LTCC

Holomorphic Dynamics and Hyperbolic Geometry (February-March 2013)

Week 3 Exercises

1. Recall that the *quaternions* are quadruples $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ written in the form $x_0+x_1i+x_2j+x_3k$ and equipped with the (non-commutive) product given by $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. An extension of the action of $PSL(2,\mathbb{C})$ from $\hat{\mathbb{C}}$ to $\mathcal{H}^3_+$ can be defined explicitly as follows. Regard $\mathbb{R}^3$ as *quaternions* of the form $x+yi+tj(+0k)$. Then, provided our matrix in $PSL(2,\mathbb{C})$ has been normalised to a form in which $ad-bc \in \mathbb{R}^{>0}$, the map

$$z + tj \to \frac{a(z+tj)+b}{c(z+tj)+d}$$

sends the half-space

$$\mathcal{H}^3_+ = \{z + tj : z \in \hat{\mathbb{C}}, t \in \mathbb{R}^{>0}\}$$

to itself, extending the bijection

$$z \to \frac{az+b}{cz+d}$$

on the boundary plane $\hat{\mathbb{C}}$.
Verify that the expression above agrees with the formulae (i),(ii) and (iii) for the action of $PSL(2,\mathbb{C})$ on $\mathcal{H}^3_+$ (at the start of Section 5).

HINT: To express

$$\frac{a(z+tj)+b}{c(z+tj)+d}$$

in the standard quaternion form $e + fi + gj + hk$ *with* $e, f, g, h \in \mathbb{R}$ *one should multiply both top and bottom by* $\bar{c}(\bar{z}+\bar{d}) - ctj$.

2. Prove that a Möbius transformation of the form

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \quad ad - b^2 = 1$$

satisfies $JAJ = A^{-1}$ where $J(z) = -z^{-1}$. Deduce that if a discrete group G is generated by elements of this form then $\Lambda(G)$ is invariant under J.

3. Let $h(z) = (az+b)/(cz+d)$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.
(i) Show that $Im(h(z)) = Im(z)/|cz+d|^2$.
(ii) Let $\Delta = \{z : Im(z) > 0,\ |Re(z)| < 1/2,\ |z| > 1\}$. If $z \in \Delta$ show that $|cz+d|^2 > (|c|-|d|)^2 + |cd|$ and deduce from (i) that if $a, b, c, d \in \mathbb{Z}$ then $Im(h(z)) < Im(z)$.
(iii) By applying (ii) to h and to h^{-1} deduce that if $z \in \Delta$ then there is no $I \neq h \in PSL(2,\mathbb{Z})$ such that $h(z) \in \Delta$.
(iv) Let $v \in \mathbb{R}$, $v > 1$. Deduce from (iii) that $\Delta = \{z \in \mathcal{H}^2_+ :\ d(z, iv) < d(z, g(iv))\ \forall I \neq g \in PSL(2,\mathbb{Z})\}$ (where d is the Poincaré metric).