**Holomorphic Dynamics and Hyperbolic Geometry (February-March 2013)**

# Week 3 Exercises - SOLUTIONS

1. Recall that the *quaternions* are quadruples $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ written in the form $x_0 + x_1 i + x_2 j + x_3 k$ and equipped with the (non-commutive) product given by $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. An extension of the action of $PSL(2, \mathbb{C})$ from $\hat{\mathbb{C}}$ to $\mathcal{H}^3_+$ can be defined explicitly as follows. Regard $\mathbb{R}^3$ as *quaternions* of the form $x + yi + tj (+0k)$. Then, provided our matrix in $PSL(2, \mathbb{C})$ has been normalised to a form in which $ad - bc \in \mathbb{R}^{>0}$, the map

$$z + tj \to \frac{a(z + tj) + b}{c(z + tj) + d}$$

sends the half-space

$$\mathcal{H}^3_+ = \{z + tj : z \in \hat{\mathbb{C}}, t \in \mathbb{R}^{>0}\}$$

to itself, extending the bijection

$$z \to \frac{az + b}{cz + d}$$

on the boundary plane $\hat{\mathbb{C}}$.
Verify that the expression above agrees with the formulae (i),(ii) and (iii) for the action of $PSL(2, \mathbb{C})$ on $\mathcal{H}^3_+$ (at the start of Section 5).

*HINT: To express*

$$\frac{a(z + tj) + b}{c(z + tj) + d}$$

*in the standard quaternion form* $e + fi + gj + hk$ *with* $e, f, g, h \in \mathbb{R}$ *one should multiply both top and bottom by* $\bar{c}(\bar{z} + \bar{d}) - ctj$.

**SOLUTION**:
We interpret a quotient of quaternions $q_1/q_2$ as $q_1 q_2^{-1}$ (order of multiplication matters!).

(i) $z \to z + \lambda$ $(\lambda \in \mathbb{C})$ has matrix

$$A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

so the formula in the question gives the action on quaternions as:

$$z + tj \to z + tj + \lambda = (z + \lambda) + tj$$

i.e. the j-component is preserved and $z$ is translated by $\lambda$, as required.

(ii) $z \to \lambda z$ $(\lambda \in \mathbb{C})$ has matrix

$$A = \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}$$

Choosing a square root $\mu$ of $\lambda$, the diagonal entries of $A$ become $\mu$ and $\bar{\mu}/|\lambda|$, and the formula for $z \to \lambda z$ becomes

$$z + tj \to |\lambda| \frac{\mu(z + tj)}{\bar{\mu}} = \mu(z + tj)\mu = \lambda z + t\mu j \mu = \lambda z + t|\mu|^2 j = \lambda z + t|\lambda| j$$

as required. (Here we have used the fact that if we write the complex number $\mu$ as $\nu + i\rho$ with $\nu, \rho$ real, then $\mu j \mu = (\nu + i\rho) j (\nu + i\rho) = \nu^2 j + \rho\nu ij + \rho\nu ji + \rho^2 iji = (\nu^2 + \rho^2) j = |\mu|^2 j$.)

(iii)

$$-\frac{1}{z + tj} = -\frac{1}{z + tj} \frac{(\bar{z} - tj)}{(\bar{z} - tj)} = \frac{-\bar{z} + tj}{z\bar{z} + t^2}$$

as required.

2. Prove that a Möbius transformation of the form

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \quad ad - b^2 = 1$$

satisfies $JAJ = A^{-1}$ where $J(z) = -z^{-1}$. Deduce that if a discrete group $G$ is generated by elements of this form then $\Lambda(G)$ is invariant under $J$.

**SOLUTION**

$$J(AJ(z)) = J\left(\frac{-a/z + b}{-b/z + d}\right) = \frac{b/z - d}{-a/z + b} = \frac{dz - b}{-bz + a} = A^{-1}(z)$$

If $z \in \Lambda(G)$ then for any point $z' \in \hat{\mathbb{C}}$ there is a sequence of $g_n \in G$ with $g_n(z')$ converging to $z$. But then, noting that $J^{-1} = J$, and conjugating by $J$ (in other words 'applying $J$ as a change of coordinate'), we deduce that the points $(Jg_nJ)(J(z')$ converge to $J(z)$. But $J(g_n)J \in G$, as $J(g_n)J = g_n^{-1}$, so this means that $J(z)$ is an accumulation point of a $G$ orbit. Hence $J(z) \in \Lambda(G)$.

3. Let $h(z) = (az + b)/(cz + d)$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.
(i) Show that $Im(h(z)) = Im(z)/|cz + d|^2$.
(ii) Let $\Delta = \{z : Im(z) > 0,\ |Re(z)| < 1/2,\ |z| > 1\}$. If $z \in \Delta$ show that $|cz + d|^2 > (|c| - |d|)^2 + |cd|$ and deduce from (i) that if $a, b, c, d \in \mathbb{Z}$ and $h \neq I$ then $Im(h(z)) < Im(z)$.
(iii) By applying (ii) to $h$ and to $h^{-1}$ deduce that if $z \in \Delta$ then there is no $I \neq h \in PSL(2, \mathbb{Z})$ such that $h(z) \in \Delta$.
(iv) Let $v \in \mathbb{R}$, $v > 1$. Deduce from (iii) that $\Delta = \{z \in \mathcal{H}^2_+ :\ d(z, iv) < d(z, g(iv))\ \forall I \neq g \in PSL(2, \mathbb{Z})\}$ (where $d$ is the Poincaré metric).

**SOLUTION**
(i)

$$h(z) = \frac{az + b}{cz + d} = \frac{az + b}{cz + d}.\frac{c\bar{z} + d}{c\bar{z} + d}$$

so

$$Im(h(z)) = \frac{(ad - bc)Im(z)}{|cz + d|^2}.$$

(ii)

$$\begin{aligned} |cz + d|^2 &= (cz + d)(c\bar{z} + d) = |c|^2\bar{z}^2 + |d|^2 + 2cd.Re(z) \\ &> |c|^2 + |d|^2 - |cd| = (|c| - |d|)^2 + |cd| \end{aligned}$$

since $|z| > 1$ and $Re(z) > -1/2$.
Finally $(|c| - |d|)^2 + |cd| \geq 1$ by considering all the possibilities for $c, d \in \mathbb{Z}$ having at least one of $c, d$ non-zero, so $Im(h(z)) < Im(z)$ by (i).

(iii) If both $z$ and $h(z) = z'$ lie in $\Delta$ then by (ii) applied to $z$ we have $Im(z') < Im(z)$, and by $(ii)$ applied to $z'$ we have $Im(z) < Im(z')$ (since $z = h^{-1}(z')$).

(iv) But from the way it is defined, $\Delta$ is the set $\{z \in \mathcal{H}^2_+ :\ d(z, iv) < d(z, T(iv)), d(z, iv) < d(z, S(iv))\}$ (where $T(z) = z + 1$ and $S(z) = -1/z$), so $\Delta$ contains the Dirichlet domain for $PSL(2, \mathbb{Z})$ defined with respect to the point $iv$. So *either* $\Delta$ is the Dirichlet domain and its images tile the upper half-plane *or* $\Delta$ properly contains the Dirichlet domain, in which case there is a $g(\Delta)$ which overlaps $\Delta$. But from (ii) we know that $g(\Delta) \cap \Delta = \emptyset$ for *all* non-identity elements of $PSL(2, \mathbb{Z})$. So $\Delta$ *is* the relevant Dirichlet domain for $PSL(2, \mathbb{Z})$.
*Comment: It now follows from Poincaré's Polygon Theorem that* $PSL(2, \mathbb{Z})$ *is generated by* $S$ *and* $T$ *subject only to the relations* $S^2 = I$ *and* $(ST)^3 = I$.