LTCC

Holomorphic Dynamics and Hyperbolic Geometry (Feb-March 2013)

Week 4 Exercises

These exercises are more open-ended. An answer to any one of questions 2,3 or 4 is an acceptable alternative to the set of assessment exercises (an answer to question 1 is too easy to find in a textbook).

1. Prove that the area of a hyperbolic triangle with angles α, β, γ is $\pi - (\alpha + \beta + \gamma)$. Deduce a formula for the area of a hyperbolic polygon with a finite number of sides.

[HINT: In the half-plane model the area of a triangle A is $\int\int_A \frac{dxdy}{y^2}$. Start by calculating the area of a triangle which has an ideal vertex, in which case you can assume that this vertex is at ∞. If it makes the calculation easier assume the triangle has angle $\pi/2$ at one vertex. A general triangle can be expressed as the difference of two triangles which have an ideal vertex.]

2(a) Consider a configuration of three circles of equal radii, C_1, C_2 and C_3 in the plane, touching in pairs and having disjoint interiors. Let R_j denote reflection in circle C_j, fixing it pointwise and exchanging its interior with its exterior. Show that the group H of orientation-preserving conformal automorphisms generated by R_2R_1 and R_3R_2 has limit set a circle, and that each of the discs in $\hat{\mathbb{C}}$ bounded by this circle is mapped to itself by H. Deduce that H is conjugate in $Aut(\hat{\mathbb{C}})$ to a Fuchsian group.

(b) Now add a fourth circle, C_4, which touches C_1, C_2 and C_3 and has interior disjoint from the their interiors. Show that the limit set of the subgroup G of $Aut(\hat{\mathbb{C}})$ generated by the R_jR_k $(j, k \in \{1, 2, 3, 4\})$ is an Apollonian circle-packing (a circle-packing obtained from three pairwise touching circles by iteratively adding a new circle of maximal radius in each space bounded by three pairwise touching arcs of circles already drawn). Can you interpret G as a 'truncated tetrahedron group'?

3. (a) Show (or explain why) the external ray corresponding to the Feigenbaum point on the Mandelbrot set (the period-doubling limit) has angle given by the Morse-Thue sequence: the sequence generated from the single digit 0 by iteratively replacing 0 by 01 and 1 by 10.

(b) Show that the external ray associated to the golden mean ($\gamma = (\sqrt{5} - 1)/2$) on the boundary of the main cardioid M_0 of the Mandelbrot set has angle θ_γ given by the sequence generated from the single digit 1 by iteratively replacing each 1 with 10 and each 0 with 1.

(c) Find an algorithm generalising that in (b) to generate θ_ν for every *noble* irrational ν (a noble irrational is one for which the continued fraction expansion ends in an infinite sequence of 1's). Consider possible generalisations.

4(a) *(Shimuzu's Lemma)* Prove that if G is Fuchsian (i.e. a discrete subgroup of $PSL(2, \mathbb{R})$) and

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \quad \text{then :}$$

(i) either $c = 0$ or $|c| \geq 1$, and hence
(ii) $|tr(ABA^{-1}B^{-1}) - 2| \geq 1$.
[HINT: For part (i) let $B_0 = B$ and $B_{n+1} = B_nAB_n^{-1}$. Compute the entries $a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}$ in terms of a_n, b_n, c_n, d_n and deduce that if $|c| < 1$ then $B_n \to A$ as $n \to \infty$, contradicting discreteness. For part (ii) compute the trace and apply part (i).]

(b) *(Jorgenson's inequality)* Prove that for any elements A, B in a non-elementary discrete subgroup of $SL(2, \mathbb{C})$:

$$|tr^2(A) - 4| + |tr(ABA^{-1}B^{-1}) - 2| \geq 1$$

[HINT: In the case that A is parabolic this is part (a). If A is elliptic or hyperbolic, then without loss of generality we may assume A is diagonal. Now consider the same sequence $\{B_n\}$ as in part (a). You will find there are various possibilities to consider and that some are easier than others.] SB 11/3/13