by where the initial four ideal hexagons $\mathcal{H}_i$ are mapped. Thus Cov stabilises Ω_1 and sends each of $\Omega_2, \Omega_3, \Omega_4$ to the other two, and J exchanges Ω_1 with Ω_2 and also exchanges Ω_3 with Ω_4 (see Figure 7 for the labelling of the components). It is easily checked that Ω_2 is stabilised by $J \circ Cov_0 \circ J$ (which is a pair of rotations of Ω_2 of order three, inverse to one another) and by the two branches of $Cov \circ J \circ Cov$ (inverse to one another) which fix the point -2 on the boundary of Ω_2. (We remark that the latter branches are conjugate to the branches of $J \circ Cov$ and $Cov \circ J$ which fix $+1$; these are parabolic transformations, inverse to one another, stabilising both Ω_3 and Ω_4.) Furthermore, since the images of $\mathcal{H}_2$ under combinations of these elliptic and parabolic transformations fully tile Ω_2, we deduce that they freely generate the set of branches of iterates of $\mathcal{F}_{a_{1/3}}$ which stabilise Ω_2, and that this set of branches forms a group isomorphic to $C_3 * C_\infty$. $\square$

Remarks

1. In the proof above our choices of Δ_{Cov} and Δ_J had boundaries meeting along an arc. This was convenient for the proof, but it is possible to modify Δ_J, keeping Δ_{Cov} unchanged and retaining the condition that $\overline{\Delta}_{Cov} \cup \overline{\Delta}_J = \hat{\mathbb{C}}$, in such a way that the boundary of the new Δ_J meets that of Δ_{Cov} at just four points, namely the point 1 and the points of the 3-cycle.

2. The limit set of $\mathcal{F}_{a_{1/3}}$ (the complement of Ω) can be described combinatorially as the quotient of the unpinched limit set (a Cantor set) under the identifications induced by contracting the leaves of $\Gamma_{1/3}$ to points. Alternatively, if we view $\mathcal{F}_{a_{1/3}}$ on $\hat{\mathbb{C}}$ as a form of generalised mating of four copies of the action of $C_3 * C_\infty$ on the complex upper half-plane, mated together along their boundaries, then the limit set becomes a quotient of the four boundary circles.

3. In this article we have considered just one example of a correspondence at the boundary of the Klein combination locus. At other boundary points the behaviour can be very different. Further examples, and conjectures concerning the overall structure of the Klein combination locus, will be presented elsewhere.

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