Answer All Seven Questions. The mark each question carries is shown beside the question. (The test is out of 100.)

Calculators are not permitted in this examination.

Write your answers on the question paper. If you need more space then use the blank pages at the end of the booklet and be sure to number your answers clearly.

Do NOT turn over until instructed to do so.

NAME:

STUDENT NUMBER:

For Examiner’s use only

<table>
<thead>
<tr>
<th>Question</th>
<th>Mark</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
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<td>4</td>
<td></td>
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<tr>
<td>5</td>
<td></td>
</tr>
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<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
</tr>
</tbody>
</table>
Question 1. (20 marks)

State the Completeness Axiom for the field of real numbers $\mathbb{R}$.

Find, with proof, an upper bound for the set $A = \{x \in \mathbb{R} : x^2 + x < 22\} \subset \mathbb{R}$, and deduce that this set has a least upper bound.

Is the set $A$ bounded below? [Answer ‘YES’ or ‘NO’ and write down a lower bound if there is one, but no further justification is expected.]

Solution:

Completeness Axiom: Every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound.

5 marks

For example, $A$ cannot contain any real number $y > 5$, since if $y > 5$ then $y^2 + y > 25 + 5 = 30$. Hence 5 is an upper bound for $A$. But $A$ is non-empty (for example $0 \in A$) and therefore $A$ has a least upper bound by the Completeness Axiom.

10 marks

(of which 3 marks for an upper bound, 3 marks for a proof it is an upper bound, 2 marks for a proof $A$ is non-empty, and 2 marks for applying the Completeness Axiom to deduce that $A$ has a least upper bound).

The set $A$ is bounded below, for example by $-6$, since if $y < -6$ then $y^2 + y > 30$ (because $y^2 + y = y(y + 1)$).

5 marks

(of which 3 marks for ‘YES’ and 2 marks for a value of a lower bound).

COMMENT: Too many people just proved that $5 \notin A$ and then said that this means that $5$ is an upper bound for $A$. To see why this is wrong, just think about the integers $\mathbb{Z} \subset \mathbb{R}$. Obviously $\sqrt{2} \notin \mathbb{Z}$, but this does not imply that $\sqrt{2}$ is an upper bound for $\mathbb{Z}$.

To prove that $x \in \mathbb{R}$ is an upper bound for a given subset $A \subset \mathbb{R}$ you have to show that every element of $A$ is less than or equal to $x$. Or, equivalently, you have to show that any element of $\mathbb{R}$ which is greater than $x$ cannot lie in $A$. 

2
Question 2. (20 marks)

Define what it means for a sequence of real numbers \((x_n)_{n=0}^{\infty}\) to converge to zero.

Using this definition (but not any Lemmas or Theorems from the course) prove that both of the following sequences converge to zero:

(a) \(x_n = \frac{1}{2n}\).

(b) \(y_n = \frac{1}{\sqrt{n}}\).

Solution:

\((x_n)_{n=1}^{\infty}\) converges to zero if

\[\forall \epsilon > 0 \ \exists N \ \forall n > N : |x_n| < \epsilon.\] 6 marks

Proof that \(x_n = \frac{1}{2n}\) converges to zero:

Given \(\epsilon > 0\) take \(N = \lceil \frac{1}{2\epsilon} \rceil\), so \(N \geq \frac{1}{2\epsilon}\) so \(\frac{1}{2N} \leq \epsilon\). Now given any \(n > N\) we have:

\[|x_n| = \frac{1}{2n} < \frac{1}{2N} \leq \epsilon,\]  as required. □ 7 marks

(Alternatively take \(N = \lceil \frac{1}{\epsilon^2} \rceil\) and observe that if \(n > N\) then \(|x_n| = \frac{1}{2n} < \frac{1}{2N} < \frac{1}{N} \leq \epsilon\).)

Proof that \(x_n = \frac{1}{\sqrt{n}}\) converges to zero:

Given \(\epsilon > 0\) take \(N = \lceil \frac{1}{\epsilon^2} \rceil\), so \(N \geq \frac{1}{\epsilon^2}\) so \(\frac{1}{\sqrt{N}} \leq \epsilon\). Now given any \(n > N\) we have:

\[|x_n| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} \leq \epsilon,\] as required. □ 7 marks

COMMENT: This question was done very well by most people. The most common mistake was to try \(N = \lceil \frac{1}{\epsilon} \rceil\) in part (b) as well as part (a), and then to try to make the proof work by writing \(\frac{1}{\sqrt{n}} < \frac{1}{n}\), which is obviously not true.
Question 3. (10 marks)

Consider the following two statements:

(a) \( \forall x \in \mathbb{R} \ \exists y \in \mathbb{R} : y^2 = x. \)
(b) \( \forall a \in \mathbb{R} \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in \mathbb{R}, |x - a| < \delta : |2x - 2a| < \varepsilon. \)

Write down the negation of each of these statements.

For each of (a) and (b), say whether the statement or its negation is true. Give a brief justification of your answer in each case.

Solution:

(a) Negation: \( \exists x \in \mathbb{R} \ \forall y \in \mathbb{R} : y^2 \neq x. \) 3 marks.
    Negation is TRUE (1 mark) as \( x \) can be negative (1 mark).

(b) Negation: \( \exists a \in \mathbb{R} \ \exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x \in \mathbb{R}, |x - a| < \delta : |2x - 2a| \geq \varepsilon. \) 3 marks.
    This time, the original statement is TRUE (1 mark) since we may take \( \delta \) to be \( \varepsilon/2 \) (1 mark).

COMMENT: Stating the negations is easy and was done correctly by most people. It is harder to figure out whether a statement or its negation is true. A good strategy is to try out as many cases you can (for example trying \( x \) negative in (a) makes the answer obvious). Several people observed that statement (b) is true - and proved it. Well done those who did!
Question 4. (10 marks)

For each of the following statements concerning sequences \((x_n)_{n=1}^{\infty}\) and \((y_n)_{n=1}^{\infty}\) of real numbers, say whether the statement is true or false. [Answer ‘TRUE’ or ‘FALSE’ in each case. You are not asked to provide any justification.]

(a) If \(y_n \leq x_n\) for all \(n\), and \((x_n)_{n=1}^{\infty}\) converges to zero, then \((y_n)_{n=1}^{\infty}\) converges to zero.

(b) If \(|y_n| \leq |x_n|\) for all \(n\), and \((x_n)_{n=1}^{\infty}\) converges to zero, then \((y_n)_{n=1}^{\infty}\) converges to zero.

(c) If \((x_n)_{n=1}^{\infty}\) converges to a real number then \((x_{n+1} - x_n)_{n=1}^{\infty}\) converges to zero.

(d) If \((x_{n+1} - x_n)_{n=1}^{\infty}\) converges to zero then \((x_n)_{n=1}^{\infty}\) converges to a real number.

(e) If \((x_n)_{n=1}^{\infty}\) is bounded above and montone decreasing then \((x_n)_{n=1}^{\infty}\) converges to a real number.

Solution:

Explanations are given below but are not needed for marks as they were not asked for in this question.

(a) FALSE. 2 marks \((y_n)\) could tend to minus infinity.

(b) TRUE. 2 marks (Lemma 8 of the course.)

(c) TRUE. 2 marks (Theorem 19 of the course.)

(d) FALSE. 2 marks (E.g. \(x_n = 1 + 1/2 + 1/3 + \ldots + 1/n\).)

(e) FALSE. 2 marks (As it is decreasing it could tend to minus infinity.)

COMMENT: This question was done well by most people. Most got 4 right and many got all 5 right - though a few people managed to get all 5 wrong. My marking system was generous (2 marks for each correct answer and no penalty for wrong answers). In an end-of-year examination the marking system would not allow you to get half marks by making random guesses!
Question 5. (20 marks)

Define what it means for a sequence \( (x_n)_{n=1}^{\infty} \) to converge to a real number \( x \).

Find the limits of the following sequences \( (x_n)_{n=1}^{\infty} \). Give brief justification.
[You may use any results from the course but you should state which result you are using.]

(a) \( x_n = \frac{2}{n} - 3 \).

(b) \( x_n = \left( \frac{5n^2 + 1}{n^2} \right) \left( \frac{n - \cos n\pi}{n} \right) \).

(c) \( x_n = \frac{n^3 + n^2}{4n^3 - 7} \).

Solution:

\( (x_n)_{n=1}^{\infty} \) converges to \( x \) if \( (x - x_n)_{n=1}^{\infty} \) converges to zero (or, equivalently, if \( \forall \epsilon > 0 \exists N : \forall n > N : |x - x_n| < \epsilon \)). 5 marks

(a) \( x_n = \frac{2}{n} - 3 \rightarrow -3. 3 \text{ marks} \)

To show this, note that \( 2/n \rightarrow 0 \) (by Lemma 10, since \( 1/n \rightarrow 0 \)), and the constant sequence \(-3\) tends to \(-3\). So the result follows by the part of Theorem 16 about sums of sequences. 2 marks

(b) \( x_n = \left( \frac{5n^2 + 1}{n^2} \right) \left( \frac{n - \cos n\pi}{n} \right) \rightarrow 5. 3 \text{ marks} \)

To show this, note firstly that \( \left( \frac{5n^2 + 1}{n^2} \right) = 5 + \frac{1}{n^2} \), which converges to 5 by Theorem 16 (since \( 1/n^2 < 1/n \) and hence \( 1/n^2 \) converges to zero by Lemma 8). Next note that \( \left( \frac{n - \cos n\pi}{n} \right) = 1 - \frac{\cos n\pi}{n} \), which converges to 1 by Theorem 16 (since \( \left| \frac{\cos n\pi}{n} \right| = 1/n \) which converges to zero). Finally, by the part of Theorem 16 about products of sequences, deduce that \( x_n \rightarrow 5. 2 \text{ marks} \)

(c) \( x_n = \frac{n^3 + n^2}{4n^3 - 7} = \frac{1 + 1/n}{4 - 7/n^3} \rightarrow 1/4. 3 \text{ marks} \)

This follows from the part of Theorem 16 about quotients, once we have shown that the numerator converges to 1 and the denominator converges 4 by the same methods we have employed for the factors in (b). 2 marks

COMMENT: This question was done very well by most people. But please don’t say things like \( \cos n\pi \leq 1 \) when what you mean is \( |\cos n\pi| \leq 1. \)
Question 6. (10 marks)

Let \( x_n = (-1)^n \left( 1 + \frac{1}{n} \right) \).

(a) Write down the first four terms of the sequence \( (x_n)_{n=1}^\infty \).

(b) Is the sequence monotonic?

(c) Is it bounded?

(d) Does it converge?

Give brief reasons for your answers.

Solution:

(a) \( x_1 = -2 \); \( x_2 = +3/2 \); \( x_3 = -4/3 \); \( x_4 = +5/4 \); 4 marks

(b) Not monotonic (alternates positive and negative). 2 marks

(c) Bounded (all terms are \( \geq -2 \) and \( \leq +3/2 \) so these are lower and upper bounds). 2 marks

(d) Not convergent (even-numbered terms converge to +1 and odd-numbered terms converge to -1). 2 marks

COMMENT: This question was done well in general. But several people said in part (d) that the sequence \( (x_n) \) ‘converges to +1 and to −1’. This is wrong: the definition of convergence only allows one limit - we proved this in lectures!
**Question 7.** (10 marks)

Define what it means for the sequence $(x_n)_{n=1}^{\infty}$ to tend to infinity.

Prove directly from the definition that the sequence given by $x_n = \frac{n}{2}$ tends to infinity.

**Solution:**

$(x_n)_{n=1}^{\infty}$ tends to infinity if

$\forall K \in \mathbb{R}, K > 0 \; \exists N \forall n > N : x_n > K$. 4 marks

(Equally acceptable: $\forall K \in \mathbb{R} \; \exists N \forall n > N : |x_n| > K$.)

**Proof** that $x_n = \frac{n}{2}$ tends to infinity:

Given any $K \in \mathbb{R}$, with $K > 0$, let $N = \lceil 2K \rceil$ so $N \geq 2K$ and hence $N/2 \geq K$.

Now for all $n > N$ we have: $x_n = \frac{n}{2} > \frac{N}{2} \geq K$, so $x_n > K$.

Hence $x_n$ tends to infinity. □ 6 marks

(If $K \leq 0$ is allowed by the definition, set $N = 0$ when $K \leq 0$, and $N = \lceil 2K \rceil$ when $K > 0$. Then $\forall K \in \mathbb{R}$ we have $N/2 \geq K$.)

**COMMENT:** This question was done well by most people. The most common mistake was to put '$|x_n| > K$' rather than ' $x_n > K$' in the definition of 'tends to infinity'. If we put '$|x_n| > K$' in the definition then the sequence $x_n = (-1)^n n$ 'tends to infinity', whereas what we really want to say about this sequence is that the odd-numbered terms tend to $-\infty$ and the even-numbered terms tend to $+\infty$, but the sequence itself does not tend to anything. However it’s an understandable mistake and putting '$|x_n| > K$' only lost one mark.