Chapter 6

The Vector Product

6.1 Parallel vectors

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors. We say that $\mathbf{u}$ and $\mathbf{v}$ are *parallel*, and write $\mathbf{u} \parallel \mathbf{v}$, if $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$ (which will also force $\mathbf{v}$ to be a scalar multiple of $\mathbf{u}$). Note that $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if they have the same or opposite directions, which happens exactly when $\mathbf{u}$ and $\mathbf{v}$ are at an angle of 0 or π.

Example. The vectors $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix}$ are parallel, but $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}$ are not parallel.

The relation of parallelism has the following properties, where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are *nonzero* vectors.

1. $\mathbf{u} \parallel \mathbf{u}$ for all vectors $\mathbf{u}$. [This property is called *reflexivity*.]

2. $\mathbf{u} \parallel \mathbf{v}$ implies that $\mathbf{v} \parallel \mathbf{u}$. [This property is called *symmetry*.]

3. $\mathbf{u} \parallel \mathbf{v}$ and $\mathbf{v} \parallel \mathbf{w}$ implies that $\mathbf{u} \parallel \mathbf{w}$. [This property is called *transitivity*.]

A relation that is reflexive, symmetric and transitive is called an *equivalence relation*. This concept will be introduced in MTH4104: Introduction to Algebra. Thus, parallelism is a equivalence relation on the set of nonzero vectors

Note. A wording like "$\mathbf{u}$ and $\mathbf{v}$ are parallel if ..." presumes that the property of parallelism is symmetric between $\mathbf{u}$ and $\mathbf{v}$. This may be obvious from the definition that follows, or else a proof would need to be supplied. If you want to define a concept that is asymmetric (or not obviously symmetric), a wording like "$\mathbf{u}$ is parallel to $\mathbf{v}$ if ..." is more appropriate. [Can you prove that the relations of parallelism and collinearity (see below) are symmetric? Be careful of the zero vector for the latter relation.]

Also, when one provides more than one definition of a concept, one should prove the equivalence of the definitions. Can you prove the various definitions of collinear to be equivalent?

6.2 Collinear vectors

It is useful to extend the notion of parallelism to pairs of vectors involving the zero vector. However, we shall give this notion a different name, and call it collinearity. We say that two vectors $\mathbf{u}$ and $\mathbf{v}$ are *collinear* if $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ or $\mathbf{u}$ and $\mathbf{v}$ are parallel (this includes the case $\mathbf{u} = \mathbf{v} = \mathbf{0}$).

An equivalent definition of collinearity is that $\mathbf{u}$ and $\mathbf{v}$ are collinear if there exist [real] numbers (scalars) α and β not both zero such that

$$\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}.$$

(This covers the case when one or both of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{0}$, as well as the general case when $\mathbf{u}$ and $\mathbf{v}$ are parallel.) The relation of collinearity is reflexive and symmetric (in all dimensions), but is not transitive in dimension at least 2. However, it is nearly transitive [not a technical term]: if $\mathbf{u}$ and $\mathbf{v}$ are collinear, and $\mathbf{v}$ and $\mathbf{w}$ are collinear, then either $\mathbf{u}$ and $\mathbf{w}$ are collinear or $\mathbf{v} = \mathbf{0}$ (or both).

Another definition of colinearity is that $\mathbf{u}$ and $\mathbf{v}$ are collinear if O, U and V all lie on some [straight] line, where $\overrightarrow{OU}$ represents $\mathbf{u}$ and $\overrightarrow{OV}$ represents $\mathbf{v}$. This line is unique except in the case $\mathbf{u} = \mathbf{v} = \mathbf{0}$, that is $O = U = V$.

Note. The word collinear is logically made up of two parts: the prefix *co-*, and the stem *linear*. Thus the logical spelling of collinear should be as *colinear*, but the standard spelling has *-ll-*, for some reason.

6.3 Coplanar vectors

We say that vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are *coplanar* if the points O, U, V, W all lie on some plane, where $\overrightarrow{OU}$, $\overrightarrow{OV}$, $\overrightarrow{OW}$ have free vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ respectively. (Note that this definition is symmetric in $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.) Thus we see that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are coplanar if and only if either $\mathbf{u}$ and $\mathbf{v}$ are collinear or there exist scalars λ and μ such that $\mathbf{w} = \lambda\mathbf{u} + \mu\mathbf{v}$.

From the above, we get the following symmetrical algebraic formulation of coplanarity: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are *coplanar* if there exist [real] numbers (scalars) α, β, γ such that

$$\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = \mathbf{0} \quad \text{and} \quad (\alpha, \beta, \gamma) \neq (0, 0, 0). \tag{6.1}$$

Note that to determine whether a particular triple $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is coplanar, we can use the above equation $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = \mathbf{0}$, which induces 3 linear equations in the unknowns α, β, γ (or n equations if $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ rather than $\mathbb{R}^3$). These equations always have the solution $\alpha = \beta = \gamma = 0$, and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are coplanar if and only if the equations have a solution *other than* $\alpha = \beta = \gamma = 0$.

Example 1. If $\mathbf{u} = \mathbf{0}$, $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ then $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are coplanar. For example, if $\mathbf{u} = \mathbf{0}$, we can take $\alpha = 1$, $\beta = \gamma = 0$ in Equation 6.1. Geometrically, the points O, U, V, W consist of at most 3 distinct points, and any three points (in $\mathbb{R}^3$) lie on at least one plane.

Example 2. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are (nonzero and) parallel. Then $\mathbf{v} = \lambda\mathbf{u}$ for some scalar λ. Therefore we have $\lambda\mathbf{u} - \mathbf{v} = \mathbf{0}$, and so in Equation 6.1 we can take $\alpha = \lambda$, $\beta = -1$, $\gamma = 0$. Geometrically, O, U, V lie on one line (which is unique since $U, V \neq O$), and so O, U, V, W lie on at least one plane (which is unique, except when W lies on the line determined by O, U, V).

Example 3. We let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$. Then Equation 6.1 yields the following linear equations in α, β, γ:

$$\left.\begin{aligned} \alpha + 4\beta + 7\gamma &= 0 \\ 2\alpha + 5\beta + 8\gamma &= 0 \\ 3\alpha + 6\beta + 9\gamma &= 0 \end{aligned}\right\}.$$

Echelonisation gives the following system of equations:

$$\left.\begin{aligned} \alpha + 4\beta + 7\gamma &= 0 \\ -3\beta - 6\gamma &= 0 \\ 0 &= 0 \end{aligned}\right\}.$$

Back substitution then gives the solution $\alpha = t$, $\beta = -2t$, $\gamma = t$, where t can be any real number. Setting $t = 1$ (any nonzero value will do), we see that $\mathbf{u} - 2\mathbf{v} + \mathbf{w} = \mathbf{0}$, so that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are coplanar.

Example 4. We let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. Then Equation 6.1 yields the following linear equations in α, β, γ:

$$\left.\begin{aligned} \alpha + 2\beta + 3\gamma &= 0 \\ 2\alpha + 3\beta + \gamma &= 0 \\ 3\alpha + \beta + 2\gamma &= 0 \end{aligned}\right\}.$$

Echelonisation gives the following system of equations:

$$\left.\begin{aligned} \alpha + 2\beta + 3\gamma &= 0 \\ -\beta - 5\gamma &= 0 \\ 18\gamma &= 0 \end{aligned}\right\}.$$

Thus we see that the only solution to this system of equations is $\alpha = \beta = \gamma = 0$. Therefore $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are *not* coplanar.

6.4 Right-handed and left-handed triples

Suppose now that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are not coplanar. Pick an origin O in 3-space, and define U, V, W by the condition that $\overrightarrow{OU}$, $\overrightarrow{OV}$, $\overrightarrow{OW}$ represent $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ respectively. Using your right hand put your thumb in the direction of $\mathbf{u}$, and your first [index] finger in the direction of $\mathbf{v}$. If W lies on the side of the plane through O, U, V indicated by your second [middle] finger, we call $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ a *right-handed triple*; otherwise it is a *left-handed triple*.

For any triple $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ of vectors, precisely one of the following properties holds: it is coplanar, it is right-handed, or it is left-handed.

Examples. $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is a right-handed triple. $\mathbf{i}$, $\mathbf{j}$, $-\mathbf{k}$ and $\mathbf{k}$, $\mathbf{j}$, $\mathbf{i}$ are both left-handed triples.

Note. The vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ forming a right-handed or left-handed triple need not be mutually orthogonal, but they *must not* be coplanar.

Another way to determine the handedness of a triple $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is as follows. We let the plane determined by O, U, V be the page of these notes. We then look at the plane from the side such the angle from $\mathbf{u}$ to $\mathbf{v}$ *proceeding anticlockwise* is between 0 and π. (If the anticlockwise angle lies between π and 2π then look at the plane from the other side.) If $\mathbf{w}$ points 'towards' you, that is W is the same side of the O, U, V-plane as you are, then $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is right-handed. If $\mathbf{w}$ points 'away from' you then $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is left-handed.

6.4.1 Some finger exercises (for you to do)

If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are coplanar, then so is any triple that is a permutation of $\pm\mathbf{u}$, $\pm\mathbf{v}$, $\pm\mathbf{w}$, for any of the 8 possibilities for sign. From now on, we assume that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are not coplanar. The basic operations we can do to a triple of vectors are:

1. multiply one the vectors by a constant $\lambda > 0$;
2. negate one of them; or
3. swap two of them.

The first operation preserves the handedness of a triple, but the latter two operations send right-handed triples to left-handed triples and vice versa. (Each of these operations sends coplanar triples to coplanar triples.) Combining these operations, we see that negating an even number of the vectors preserves handedness, while negating an odd number of the vectors changes the handedness. One can convert $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ to each of $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{w}$, $\mathbf{u}$, $\mathbf{v}$ using two swaps. Thus one sees that (in 3 dimensions) cycling $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ does not change the handedness of the system. We have the following.

- If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a right-handed triple, then the following triples are also right-handed: $\mathbf{u},\mathbf{v},\mathbf{w}$; $\mathbf{v},\mathbf{w},\mathbf{u}$; $\mathbf{w},\mathbf{u},\mathbf{v}$; $\mathbf{u},-\mathbf{v},-\mathbf{w}$; $-\mathbf{u},\mathbf{v},-\mathbf{w}$; $-\mathbf{u},-\mathbf{v},\mathbf{w}$; $\mathbf{u},-\mathbf{w},\mathbf{v}$; $\mathbf{u},\mathbf{w},-\mathbf{v}$; $-\mathbf{u},\mathbf{w},\mathbf{v}$; $\mathbf{w},\mathbf{v},-\mathbf{u}$; $\mathbf{w},-\mathbf{v},\mathbf{u}$; $-\mathbf{w},\mathbf{v},\mathbf{u}$; $-\mathbf{w},-\mathbf{v},-\mathbf{u}$; and so on.
- If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a right-handed triple, then the following triples are left-handed: $\mathbf{u},\mathbf{w},\mathbf{v}$; $\mathbf{w},\mathbf{v},\mathbf{u}$; $\mathbf{v},\mathbf{u},\mathbf{w}$; $-\mathbf{u},\mathbf{v},\mathbf{w}$; $\mathbf{u},-\mathbf{v},\mathbf{w}$; $\mathbf{u},\mathbf{v},-\mathbf{w}$; $-\mathbf{u},-\mathbf{v},-\mathbf{w}$; $-\mathbf{v},-\mathbf{w},-\mathbf{u}$; $-\mathbf{w},-\mathbf{u},-\mathbf{v}$; $\mathbf{w},-\mathbf{v},-\mathbf{u}$; $-\mathbf{w},\mathbf{v},-\mathbf{u}$; $-\mathbf{w},-\mathbf{v},\mathbf{u}$; and so on.

6.5 The vector product

We now describe a method of multiplying two vectors to obtain another vector.

Definition 6.1. Suppose that $\mathbf{u}$, $\mathbf{v}$ are nonzero non-parallel vectors (of $\mathbb{R}^3$) at angle θ. Then the *vector product* or *cross product* $\mathbf{u} \times \mathbf{v}$ is defined to be the vector satisfying:

(i) $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$ (note that $\sin\theta > 0$ here, since $0 < \theta < \pi$);

(ii) $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$; and

(iii) $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ is a right-handed triple.

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ or $\mathbf{u}$, $\mathbf{v}$ are parallel, then the vector product $\mathbf{u} \times \mathbf{v}$ is defined to be $\mathbf{0}$.

Note. Occasionally, you will see the cross product $\mathbf{u} \times \mathbf{v}$ written as $\mathbf{u} \wedge \mathbf{v}$. However, this wedge product properly means something else, so you should not use such a notation.

We note that if $\mathbf{u}$, $\mathbf{v}$ are nonzero and non-parallel, then $|\mathbf{u}\times\mathbf{v}| > 0$ (so that $\mathbf{u}\times\mathbf{v} \neq \mathbf{0}$); otherwise $|\mathbf{u} \times \mathbf{v}| = 0$ (so that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$). In particular, for all vectors $\mathbf{u}$, we have $\mathbf{u} \times \mathbf{u} = \mathbf{0}$. Further, we note that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, since:

(i) $|\mathbf{k}| = 1 = |\mathbf{i}||\mathbf{j}| \sin \frac{\pi}{2}$;

(ii) $\mathbf{k}$ is orthogonal to both $\mathbf{i}$ and $\mathbf{j}$; and

(iii) $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is a right-handed triple.

Similarly, we get the following table of cross products:

	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{0}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$-\mathbf{k}$	$\mathbf{0}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$\mathbf{0}$

where the entry in the $\mathbf{u}$ row and $\mathbf{v}$ column is $\mathbf{u} \times \mathbf{v}$.

6.6 The vector product in coördinates

We now use the rules for the vector product we have proved to find a formula for the vector product of two vectors given in coördinates.
Note that these proofs are not written up yet.

Theorem 6.2. Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$. Then we have:

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

Proof. We have $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Therefore, the distributive and scalar multiplication laws give us:

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= (u_1\mathbf{i} \times v_1\mathbf{i}) + (u_1\mathbf{i} \times v_2\mathbf{j}) + (u_1\mathbf{i} \times v_3\mathbf{k}) + (u_2\mathbf{j} \times v_1\mathbf{i}) \\
&\qquad + (u_2\mathbf{j} \times v_2\mathbf{j}) + (u_2\mathbf{j} \times v_3\mathbf{k}) + (u_3\mathbf{k} \times v_1\mathbf{i}) + (u_3\mathbf{k} \times v_2\mathbf{j}) + (u_3\mathbf{k} \times v_3\mathbf{k}) \\
&= u_1v_1(\mathbf{i} \times \mathbf{i}) + u_1v_2(\mathbf{i} \times \mathbf{j}) + u_1v_3(\mathbf{i} \times \mathbf{k}) + u_2v_1(\mathbf{j} \times \mathbf{i}) \\
&\qquad + u_2v_2(\mathbf{j} \times \mathbf{j}) + u_2v_3(\mathbf{j} \times \mathbf{k}) + u_3v_1(\mathbf{k} \times \mathbf{i}) + u_3v_2(\mathbf{k} \times \mathbf{j}) + u_3v_3(\mathbf{k} \times \mathbf{k}) \\
&= \mathbf{0} + u_1v_2\mathbf{k} + u_1v_3(-\mathbf{j}) + u_2v_1(-\mathbf{k}) + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} + u_3v_2(-\mathbf{i}) + \mathbf{0} \\
&= (u_2v_3 - u_3v_2)\mathbf{i} + (-u_1v_3 + u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \\
&= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k},
\end{aligned}$$

which is the result we wanted. □

There is a more useful way to remember this formula for $\mathbf{u} \times \mathbf{v}$. We define:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

which is the *determinant* of the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$.
Then we have:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \mathbf{i} + \begin{vmatrix} u_3 & v_3 \\ u_1 & v_1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \mathbf{k} = \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \mathbf{k}.$$

TO BE COMPLETED