Holomorphic Dynamics and Hyperbolic Geometry

Shaun Bullett, QMUL

Overview

Rational maps are self-maps of the Riemann sphere of the form $z \to p(z)/q(z)$ where $p(z)$ and $q(z)$ are polynomials. Kleinian groups are discrete subgroups of $PSL(2, \mathbb{C})$, acting as isometries of 3-dimensional hyperbolic space and as conformal automorphisms of its boundary, the Riemann sphere. Both theories experienced remarkable advances in the last two decades of the 20th century and are very active areas of continuing research. The aim of the course is to introduce some of the main techniques and results in the two areas, emphasising the strong connections and parallels between them.

Topics to be covered in 5 two-hour lectures

(Added April 2013. The lecture notes that follow are divided into 8 chapters. They comprise the material that was covered during the 5 week lecture course in February-March 2013. This was essentially the list of topics below, with some covered in more depth than others.)

1. Dynamics of rational maps: The Riemann sphere and rational maps (basic essentials from complex analysis); conformal automorphisms of the sphere, plane and disc; Schwarz's Lemma; the Poincaré metric on the upper half-plane and unit disc; conjugacies, fixed points and periodic orbits (basic essentials from dynamical systems); spherical metric; equicontinuity; Fatou and Julia sets (definition).

2. Fatou and Julia sets: Normal families and Montel's Theorem; characterisations and proporties of Fatou and Julia sets; types of Fatou component; linearization theorems (Koenigs, Böttcher, Siegel, Brjuno, Yoccoz).

3. Hyperbolic 3-space and Kleinian groups: Hyperbolic 3-space and its isometry group; Kleinian groups; ordinary sets and limit sets; fundamental domains, Poincaré's polyhedron theorem; examples of Fuchsian and Kleinian groups and their limit sets.

4. Quadratic maps and the Mandelbrot set: The Mandelbrot set and its connectivity; geography of the Mandelbrot set: internal and external rays; introduction to kneading theory (Milnor-Thurston); open questions.

5. Further topics (selection from the following): The Measurable Riemann Mapping Theorem and its applications to holomorphic dynamics and Kleinian groups; polynomial-like mappings and renormalisation theory; Thurston's Theorem (characterizing topological branched-covering maps equivalent to rational maps); conformal surgery, matings; the 'Sullivan Dictionary' between holomorphic dynamics and Kleinian groups.

Prerequisites

Undergraduate complex analysis, linear algebra and elementary group theory.

1 Introduction

1.1 Overview

The objective of these lecture notes is to develop some of the main themes in the study of iterated *rational maps*, that is to say maps of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ to itself of the form

$$z \to \frac{p(z)}{q(z)}$$

(where p and q are polynomials with complex coefficients), and the study of *Kleinian groups*, discrete groups of maps from $\hat{\mathbb{C}}$ to itself, each of the form

$$z \to \frac{az+b}{cz+d}$$

(where a, b, c, d are complex numbers, with $ad - bc \neq 0$).

We shall develop these studies in parallel: although there is no single unifying theory encompassing both areas there are tantalizing similarities between them, and results in one field frequently suggest what we should look for in the other (*Sullivan's Dictionary*).

The study of iterated rational maps had its first great flowering with the the work of the French mathematicians Julia and Fatou around 1918-20, though its origins perhaps lie earlier, in the late 19th century, in the more geometric work of Schottky, Poincaré, Fricke and Klein. It has had its second great flowering over the last 30 years, motivated partly by the spectacular computer pictures which started to appear from about 1980 onwards, partly by the explosive growth in the subject of chaotic dynamics which started about the same time, and not least by the revolutionary work in three-dimensional hyperbolic geometry initiated by Thurston in the early 1980's. In the intervening period Siegel (in the 1940s) had proved key results concerning local linearisability of holomorphic maps, and Ahlfors and Bers (in the 1960s) following pioneering work of Teichmüller (in the late 1930s) had developed quasiconformal deformation theory for Kleinian groups: the stage was set for an explosion of interest, both experimental and analytical. Some of the names associated with this second great wave of activity are Mandelbrot, Douady, Hubbard, Sullivan, Herman, Milnor, Thurston, Yoccoz, McMullen and Lyubich. Both subjects are still very active indeed: as we shall see, some of the major conjectures are still waiting to be proved. But the remarkable mixture of complex analysis, hyperbolic geometry and symbolic dynamics that constitutes the subject of holomorphic dynamics yields powerful methods for problems which at first sight might appear only to concern only real mathematics. For example the most conceptual proof of the universality of the Feigenbaum ratios for period doubling renormalisation of real unimodal maps is that of Sullivan (1992) using complex analysis.

We start our study of rational maps and Kleinian groups - as we mean to go on - with motivating examples.

1.2 The family of maps $z \to z^2 + c$

(i) $c = 0$

Here the dynamical behaviour is straightforward. When we iterate $z \to z^2$ any orbit started inside the unit circle heads towards the point 0, any orbit started outside the unit circle heads towards ∞, and any orbit started on the unit circle remains there. The two components of $\{z : |z| \neq 1\}$ are known as the *Fatou set* of the map and the circle $|z| = 1$ is called the *Julia set*. On the unit circle itself the dynamics are those of the *shift*, namely if we parametrise the circle by $t \in [0, 1) \subset \mathbb{R} \quad (t = arg(z)/2\pi)$: then $z \to z^2$ sends $t \to 2t$ mod 1.

Any $t \in [0, 1)$ of the form $t = m/(2^n - 1)$ (for $0 \leq m < 2^n - 1$ integer) is periodic, of period n (exercise: prove this). Hence the periodic points form a dense set on the unit circle. Moreover the map $z \to z^2$ has *sensitive dependence on initial condition*, since the map on the unit circle doubles distance.

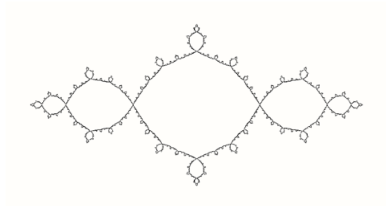

Figure 1: Julia set for $z \to z^2 - 1$

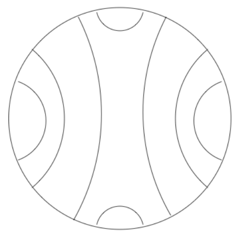

Figure 2: Lamination for $z \to z^2 - 1$ (first few leaves).

(ii) $c = -1$

When we vary c just a little from 0 the dynamical picture remains like that for $z \to z^2$. There is a single attractive fixed point (but this is no longer 0 itself), the Fatou set is a pair of (topological) discs, the basins of attraction of the finite fixed point and ∞ respectively, and the Julia is a (fractal) topological circle separating these discs. However as $|c|$ becomes larger the Julia set becomes more and more distorted and eventually self-intersects. For example once c has reached -1 the dynamical behaviour is rather more complicated to describe (see Figure 1). The Fatou set now has infinitely many components. There is a fixed point at ∞ to which every orbit started in the component of the Fatou set outside the 'filled Julia set' is attracted, and a period 2 cycle $0 \to -1 \to 0 \to -1 \to ...$ towards which every orbit started in any other component of the Fatou set is attracted. An orbit which starts on the common boundary of the two attractors (the 'Julia set', which we shall define formally soon) remains on that boundary. Combinatorially, the Julia set in this example is a *quotient* of the circle, and the dynamics are those of the corresponding *quotient* of the shift. Figure 2 shows the first few identifications on the unit circle in the construction of this quotient: contracting the leaves on the closed unit disc gives a model of the *filled Julia set* for $z \to z^2 - 1$.

(iii) $c = i$

See Figure 3. Note that the point 0 is *preperiodic* for this map ($0 \to i \to -1 + i \to -i \to -1 + i ...$). It can be

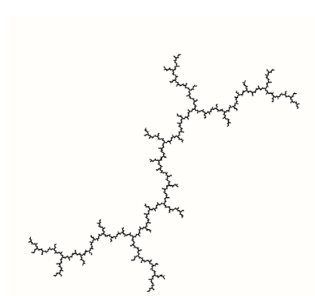

Figure 3: Julia set for $z \to z^2 + i$

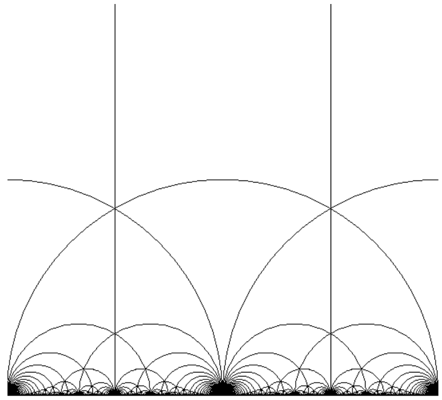

Figure 4: The modular group action on the upper half-plane

proved that whenever c is such that the critical point 0 of $z \to z^2 + c$ is preperiodic but not periodic, the Julia set is a *dendrite* (that is a connected, simply-connected set with empty interior).

(iv) $c = -2$

Here again 0 is preperiodic, and the dendrite (not drawn here) is the real interval $[-2, 2]$.

Exercise Show that $h : z \to z + 1/z$ is a semiconjugacy from $f : z \to z^2$ to $g : z \to z^2 - 2$ (that is, h is a surjection satisfying $hf = gh$) and that h sends the Julia set of f (the unit circle) onto the real interval $[-2, +2]$.

For $|c|$ sufficiently large the Julia set becomes disconnected - in fact it becomes a Cantor set. The set of all values of $c \in \mathbb{C}$ such the Julia set is connected is known as the *Mandelbrot Set*.

1.3 The modular group $PSL(2, \mathbb{Z})$

The *modular group* $PSL(2, \mathbb{Z})$ is the group of Möbius transformations

$$z \to \frac{az + b}{cz + d}$$

such that a, b, c, d are *integers* with $ad - bc = 1$. It is easy to see that $PSL(2, \mathbb{Z})$ maps the open upper half $\mathcal{H}_+$ of the complex plane to itself, the open lower half plane $\mathcal{H}_-$ to itself and the extended real axis $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$ to itself (see Figure 4).

We remark that the modular group is generated by $S : z \to -1/z$ and $T : z \to z+1$. All relations in the group are consequences of the pair of relations $S^2 = I$, $(ST)^3 = I$. The region $\Delta = \{z : |z| \leq 1, Re(z) \leq 1/2, Im(z) > 0\}$ is a *fundamental domain* for the action of $PSL(2, \mathbb{Z})$ on the upper half plane: $\mathcal{H}_+$ is 'tiled' by the translates of Δ under elements of the group. Similarly $\mathcal{H}_-$ is tiled by the mirror image of Δ and its translates. Both sets of tiles accumulate on $\hat{\mathbb{R}}$. Just as is the case for rational maps, the action of a Kleinian group G partitions the Riemann sphere into two disjoint completely invariant subsets, an *ordinary set* $\Omega(G)$ (in the case of the modular group this is $\mathcal{H}_+ \cup \mathcal{H}_-$), and a *limit set* $\Lambda(G)$ (in this case $\bar{\mathbb{R}}$) on which the system exhibits *sensitive dependence on initial conditions*: arbitrarily close to any point in $\Lambda(G)$ we can find another point and an element of G sending the two points arbitrarily far apart.

2 Dynamics of rational maps

2.1 The Riemann sphere

The *extended complex plane* is $\mathbb{C}$ together with an extra point '∞'. The topology on $\mathbb{C} \cup \{\infty\}$ can be described as follows. Let S^2 denote the unit sphere in $\mathbb{R}^3$, regard $\mathbb{C}$ as the plane $\mathbb{R}^2 \subset \mathbb{R}^3$ (which cuts through S^2 at its equator), and let $N = (0,0,1)$ denote the 'north pole' of S^2. Stereographic projection from N defines a homeomorphism $\pi : S^2 - \{N\} \to \mathbb{C}$. Extending π to send N to ∞ we obtain a homeomorphism from S^2 to $\mathbb{C} \cup \infty$, where the latter is topologised by taking as neighbourhoods of ∞ the sets $\{z : |z| > R\} \cup \infty$. However we need more than just a *topology* on $\mathbb{C} \cup \infty$: we give S^2 the structure of a *Riemann surface* by equipping it with charts (homeomorphisms) $\phi_1 : \mathbb{C} \to S^2 - \{N\}$ and $\phi_2 : \mathbb{C} \to S^2 - \{S\}$ such that $\phi_2^{-1}\phi_1$ is an analytic bijection on the overlap. We may take ϕ_1 to be the inverse π^{-1} of stereographic projection from the north pole and ϕ_2 to be the inverse of sterographic projection from the south pole, followed by complex conjugation. The overlap $\phi_2^{-1}\phi_1$ is then $z \to \bar{z}/|z|^2 = 1/z$.

Equivalently we can put a complex structure on $\mathbb{C} \cup \infty$ by regarding it as the *complex projective line*

$$\mathbb{CP}^1 = \{\mathbb{C}^2 - (0,0)\}/\mathcal{R}$$

where $\mathcal{R}$ is the relation $(z,w) \sim (\lambda z, \lambda w)$ for $\lambda \in \mathbb{C} - 0$. An equivalence class $[z,w]$ contains $(z/w, 1)$ if $w \neq 0$ or $(1, w/z)$ if $z \neq 0$, so we may think of $\mathbb{CP}^1$ as the union of two copies of the complex plane glued together, $\mathbb{C}_1 \cup \mathbb{C}_2/(z_1 \sim 1/z_2)$. The bijection

$$\mathbb{CP}^1 \leftrightarrow \hat{\mathbb{C}}$$

is given by $[z,w] \leftrightarrow z/w$ when $w \neq 0$ and $[z,0] \leftrightarrow \infty$. We shall use the term *Riemann sphere* interchangeably for $\hat{\mathbb{C}}$ or $\mathbb{CP}^1$, but we shall tend to use the notation $z \in \hat{\mathbb{C}}$ rather than $[z,w]$ for an individual point, just for convenience: all polynomial expressions in the former form can if necessary be re-written in the latter form simply by introducing a homogenising variable w.

2.2 Basic essentials from complex analysis

Definitions An open connected set $\Omega \subset \mathbb{C}$ is called a *domain.*

$f : \Omega \to \mathbb{C}$ is said to be *differentiable* at $z_0 \in \Omega$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

$f : \Omega \to \mathbb{C}$ is said to be *holomorphic* if f is differentiable at all $z_0 \in \Omega$.

Theorem 2.1 *Let f be holomorphic on the domain $\Omega \subset \mathbb{C}$ and let $z_0 \in \Omega$. Let R denote the radius of the largest disc which has centre z_0 and is contained in Ω. Then for all z with $|z - z_0| < R$ the Taylor series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ for f at z_0 converges absolutely to the value $f(z)$.*

This is a classical theorem of complex analysis. The coefficients a_n are given by the formulae

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

where C is a positively-oriented circle around z_0, or equivalently by

$$a_n = \frac{f^n(z_0)}{n!}$$

A function expressible as a power series is called *analytic.* Thus Theorem 2.1 says that a holomorphic function on a domain $\Omega \subset \mathbb{C}$ is analytic. The converse is also a well known result: every function expressible as a power series is holomorphic on the disc of convergence of the series, and its derivative is given by term-by-term differentiation.

There is a geometric interpretation for the statement that a function f is differentiable at z_0. If $f'(z_0) \neq 0$, then near z_0 we have $f(z) - f(z_0) \sim f'(z_0)(z - z_0)$ so f acts on $z - z_0$ by multiplying it by the scaling factor $|f'(z_0)|$ and turning it through an angle $arg(f'(z_0))$. Thus in particular if $f'(z_0) \neq 0$ the function f is *conformal* (angle-preserving) at z_0. If $f'(z_0) = 0$, then on a small disc centred at z_0 we have $f(z) \sim f(z_0) + a_n(z - z_o)^n$ for the first coefficient $a_n \neq 0$ and f acts on this disc as an *n-to-1 branched covering map* (branched at z_0): note that f is then *not* conformal at z_0, indeed it multiplies angles at z_0 by n.

If $f : \Omega \to \mathbb{C}$ is holomorphic except at *isolated singularities* (isolated points where f is undefined or not differentiable) then we say that f is *meromorphic* if all these singularities are either *removable* or *poles*, or equivalently if for each $z_0 \in \Omega$ there is a disc neighbourhood D of z_0 such that the Laurent series for f in the punctured disc $D - \{z_0\}$ has the form $\sum_{n=-m}^{+\infty} a_n(z - z_0)^n$. Recall that z_0 is said to be a *pole of order m* if $m > 0$ is such that $a_{-m} \neq 0$ but $a_{-n} = 0$ for all $n > m$, and that z_0 is said to be *removable* if $a_{-n} = 0$ for all $n > 0$. When z_0 is a removable singularity we can set $f(z_0) = a_0$ and thereby extend f to a function differentiable at z_0, and when z_0 is a pole $\lim_{z \to z_0} f(z) = \infty$ so we can extend the definition of f by setting $f(z_0) = \infty$ and regard f as a continuous function $f : \Omega \to \hat{\mathbb{C}} = \mathbb{C} \cup \infty$. This extension is generally called *meromorphic* too. (Note that if our original $f : \Omega \to \mathbb{C}$ has any *essential singularities* there is no way to assign values at these singularities to obtain a continuous extension $f : \Omega \to \mathbb{C} \cup \infty$ since in any neighbourhood of an essential singularity f takes values arbitrarily close to any given value.)

There is a nice way to characterise a meromorphic function $f : \Omega \to \hat{\mathbb{C}}$ (Ω a domain in $\mathbb{C}$), making use of the 'duality' between '0' and '∞'. Let σ denote the function $z \to 1/z$. Then around any pole z_0 of f the function σf is analytic, since $f(z)$ has an expression as a Laurent series

$$f(z) = (z - z_0)^{-m} \sum_{n=0}^{\infty} b_n(z - z_0)^n \quad (b_0 \neq 0)$$

and taking the reciprocal of this expression we obtain for $\sigma f(z)$ a series of the form

$$\sigma f(z) = (z - z_0)^m \sum_{n=0}^{\infty} c_n(z - z_0)^n$$

where $c_0 = 1/b_0$. It follows that $f : \Omega \to \hat{\mathbb{C}}$ is meromorphic if and only if f is analytic at those points z_0 where $f(z_0) \neq \infty$ and σf is analytic at those where $f(z_0) \neq 0$.

Finally, for full generality, we allow Ω to be a domain in $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ and not just in $\mathbb{C}$ and we say that $f : \Omega \to \hat{\mathbb{C}}$ is *meromorphic at* ∞ if $f\sigma$ is meromorphic at 0. The class of functions $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which are meromorphic on $\mathbb{C}$ and at ∞ are precisely the functions we are interested in: they are the functions which, provided we replace f by σf, $f\sigma$ or $\sigma f \sigma$ as appropriate, have a Taylor series expansion at every point of $\hat{\mathbb{C}}$.

Definition $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is *holomorphic* if f is meromorphic at every point of $\mathbb{C}$ and at ∞.

2.3 Rational maps and critical points

Theorem 2.2 *$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic if and only if f is a rational function, that is to say there exist polynomials $p(z), q(z)$, with complex coefficients, such that $f(z) = p(z)/q(z)$ for all $z \in \hat{\mathbb{C}}$.*

Proof It is an elementary exercise to show that any rational map f is meromorphic both at points of $\mathbb{C}$ and at ∞, since by the Fundamental Theorem of Algebra f has the form

$$f(z) = c\frac{(z - \alpha_1)^{m_1} \ldots (z - \alpha_r)^{m_r}}{(z - \beta_1)^{n_1} \ldots (z - \beta_s)^{n_s}}$$

For the converse, let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be holomorphic. Then f has finitely many poles (else $1/f$ has a convergent sequence of zeros, which, by Theorem 2.1, is only possible if $1/f$ is identically zero). Let these poles be $\beta_1, ..., \beta_s$, of order $n_1, ...n_s$ respectively. Then

$$g(z) = (z - \beta_1)^{n_1} ... (z - \beta_s)^{n_s} f(z)$$

is analytic on $\mathbb{C}$, and so g can be written in the form

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

Since f is meromorphic at ∞ so is g. Thus $g\sigma$ is meromorphic at 0. In other words $\sum_{n=0}^{\infty} a_n z^{-n}$ has a pole or a removable singularity at $z = 0$. It follows that only finitely many of the a_n are non-zero and hence g is a polynomial. QED

Comments

1. This is a very powerful result: it tells us that any holomorphic $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is determined by a *finite* set of data, for example the poles and zeros of f together with the value of f at one other point.

2. We can write a rational map $f(z) = p(z)/q(z) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ in terms of *homogeneous coordinates* on $\mathbb{CP}^1$ as follows: write

$$p(z) = \sum_{m=0}^{d} a_m z^m; \qquad q(z) = \sum_{m=0}^{d} b_m z^m$$

(where if necessary extra zero coefficients have been added to give p and q the same degree). Now define

$$f([z, w]) = [\sum_{m=0}^{d} a_m z^m w^{d-m}, \sum_{m=0}^{d} b_m z^m w^{d-m}]$$

Let $f(z) = p(z)/q(z)$, where p and q are polynomials of degree d_p and d_q respectively, with no common zeros. Then a general point $\zeta \in \hat{\mathbb{C}}$ has $max(d_p, d_q)$ inverse images (just consider the equation $\zeta = p(z)/q(z)$, that is to say $p(z) - \zeta q(z) = 0$: this has $max(d_p, d_q)$ solutions z for any ζ in general position). We define the *degree* of f to be $max(d_p, d_q)$. Thus, for example, rational maps of degree 1 have $f(z) = p(z)/q(z)$ where $p(z) = az + b$ and $q(z) = cz + d$ (but $ad - bc \neq 0$ else p is a constant multiple of q).

Definition A *critical point* of a rational map f is a point z_0 where the degree one term of the Taylor series for f vanishes, in other words the derivative $f'(z_0)$ vanishes.

As usual we replace f by $f\sigma$ here if $z_0 = \infty$, by σf if $f(z_0) = \infty$ and by $\sigma f \sigma$ if both are ∞, so that an appropriate Taylor series exists. Looked at topologically a critical point of f is a *branch point* of f, a point z_0 such that $f(z) - f(z_0)$ has a factor $(z - z_0)^n$ for some $n > 1$, and thus in particular where $f^{-1}f(z_0)$ consists of less than d distinct points. (But for $d > 2$ it does not follow that z_0 is a critical point just because $f^{-1}f(z_0)$ consists of less than d distinct points (exercise!).) Writing $f(z) = p(z)/q(z)$, we see that $f'(z) = 0 \Leftrightarrow q'(z)p(z) - p'(z)q(z) = 0$.

Proposition 2.3 *A degree d rational map has $2d - 2$ critical points (counted with multiplicity)*

Proof In the generic case both p and q have degree d and $q'(z)p(z) - p'(z)q(z)$ is generically a polynomial of degree $2d - 2$ (since $q'(z)p(z)$ and $p'(z)q(z)$ have the same degree $2d - 1$ term). In the non-generic case we obtain the same result if we adopt the right notion of 'multiplicity': this is best proved topologically using an argument based on Euler characteristics (see later for a full proof). QED

2.4 Conformal automorphisms of $\hat{\mathbb{C}}$, $\mathbb{C}$ and $\mathbb{D}$

The invertible holomorphic maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ are the *conformal automorphisms* of the Riemann sphere. They form a group $Aut(\hat{\mathbb{C}})$.

Proposition 2.4 *The conformal automorphisms of* $\hat{\mathbb{C}}$ *are the rational maps of form*

$$f(z) = \frac{az+b}{cz+d}$$

having $a, b, c, d \in \mathbb{C}$ *and* $ad \neq bc$.

Proof By Theorem 2.2 for f to be holomorphic it must be rational, but to be injective it must have degree 1. Conversely, any f of this form is invertible since it has inverse $f^{-1}(z) = (dz - b)/(-cz + a)$. QED

Maps of the form $f(z) = (az+b)/(cz+d)$ having $a, b, c, d \in \mathbb{C}$ and $ad \neq bc$ are called *fractional linear* or *Möbius* transformations.

Properties of Möbius transformations

1. Any invertible linear map $\alpha : \mathbb{C}^2 \to \mathbb{C}^2$ has the form

$$\begin{pmatrix} z \\ w \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} az + bw \\ cw + dw \end{pmatrix}$$

and passes to a map $\mathbb{CP}^1 \to \mathbb{CP}^1$ which in our coordinate z/w on $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ is

$$z/w \to \frac{az + bw}{cz + dw} = \frac{az/w + b}{cz/w + d}$$

(where $(a\infty + b)/(c\infty + d)$ is to be interpreted as a/c and so on).

2. Composition of linear maps passes to composition of Möbius transformations. The group of all Möbius transformations is therefore

$$PGL(2, \mathbb{C}) = \frac{GL(2, \mathbb{C})}{\{\lambda I; \lambda \in \mathbb{C} - \{0\}\}} = \frac{SL(2, \mathbb{C})}{\{\pm I\}} = PSL(2, \mathbb{C})$$

where $GL(2, \mathbb{C})$ denotes the group of all invertible 2×2 matrices and $SL(2, \mathbb{C})$ denotes those of determinant 1.

3. Given any three distinct points $P, Q, R \in \hat{\mathbb{C}}$, there exists a unique Möbius transformation sending $P \to \infty, Q \to 0, R \to 1$, given by

$$\alpha(z) = \frac{(P - R)(Q - z)}{(Q - R)(P - z)}$$

(Uniqueness follows from the easy exercise that the only Möbius transformation fixing $0, 1$ and ∞ is the identity.) It follows that given any other three distinct points $P', Q', R' \in \hat{\mathbb{C}}$ there exists a unique Möbius transformation sending $P \to P'$, $Q \to Q'$ and $R \to R'$, for if α is as above and β sends $P' \to \infty, Q' \to 0, R' \to 1$ then $\beta^{-1}\alpha$ has the required property.

4. Given any four distinct points $P, Q, R, S \in \hat{\mathbb{C}}$, their *cross-ratio* is defined to be

$$(P, Q; R, S) = \frac{(P - R)(Q - S)}{(Q - R)(P - S)} \in \hat{\mathbb{C}} - \{0.1, \infty\}$$

(**Warning:** There are several different definitions of a cross-ratio in common use.) It follows from the preceding remark that $(P, Q; R, S) = \alpha(S)$, where α is the unique Möbius transformation sending $P \to \infty$, $Q \to 0$ and $R \to 1$. Hence if γ is a Möbius transformation then $(\gamma(P), \gamma(Q); \gamma(R), \gamma(S)) = (P, Q; R, S)$, for $\alpha\gamma^{-1}$ is then a Möbius transformation sending $\gamma(P) \to \infty$, $\gamma(Q) \to 0$, $\gamma(R) \to 1$ and has $(\alpha\gamma^{-1})\gamma(S) = \alpha(S) = (P, Q; R, S)$. Thus *cross-ratios are preserved by Möbius transformations.*

Möbius transformations are conformal (since they are invertible and therfore have non-zero derivative everywhere). But conformality is just a local property and we can prove a much stronger result:

Proposition 2.5 *Möbius transformations send circles in* $\hat{\mathbb{C}}$ *to circles in* $\hat{\mathbb{C}}$ *(where a 'circle through* ∞*' is a straight line in* $\mathbb{C}$*).*

Proof Any 'circle' in $\hat{\mathbb{C}}$ (including those through ∞) has the form

$$\alpha(x^2+y^2)+2\beta x+2\gamma y+\delta=0 \quad (\alpha,\beta,\gamma,\delta\in\mathbb{R})$$

in other words

$$Az\bar{z}+Bz+\bar{B}\bar{z}+C=0$$

where $A=\alpha\in\mathbb{R}, B=\beta-i\gamma\in\mathbb{C}, C=\delta\in\mathbb{R}$. Let

$$z=\frac{aw+b}{cw+d}$$

Now a direct substitution for z in the equation above gives an equation of the same form for w once the denominator has been cleared. QED

Corollary 2.6 *Any four distinct points $P,Q,R,S\in\hat{\mathbb{C}}$ lie on a common circle if and only if their cross ratio $(P,Q;R,S)$ is real.*

Proof Send P,Q,R to $\infty,0,1$ by a Möbius transformation. QED

Proposition 2.7 *The conformal automorphisms of $\mathbb{C}$ are the maps $f(z)=az+b$ having $a,b\in\mathbb{C}$ and $a\neq 0$.*

Proof Let f be a conformal automorphism of $\mathbb{C}$. Then $\lim_{z\to\infty} f(z)=\infty$ (this follows from the fact that f is a homeomorphism). Hence $\sigma f\sigma$ has a removable singularity at 0 and so f extends to a conformal automorphism of $\hat{\mathbb{C}}$. The result follows by Proposition 2.4. QED

We next identify the conformal automorphisms of $\mathbb{D}$. The neatest method is via Schwarz's Lemma, which will be an important tool for us later for other purposes.

Lemma 2.8 (Schwarz's Lemma) *If f is holomorphic $\mathbb{D}\to\mathbb{D}$ and $f(0)=0$ then $|f'(0)|\leq 1$. If $|f'(0)|=1$ then $f(z)=\mu z$ for some $\mu\in\mathbb{C}$ with $|\mu|=1$. If $|f'(0)|<1$ then $|f(z)|<|z|$ for all $0\neq z\in\mathbb{D}$.*

Proof Let $f(z)$ have Taylor series $a_1z+a_2z^2+...$ on $\mathbb{D}$, and set $g(z)=a_1+a_2z+...(=f(z)/z)$. Then g is holomorphic $\mathbb{D}\to\mathbb{C}$ and on the circle $\mathcal{C}$ having centre 0 and radius ρ we see that

$$|g(z)|=\frac{|f(z)|}{|z|}\leq\frac{1}{\rho}$$

so by the Maximum Modulus Principle $|g(z)|$ has the bound $1/\rho$ for all z inside $\mathcal{C}$ too. Letting ρ tend to 1 (from below) we deduce that $|g(z)|\leq 1$ for all $z\in\mathbb{D}$, and in particular $|g(0)|\leq 1$, that is $|f'(0)|\leq 1$. If there is any $z_0\in\mathbb{D}$ with $|g(z_0)|=1$ (for example if $|g(0)|=1$), then $|g(z)|=1$ for all $z\in\mathbb{D}$ (again by the Maximum Modulus Principle) in which case g must be constant, say $g(z)=\mu$, with $|\mu|=1$. If there is no such z_0 then $|g(z)|<1$ for all $z\in\mathbb{D}$, i.e. $|f(z)|<|z|$ for all $z\in\mathbb{D}$. QED

Proposition 2.9 *The conformal automorphisms of $\mathbb{D}$ are the maps of form*

$$f(z)=e^{i\theta}\frac{z-a}{1-\bar{a}z}, \quad \theta\in\mathbb{R}, a\in\mathbb{D}$$

Proof Let f be a conformal automorphism of $\mathbb{D}$. Then $f^{-1}(0)=a\in\mathbb{D}$. The Möbius transformation

$$g(z)=\frac{z-a}{1-\bar{a}z}$$

sends a to 0 and the unit circle to itself, so it sends $\mathbb{D}$ to itself. Thus fg^{-1} is a conformal automorphism of $\mathbb{D}$ sending 0 to 0. From Schwarz's Lemma it follows that $fg^{-1}(z)=\mu z$ for some μ with $|\mu|=1$. QED

Corollary 2.10 *The conformal automorphisms of the upper half-plane $\mathcal{H}_+$ are the Möbius transformations*

$$f(z)=\frac{az+b}{cz+d}$$

having $a, b, c, d \in \mathbb{R}$ and $ad \neq bc$.

Proof. Take any Möbius transform M which sends the upper half plane $\mathcal{H}_+$ bijectively onto $\mathbb{D}$ (exercise: write one down). The conformal automophisms of $\mathcal{H}_+$ are the maps $M^{-1}gM$ where g runs through the conformal automorphisms of $\mathbb{D}$ given by Proposition 2.9 (details: exercise). QED

We recall that not only is there is a conformal bijection between $\mathcal{H}_+$ and $\mathbb{D}$, but that the Riemann Mapping Theorem states that for every simply-connected domain $U \subset \mathbb{C}$ ($U \neq \mathbb{C}$) there is a conformal bijection between U and $\mathbb{D}$. An important generalisation of this that we shall repeatedly use explicitly or implicitly, but which we will not prove in this course, is the following (proved by Poincaré and Koebe):

Theorem 2.11 (The Uniformisation Theorem) *Every simply-connected Riemann surface is conformally bijective to one of $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$.*

2.5 The Poincaré metric on the upper half plane

Define the *infinitesimal Poincaré metric* on the upper half plane by $ds = \frac{\sqrt{(dx)^2+(dy)^2}}{y}$.

Proposition 2.12 *ds is invariant under $PSL(2, \mathbb{R})$*

Proof Every element of $PSL(2, \mathbb{R})$ can be written as a composition of transformations of the type $z \to z + \lambda$ ($\lambda \in \mathbb{R}$), $z \to \mu z$ ($\mu \in \mathbb{R}^{>0}$) and $z \to -1/z$, and it is easily checked that each preserves ds. QED

A path in $\mathcal{H}_+$ is called a *geodesic* from P to Q in $\mathcal{H}_+$ if it is a path of shortest length.

Proposition 2.13 *There is a unique geodesic between any two distinct points P and Q in $\mathcal{H}$. It is the segment between P and Q of the (unique) euclidean semicircle through P and Q which meets $\hat{\mathbb{R}}$ orthogonally. The distance between P and Q (in the Poincaré metric) is $ln(|(P, Q; A, B)|)$ where A and B are the points where the semicircle meets $\hat{\mathbb{R}}$.*

Proof In the case that P and Q are on the imaginary axis, the straight line path γ_1 from P to Q is shorter than any other path γ_2 from P to Q, since

$$\int_{\gamma_2} \frac{1}{y}\sqrt{(dx)^2 + (dy)^2} > \int_{\gamma_2} \frac{1}{y} dy = \int_{\gamma_2} \frac{1}{y} dy$$

For $P = i$ and $Q = it$ (real $t > 1$) the hyperbolic distance from P tp Q is

$$\int_1^t \frac{1}{y} dy = \ln t = \ln |(i, it; 0, \infty)|$$

The result follows, since given any P', Q' in $\mathcal{H}_+$ there is an element of $PSL(2, \mathbb{R})$ which sends P to P' and Q to Q', and moreover this Möbius transformation sends the positive imaginary axis to a semicircle with ends on the extended real axis $\hat{\mathbb{R}}$ and preserves corss-rations. QED

Corollary 2.14 *The group of conformal automorphisms of the upper half-plane, $PSL(2, \mathbb{R})$, is also the group of orientation-preserving isometries of the upper half-plane (equipped with the Poincaré metric).*

Proof (sketch) It is obvious that every element of $PSL(2, \mathbb{R})$ preserves the Poincaré metric since it preserves the upper half-plane, the real axis and cross-ratios. For the other direction, observe that an isometry of the Poincaré metric must send geodesics to geodesics, and it must send orthogonal pairs of geodesics to orthogonal pairs of geodesics (since othogonal pairs of geodesics are pairs of semicircles with end points on $\hat{\mathbb{R}}$ having cross-ratio -1). It follows that an isometry must satisfy the Cauchy-Riemann equations everywhere and is therefore a conformal automorphism. QED

We can transfer the Poincaré matric to $\mathbb{D}$, using any Möbius transformation M sending $\mathcal{H}_+ \to \mathbb{D}$.

Exercise. Show that the infinitesimal metric $\frac{2|dz|}{1-|z|^2}$ on $\mathbb{D}$ is invariant under $Aut(\mathbb{D})$, show that the distance between 0 and $t \in (0, 1) \subset \mathbb{D} \cap \mathbb{R}$ in this metric is $\ln |(0, t; -1, +1)|$, and deduce that this is the infinitesimal Poincaré metric, transferred from $\mathcal{H}_+$ to $\mathbb{D}$.

2.6 Conjugacies, fixed points and multipliers

Definition Rational maps f, g are said to be *conjugate* if there exists a Möbius transformation h such that $g = hfh^{-1}$, in other words such that the following diagram commutes:

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} \\ h\big\downarrow & & \big\downarrow h \\ \hat{\mathbb{C}} & \xrightarrow{g} & \hat{\mathbb{C}} \end{array}$$

Conjugate maps have identical dynamical behaviour (think of h as a 'change of coordinate system'). In particular h sends fixed points of f to fixed points of g, periodic points of f to periodic points of g etc, as we shall see below. We can often put a rational map into a simpler form by applying a suitable conjugacy.

Examples

1. A rational map f is conjugate to a polynomial if and only if there exists a point $z_0 \in \hat{\mathbb{C}}$ such that $f^{-1}(z_0) = \{z_0\}$. (**Proof**: Move z_0 to ∞ by a Möbius transformation h. Details: exercise.)

2. A rational map f is conjugate to a polynomial of the form $z \to z^n$ (some $n > 0$) if and only if there exist distinct points $z_0, z_1 \in \hat{\mathbb{C}}$ such that $f^{-1}(z_0) = \{z_0\}$ and $f^{-1}(z_1) = \{z_1\}$. (**Proof**: Move z_0 to ∞ and z_1 to 0 by a Möbius transformation h. Details: exercise.)

3. Every degree 2 polynomial $z \to \alpha z^2 + \beta z + \gamma$ ($\alpha \neq 0$) is conjugate to a (unique) one of the form $z \to z^2 + c$. (**Proof**: Exercise: h can be taken of the form $az + b$ since we do not have to move ∞).

Fixed points and multipliers

Definitions A *fixed point* of a rational map f is a point $z_0 \in \hat{\mathbb{C}}$ such that $f(z_0) = z_0$.

The *multiplier* of f at such a fixed point is the derivative $f'(z_0) = \lambda$. We say that z_0 is

attracting if $|\lambda| < 1$ (if $\lambda = 0$ we say z_0 is *superattracting*);

repelling if $|\lambda| > 1$;

neutral if $|\lambda| = 1$, i.e. $\lambda = e^{2\pi i\theta}$ for some $\theta \in \mathbb{R}$.

As we shall see, the dynamical behaviour around a neutral periodic point depends on whether θ is rational or irrational, and the irrational case can be further subdivided into 'linearisable' and 'non-linearisable'.

Proposition 2.15 *When the function f is conjugated by a Möbius transformation h any fixed point z_0 of f is sent to a fixed point $w_0 = h(z_0)$ of $g = hfh^{-1}$ and the multiplier of the fixed point w_0 for g is equal to the multiplier for the fixed point z_0 for f.*

Proof If z_0 is a fixed point of f and $w_0 = h(z_0)$ then

$$g(w_0) = gh(z_0) = hf(z_0) = w_o$$

and, by the chain rule for differentiation,

$$g'(w_0) = h'(w_0)f'(z_0)(h^{-1})'(w_0)$$

but since h is differentiable, has differentiable inverse and sends z_0 to w_0, we know that

$$(h^{-1})'(w_0) = \frac{1}{h'(z_0)}$$

and hence $g'(w_0) = f'(z_0)$. QED

Note that we cannot expect the *derivative* of a rational map f at a point z_0 to be a conjugacy invariant when z_0 is not a fixed point, since there is no reason to expect any relation between $h'(z_0)$ and $(h^{-1})'(hf(z_0))$. However the property of having zero derivative does turn out to be a conjugacy invariant (exercise). This should not surprise us as this is a topological property of the map: the critical points are the *branch points* of the map, that is to say the points where it fails to be locally one-to-one.

Proposition 2.15 says that a conjugacy sends a fixed point of f to a fixed point of g having the same dynamical behaviour (attractor, repellor etc). Analogous results hold for periodic orbits:

Definition A point z_0 is said to *periodic* of period n for f if $f^n(z_0) = z_0$ but $f^j(z_0) \neq z_0$ for $0 < j < n$. The *multiplier* of the periodic orbit $\{z_0, f(z_0) = z_1, f(z_1) = z_2, ..., f(z_{n-1}) = z_0\}$ is defined to be $(f^n)'(z_0)$. Note that $(f^n)'(z_0) = f'(z_0)f'(z_1)...f'(z_{n-1})$ by the chain rule.

Proposition 2.16 *When the function f is conjugated by a Möbius transformation h any orbit of period n of f is sent an orbit of period n of $g = hfh^{-1}$, and the two orbits have the same multiplier.*

Proof Denote the periodic orbit of f by $\{z_0, f(z_0) = z_1, f(z_1) = z_2, ..., f(z_{n-1}) = z_0\}$. Then $g^j h(z_0) = hf^j(z_0) = h(z_j)$. So $g^j h(z_0) \neq h(z_0)$ for $0 < j < n$ (h being injective) and $g^n h(z_0) = h(z_0)$. Hence $h(z_0)$ is periodic of period n for g. The orbits have the same multiplier by Proposition 2.15 applied to f^n. QED

2.7 The spherical metric and the Fatou and Julia sets of a rational map

We define the *spherical metric* on the unit sphere S^2 by setting the distance between two points to be the shortest Euclidean length of a great circle path between them. On the Riemann sphere, parameterised as the extended complex plane $\mathbb{C} \cup \infty$, the infinitesimal spherical metric is:

$$ds = \frac{2|dz|}{1 + |z|^2}$$

WARNING The spherical metric is not preserved by $Aut(\hat{\mathbb{C}})$, but conjugating by any particular conformal automorphism sends the spherical metric to a Lipschitz equivalent metric, since $\hat{\mathbb{C}}$ is compact.

Definition Let f be a rational map and z_0 be a point of $\hat{\mathbb{C}}$. We say that the family of iterates $\{f^n\}_{n \geq 0}$ is *equicontinuous at z_0* if given any $\epsilon > 0$ there exists $\delta > 0$ such that for all $n \geq 0$ $d(f^n(z), f^n(z_0)) < \epsilon$ whenever $d(z, z_0) < \delta$. (Here d is the spherical metric on $\hat{\mathbb{C}}$).

Think of this as saying that *'any orbit which that starts near z_0 remains close to the orbit of z_0'*.

Definitions The *Fatou set* $F(f)$ of f is the largest open subset of $\hat{\mathbb{C}}$ on which the family $\{f^n\}_{n \geq 0}$ is equicontinuous at every point. The *Julia set* $J(f)$ of f is $\hat{\mathbb{C}} - F(f)$.

The Julia set should be thought of as the set of points the orbits of which exhibit *'sensitive dependence on initial conditions'*.

Example

$f(z) = z^2$ has Fatou set $F(f) = \{z : |z| \neq 1\}$, and Julia set $J(f) = \{z : |z| = 1\}$.

Since f doubles length along the unit circle it is clear that $\{z : |z| = 1\} \subset J(f)$. It is not quite so obvious that points not on the unit circle are in $F(f)$. One can give a direct formal proof of this, but the details are a little messy in practice: the problem is that orbits started close together near (but not on) the unit circle will move apart for a large number of iterations before they start approaching each other again. For a more general method of proof, see the next chapter.

Remark If $g = hfh^{-1}$ where $h \in Aut(\hat{\mathbb{C}})$, then $F(g) = h(F(f))$ and $J(g) = h(J(f))$. This follows from the remark about Lipschitz equivalent metrics in the warning above.

3 Fatou and Julia sets

The following properties follow immediately from our definitions at the end of the previous chapter:

1. $F(f)$ is open (by definition); hence $J(f)$ is closed and therefore compact (since $\hat{\mathbb{C}}$ is compact).

2. $F(f)$ is *completely invariant*, that is $f(F(f)) = F(f) = f^{-1}(F(f))$. The fact that $f^{-1}(F(f)) \subset F(f)$ follows from the definition of $F(f)$ and the continuity of f; the converse, $F(f) \subset f^{-1}(F(f))$, is a consequence of the fact that a rational map is *open* (i.e the image of an open set is itself open).

3. $J(f)$ is completely invariant. (This follows at once from 2.)

What kinds of families $\mathcal{F}$ of analytic maps $f : \Omega \to \hat{\mathbb{C}}$ are equicontinuous ? Our first step towards an answer is the following very useful Proposition which interprets Schwarz's Lemma in the language of hyperbolic geometry:

Proposition 3.1 *If f be a holomorphic map $\mathbb{D} \to \mathbb{D}$ then f is non-increasing in the Poincaré metric.*

Proof Let z_0, z_1 be any two points in $\mathbb{D}$. Let $f(z_0) = w_0$ and $f(z_1) = w_1$. Choose isometries h, k of the Poincaré disc D (Möbius transformations) such that $h(0) = z_0$ and $k(0) = w_0$. Let $z_1' = h^{-1}z_1$ and $w_1' = k^{-1}w_1$. Now $k^{-1}fh$ is a holomorphic map of $\mathbb{D}$ to itself sending 0 to 0 and z_1' to w_1'. Hence $|w_1'| \leq |z_1'|$ by Schwarz's Lemma, and so $d(0, w_1') < d(0, z_1)$ in the Poincaré metric. But $d(w_0, w_1) = d(0, w_1')$ and $d(z_0, z_1) = d(0, z_1')$ (as h and k are isometries). QED

Corollary 3.2 *Every family of holomorphic maps $\mathbb{D} \to \mathbb{D}$ is equicontinuous.*

Proof It follows at once from Proposition 3.1 that every such family $\mathcal{F}$ is equicontinuous with respect to the Poincaré metric on $\mathbb{D}$. But we need to show that it is equicontinuous with respect to the spherical metric (where we regard $\mathbb{D}$ as $\mathbb{D} \subset \mathbb{C} \subset \hat{\mathbb{C}}$). However given any point z_0 we can find a small disc around z_0 and a constant k such that the distance between any two points z, z' in this disc in the Poincaré metric is less than k times the distance in the spherical metric. Equicontinuity at z_0 follows, since the spherical distance between the images $f(z), f(z')$ of two points under $f \in \mathcal{F}$ is less than or equal to the Poincaré distance betwen these images, this being true for any pair of points in $\mathbb{D}$. QED

Example The family $\{z \to z^{2^n}\}_{n \geq 0}$ is equicontinuous on $\mathbb{D}$: thus the Fatou set of $z \to z^2$ contains $\{z : |z| < 1\}$. Conjugating by $\sigma : z \to 1/z$ we see that the Fatou set of $z \to z^2$ also contains $\{z : |z| > 1\}$. Since every point on the unit circle is in the Julia set of $z \to z^2$, we now have a proof that the Fatou and Julia sets of this map are as we claimed at the end of the previous chapter.

It follows at once from Corollary 3.2 that every *bounded* family of holomorphic maps $\mathbb{D} \to \mathbb{C}$ is equicontinuous (again with respect to the spherical metric).

There are two approaches to defining the Fatou set of a rational map f, either as the *equicontinuity set* of the family of iterates of f, or as the *normality set* of this family. They give equivalent definitions, so it really makes no difference which route we take, but it will be convenient for us to switch back and forth.

Definition Let Ω be a domain in $\hat{\mathbb{C}}$. A family $\mathcal{F}$ of maps $\Omega \to \hat{\mathbb{C}}$ is called *normal* if every sequence in $\mathcal{F}$ contains a subsequence which converges *locally uniformly* to a map $f : \Omega \to \hat{\mathbb{C}}$ (not necessarily in $\mathcal{F}$).

Example $\{z \to z^{2^n}\}_{n \geq 0}$ are a normal family on $\mathbb{D}$, since they converge locally uniformly there to the constant map $z \to 0$.

Theorem 3.3 (Arzelà-Ascoli) *Let Ω be a domain in $\hat{\mathbb{C}}$. Any family of continuous maps $\Omega \to \hat{\mathbb{C}}$ is normal if and only if it is equicontinuous.*

For a proof, see for example Ahlfors' book 'Complex Analysis'.

Comments

1. We remind the reader that we use the *spherical metric* on $\hat{\mathbb{C}}$, both in the definition of *local uniform convergence* (used in defining the notion of a *normal family*) and in the definition of *equicontinuity*.

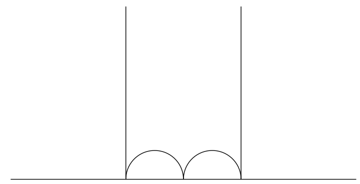

Figure 5: A fundamental domain Δ for the action of the group G on the upper half-plane (Theorem 3.4). The two vertical lines are identified by $z \to z+1$, the two semicircles are identified by $z \to z/(2z+1)$, and the quotient Δ/G is a thrice punctured sphere.

2. It is more elegant mathematically to develop the whole Fatou-Julia theory via normality rather than equicontinuity but the latter is perhaps easier to comprehend dynamically (Milnor follows the normality route).

3. It follows at once from Corollary 3.2 and Theorem 3.3 that every family of holomorphic maps from $\mathbb{D}$ to itself is normal. One can also prove Corollary 3.2 directly from the definition of a normal family (see Milnor, Thm 3.2, for a general version): at the heart of the argument is the Bolzano-Weierstrass Theorem that in any metric space a set is compact if and only if every infinite subset contains a convergent subsequence. Also relevant to this circle of ideas is the Denjoy-Wolff Theorem (1926), which states that a holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is either a conformal bijection or the iterates of f converge locally uniformly to a constant map $\mathbb{D} \to \zeta \in \bar{\mathbb{D}}$.

This brings us to a theorem central to the development of the Fatou-Julia theory of rational maps:

Theorem 3.4 (Montel, 1911) *let Ω be a domain in $\hat{\mathbb{C}}$. Every family of analytic maps $\Omega \to \hat{\mathbb{C}} - \{0,1,\infty\}$ is normal (or equivalently, by Arzelà-Ascoli, equicontinuous).*

Proof Without loss of generality assume Ω is an open disc (since equicontinuity and normality are local properties), and indeed by scaling if necessary assume $\Omega = \mathbb{D}$, the unit disc. Since $\mathbb{D}$ is simply connected, any map $f : \mathbb{D} \to \hat{\mathbb{C}} - \{0,1,\infty\}$ lifts to a map $\tilde{f}$ from $\mathbb{D}$ to the *universal cover* of $\hat{\mathbb{C}} - \{0,1,\infty\}$, which is the complex upper half plane $\mathcal{H}_+$, the group of covering translations being

$$G = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$$

acting on the upper half plane in the usual way (Figure 5).

*[**Aside** Here we recall that a universal cover of a manifold M is a simply-connected manifold $\tilde{M}$ which evenly covers M, i.e. $\tilde{M}$ is equipped with a projection $p : \tilde{M} \to M$ with the property that every $x \in M$ has a neighbourhood U such that $p^{-1}(U)$ is a disjoint union of copies of U, each mapped homeomorphically by p onto U. Given a universal cover $p : \tilde{M} \to M$ and a simply-connected space Y, every continuous $f : Y \to M$ has a lift, $\tilde{f} : Y \to \tilde{M}$ such that $p\tilde{f} = f$, and indeed there is a unique $\tilde{f}$ that lifts a chosen base point $x \in M$ to any specified point in $p^{-1}(x)$.]*

Equivalently we may take the universal cover to be the Poincaré disc model, $\mathbb{D}$, of the hyperbolic plane. Observe that the set of all lifts $\tilde{f} : \mathbb{D} \to \mathbb{D}$ of elements f of $\mathcal{F}$ forms a normal family, since these lifts are self-maps of the disc. The Poincaré metric on $\mathbb{D}$ projects under $q : \mathbb{D} \to \hat{\mathbb{C}} - \{0,1,\infty\}$ to a metric on $\hat{\mathbb{C}} - \{0,1,\infty\}$ in which the three missing points are pairwise infinitely far apart. Taking the Poincaré metric on domain and range each $\tilde{f}$ is metric non-increasing (by Proposition 3.1) and hence the same is true for each $f : \mathbb{D} \to \hat{\mathbb{C}} - \{0,1,\infty\}$. Given any $z_0 \in \mathbb{D}$, we may restrict consideration to a small disc $D' \subset \mathbb{D}$ centred on z_0 (since equicontinuity and normality are local properties). Since D' has finite diameter, say k (in the Poincaré metric), each $f(D')$ has diameter $\leq k$, and so for a small disc neighbourhood N of at least one of the three missing points in $\hat{\mathbb{C}} - \{0,1,\infty\}$, there must be an infinite sub-family $\mathcal{F}' \subset \mathcal{F}$ such that $f(D') \cap N = \emptyset$ for all $f \in \mathcal{F}'$. Since $\hat{\mathbb{C}} - N$ is a disc, the family $\mathcal{F}'$ is equicontinuous with respect to the spherical metric (by Prop. 3.2), therefore normal, and hence so is $\mathcal{F}$. QED

We can replace the points $0, 1, \infty$ in the statement of Montel's Theorem by any other three points of $\hat{\mathbb{C}}$ (just compose with a suitable Möbius transformation). Montel's Theorem is a much more powerful result than our earlier observation that any family of maps with a common bound is equicontinuous. One should perhaps compare it with Picard's Theorem that any holomorphic function $\mathbb{C} \to \hat{\mathbb{C}} - \{0, 1, \infty\}$ is constant, which is in turn is much more powerful than Liouville's Theorem that a bounded holomorphic function on $\mathbb{C}$ is constant.

Exercise Deduce Picard's Theorem from Liouville's Theorem and the fact that $\mathbb{D}$ is the universal cover of the thrice-punctured Riemann sphere.

3.1 Counting critical points, and the exceptional set

Before considering the many properties of Julia sets which follow from Montel, we make a brief excursion into topology to count critical points and derive some consequences for *finite* completely invariant sets.

Definition The *valency* of a critical point c of a rational map f is ν_c, where locally near c the map f has the form $z \to kz^{\nu_c}$ (plus higher order terms). In other words the valency is the 'degree of branching' at c.

The following result gives the delayed precise formulation and proof of Proposition 2.3.

Proposition 3.5 (Riemann-Hurwitz Formula) *If f is a rational map of degree d, then*

$$\sum_c (\nu_c - 1) = 2d - 2$$

where the sum is taken over all critical points of f.

Proof Triangulate the target copy of $\hat{\mathbb{C}}$ in such a way that the critical values of f are all vertices, and pull this triangulation back, via f, to a triangulation of the source copy of $\hat{\mathbb{C}}$. The Euler characteristic of $\hat{\mathbb{C}}$ (number of triangles minus number of edges plus number of vertices) is 2. Apart from at critical points, f is a d to one map and so we obtain the equation

$$2d - \sum_c (\nu_c - 1) = 2$$

and thus

$$\sum_c (\nu_c - 1) = 2d - 2$$

QED

Corollary 3.6 *Let f be a rational map with deg$(f) \geq 2$, and suppose E is a finite completely invariant subset of $\hat{\mathbb{C}}$. Then E contains at most 2 points.*

Proof Suppose E contains k points. Then f must permute these points (since every surjection of a finite set to itself is a bijection) and hence for some q the iterate $f^q = g$ is the identity on E. Suppose g has degree d. Each point $z \in E$ must be a critical point of g, of *valency* d, else $g^{-1}(z)$ would contain points other than z. Hence by Proposition 3.5

$$k(d-1) \leq 2d - 2$$

and therefore $k \leq 2$. QED

Definition The *exceptional set* $E(f)$ of a rational map is the union of all finite completely invariant sets. Corollary 3.6 says $|E(f)| \leq 2$. Note that if $|E(f)| = 1$ then f is conjugate to a polynomial (just conjugate by a Möbius transformation sending the exceptional point to ∞), and if $|E(f)| = 2$ then f is conjugate to some $z \to z^d$, with d a positive or negative integer (just send the two exceptional points to ∞ and 0).

3.2 Properties of Julia sets

For a rational map of degree greater or equal to two we have the following:

1. $J(f) \neq \emptyset$.

Proof. Let f be a rational map of degree $d \geq 2$. Then f^n has degree d^n (this can be proved various ways: if you know about homology groups it follows from the fact that $f_* : H_2(S^2) \to H_2(S^2)$ is the homorphism

$\times d : \mathbb{Z} \to \mathbb{Z}$). If $\{f^n\}_{n\geq 0}$ form a normal family on the whole of $\hat{\mathbb{C}}$ then some subfamily $\{f^{n_j}\}_{j\geq 1}$ converges locally uniformly to a function g, and since $\hat{\mathbb{C}}$ is compact $\exists J$ such that $\forall j > J$ and all $z \in \hat{\mathbb{C}}$ we have $d(f^{n_j}(z), g(z)) < \pi/2$ in the spherical metric. But then $\forall j, k > J$ we have $d(f^{n_j}(z), f^{n_k}(z)) < \pi$, and hence that f^{n_j} is homotopic to f^{n_k} (by the 'straight line homotopy' along the shortest great circle arc between $f^{n_j}(z)$ and $f^{n_k}(z)$). Hence $\forall j, k > J$ we have $deg(f^{n_j}) = deg(f^{n_k})$ i.e. $d^{n_j} = d^{n_k}$, contradicting $n_j \to \infty$.

2. $J(f)$ is infinite.

Proof. By Corollary 3.6 the only possibilities for finite completely invariant sets are (up to conjugacy) the set $\{\infty\}$ (for a polynomial) or $\{\infty, 0\}$ (for a map $z \to z^d$). But in both cases these exceptional sets are contained in the Fatou set.

3. $J(f)$ is the smallest completely invariant closed set containing at least three points.

Proof. The complement of a completely invariant closed set containing at least three points is an open completely invariant set omitting at least three points, hence contained in the Fatou set, by Montel's Theorem.

4. $J(f)$ is perfect, that is, every point of $J(f)$ is an accumulation point of $J(f)$.

Proof. For if we let J_0 be the set of accumulation points of J, then J_0 is non-empty (by Property 2), closed (by definition) and completely invariant (using the facts that f is continuous, open and finite-to-one). But J_0 cannot be finite since it would then be exceptional and hence contained in $F(f)$, so $J_0 = J$ by Property 3.

5. $J(f)$ is either the whole of $\hat{\mathbb{C}}$ or it has empty interior.

Proof. Write $S = \hat{\mathbb{C}} - int(J)$. Then S is the union of the Fatou Set F and the boundary ∂J of J, and either S is empty or it is an infinite closed completely invariant set, so containing J (by Property 3).

We remark in connection with Property 5 that there exist examples of rational maps f having $J(f) = \hat{\mathbb{C}}$ (e.g. the example of Lattès (1918): $z \to (z^2+1)^2/4z(z^2-1)$) but that for a *polynomial* map the Fatou set always contains the point ∞ and hence is non-empty.

3.3 Useful results for plotting $J(f)$

Proposition 3.7 *If $deg(f) \geq 2$ and U is any open set meeting $J(f)$, then $\bigcup_{n=0}^{\infty} f^n(U) \supset \hat{\mathbb{C}} - E(f)$.*

Proof If $\bigcup_{n=0}^{\infty} f^n(U)$ misses three or more points of $\hat{\mathbb{C}}$ then f^n are a normal family on U by Montel, contradicting $U \cap J \neq \emptyset$. But if a non-exceptional z lies in $\hat{\mathbb{C}} - \bigcup_{n=0}^{\infty} f^n(U)$ then for some m and n a point of $f^{-m}(z)$ must lie in $f^n(U)$ (since $\bigcup_{m\geq 0} f^{-m}(z)$ is infinite). Hence $z \in f^{m+n}(U)$. Contradiction. QED.

Corollary 3.8 *If z_0 is not in $E(f)$, then $J(f) \subset \overline{\bigcup_{n\geq 0} f^{-n}(z_0)}$.*

Proof Take any $z \in J(f)$ and neighbourhood U of z. By Proposition 3.7 the given point z_0 lies in some $f^n(U)$. Hence $f^{-n}(z_0) \cap U \neq \emptyset$. QED.

This gives us a very simple algorithm for plotting $J(f)$. One just has to start at any (non-exceptional) z_0 whatever and plot all its images under f^{-1}, then all of their images under f^{-1} etc., or alternatively plot $z_0, z_1, z_2, ...$ where each z_{j+1} is a random choice out of the (finite) set of values of $f^{-1}(z_j)$. The resulting set accumulates on the whole of $J(f)$. Even better, if one starts at a point z_0 known to be in $J(f)$ (for example a repelling fixed point) one has $J(f) = \overline{\bigcup_{n\geq 0} f^{-n}(z_0)}$, so that all points plotted are actually in the Julia set, not just accumulating there.

3.4 Julia sets and repelling periodic points

Obviously every repelling periodic point of f lies in the Julia set. However it is also true that every point of the Julia set has a periodic point arbitrarily close to it:

Theorem 3.9 *If deg$(f) \geq 2$ then $J(f)$ is contained in the closure of the set of all periodic points of f.*

Proof Let $z_0 \in J(f)$ and assume z_0 is not a critical value of f (without loss of generality, since there are only a finite set of critical values and $J(f)$ is perfect). Then z_0 has a neighbourhood U on which two distinct branches of f^{-1} are defined. Denote these by $h_1 : U \to U_1$ and $h_2 : U \to U_2$ (where the sets U_1 and U_2 are disjoint).

Suppose (for a contradiction) that U contains no periodic point of f. For each $z \in U$ set

$$g_n(z) = \frac{(f^n z - h_1 z)}{(f^n z - h_2 z)} \frac{(z - h_2 z)}{(z - h_1 z)}$$

Then for each value of n, $g_n(z) \neq 0, 1, \infty$ for $z \in U$ (else f would have a periodic point). So by Montel's Theorem the $\{g_n\}$ form a normal family. It follows (by an exercise in analysis) that the $\{f^n\}$ form a normal family, contradicting the hypothesis that $z_0 \in J(f)$. QED

Comments

1. In fact with a little more work one can establish that $J(f)$ is *equal* to the closure of the set of all *repelling* periodic points of f: this follows from Theorem 3.9 together with the observations that every repelling periodic point of f lies in the Julia set, and the result (of Fatou) that there are only finitely many non-repelling periodic orbits. What Fatou showed was that non-repelling periodic points must either have critical points in their basins of attraction or on the boundaries of their basins (we shall investigate these basins shortly). Shishikura (1987) improved Fatou's result to show that a degree d rational map has at most $2d - 2$ non-repelling periodic orbits.

2. Earlier we observed that for the map $z \to z^2$ the Julia set (the unit circle) is the closure of the set of repelling periodic points. Theorem 3.9 shows that this example typifies the general case.

3.5 The Julia set of $q_c : z \to z^2 + c$ for $|c|$ large

Lemma 3.10 *Let $|c| > 1$. Then $|q_c(z)| > |z|(|c| - 1)$ whenever $|z| \geq |c|$.*

Proof $|q_c(z)| \geq |z|^2 - |c| \geq |z|^2 - |z| = |z|(|z| - 1) \geq |z|(|c| - 1)$. QED

Thus if $|c| > 2$ the orbit $z_n = q_c^n(0)$ converges to ∞ as $n \to \infty$, since $z_1 = c$ and $|z_n| \geq |z_{n-1}|(|c| - 1)$.

Definition A *Cantor set* is a topological space homeomorphic to the space $C = \{0, 1\}^{\mathbb{N}}$ of all infinite sequences of 0's and 1's, equipped with the product topology (that is, two sequences are close if and only if they have the identical initial segments, and the longer these identical segments, the closer the points). Recall that every perfect totally disconnected compact subset of $\mathbb{R}^n$ is homeomorphic to C (an example is the Cantor set obtained by removing the open interval $(1/3, 2/3)$ from the closed unit interval on the real line, then the 'middle thirds' $(1/9, 2/9) \cup (7/9, 8/9)$ of the remaining intervals, then the middle thirds of the remaining intervals and so on).

Proposition 3.11 *For $|c|$ sufficiently large, $J(q_c)$ is homeomorphic to the Cantor set C, and the action of q_c on $J(q_c)$ is conjugate to that of the shift σ on C.*

Proof Let γ_0 be the circle $|z| = |c|$, and let $\gamma_1 = q_c^{-1}(\gamma_0)$. Then, if $|c| > 2$, γ_1 lies inside γ_0 (by Lemma 3.10) and γ_1 is a *lemniscate* (since 0 is the only critical point of q_c on $\mathbb{C}$). $q_c^{-1}(\gamma_1)$ now consists of a lemniscate inside each lobe of γ_1, and so on (Figure 6).

Let D be any disc containing γ_1 and contained in γ_0. Label the the two discs making up $q_c^{-1}(D)$ as D_0 and D_1, and label the components of $q_c^{-2}(D)$ by

$$D_{00} = D_0 \cap q_c^{-1}(D_0) \quad D_{01} = D_0 \cap q_c^{-1}(D_1) \quad D_{10} = D_1 \cap q_c^{-1}(D_0) \quad D_{11} = D_1 \cap q_c^{-1}(D_1)$$

Continue inductively, setting

$$D_{0s} = D_0 \cap q_c^{-1}(D_s) \quad D_{1s} = D_1 \cap q_c^{-1}(D_s)$$

for any finite sequence s of 0's and 1's. Set

$$\Lambda = \bigcap_{1}^{\infty} q_c^{-n}(D)$$

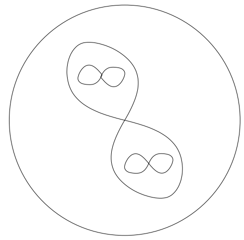

Figure 6: The circle γ_0 and its inverse images (Proposition 3.11).

To show that Λ is a Cantor set we observe that for large $|c|$ both branches of q_c^{-1} contract distances, by a definite factor $k < 1$, on both D_0 and D_1 (the details are in Comment 1 below). Λ therefore consists of points D_s, each labelled by an infinite sequence s of 0's and 1's, and the action of q_c on Λ is conjugate to the action of the shift σ on these sequences. Since Λ is a closed completely invariant set it contains $J(q_c)$; moreover since Λ contains a dense orbit (just write down an infinite sequence of 0's and 1's containing *all* finite sequences) it is a minimal closed completely invariant set and is therefore equal to $J(q_c)$. QED

Comments

1. To show that q_c^{-1} contracts on D_0 and D_1 we must show that q_c expands on their inverse images. But q_c contracts at a point z if and only if $|2z| < 1$, which is to say if and only if $|z| < 1/2$. So it will suffice to show that q_c maps the disc having centre 0, radius 1/2, to the region outside the lemniscate γ_1 (and hence outside both D_0 and D_1). But q_c maps this disc to the disc which has centre c and radius 1/4, and the largest modulus of any point of γ_1 is $|\sqrt{-2c}|$ (exercise). It follows that if c is sufficiently large (for example $|c| > 3$) the image disc lies outside γ_1 and so q_c^{-1} contracts on D_0 and D_1 as required.

2. In fact Proposition 3.11 holds whenever $q_c^n(0) \to \infty$, not just for 'large' $|c|$, but the proof requires a little more work to show that the D_s (s an infinite sequence of 0's and 1's) are points. This is best done by an argument appealing to 'moduli of annuli' (Grötzsch's inequality) or by a normal families argument applied to the branches of q_c^{-1} (see Beardon).

3. For maps in the family q_c, the Julia set $J(q_c)$ is either a Cantor set or else is connected. For if the orbit $q_c^n(0)$ does not tend to ∞ one can show that the basin of attraction of ∞ is a (topological) disc, with boundary a minimal closed completely invariant non-empty set, in other words $J(q_c)$.

4 Fatou components and linearisation theorems

4.1 Counting components

Proposition 4.1 *The Fatou set of a rational map f of degree at least two contains at most two completely invariant simply-connected components.*

Proof Any such component is homeomorphic to a disc D, and the restriction of f to D is a branched covering of degree d. Since D has Euler characteristic 1 we deduce that f has $d-1$ critical points on D (counted with multiplicity). But f has only $2d-2$ critical points. QED.

Example The Fatou set for $z \to z^2$ has exactly two such components.

Omitting the words 'completely invariant simply-connected' and just counting components, we have:

Proposition 4.2 *If $F(f)$ has more than two components, it has infinitely many components.*

Proof If $F(f)$ has only finitely many components, $D_1, ...D_k$, they must be permuted by f (since each component has image a component and inverse image a union of components). Hence there exists an m such that $g = f^m$ maps each D_j to itself. But $F(g) = F(f)$ (from the definition of a normal family) and the D_j are completely invariant for g. To apply Proposition 4.1 and complete the proof it remains to show that the D_j are simply-connected. But each D_j has boundary ∂D_j closed and completely invariant under g, and hence $\partial D_j = J(f)$. It follows that

$$\hat{\mathbb{C}} - \bar{D}_1 = \hat{\mathbb{C}} - (J(f) \cup D_1) = F(f) - D_1 = D_2 \cup ... \cup D_k$$

Hence $D_2, ...D_k$ are the components of the complement of the connected set D_1 and are therefore simply-connected. Similarly D_1 is simply-connected. QED

Examples

(i) $z \to z^2 - 1$. The basin of infinity is a completely invariant component. The components containing 0 and -1 form a periodic 2-cycle. All other components are pre-periodic (fall onto the period two cycle after a finite number of steps).

(ii) $z \to z^2 + c$ with $|c|$ large. Here $F(f)$ has a single component, the complement in $\hat{\mathbb{C}}$ of a Cantor set (but note that this component is multi-connected).

A key theorem concerning the components of $F(f)$ is:

Theorem 4.3 (Sullivan's 'No Wandering Domains Theorem' 1985) *Every component of $F(f)$ is either periodic or preperiodic*

For a proof see Sullivan (Annals 1985), or Appendix F of Milnor's book. The basic idea is that if there were a wandering domain then it would be possible to construct an infinite-dimensional family of perturbations of f, all of them rational and topologically conjugate to f, but this is impossible since f is determined by a finite set of data (as already remarked earlier). The key ingredient is the quasi-conformal deformation theory developed by Ahlfors and Bers, in particular the 'Measurable Riemann Mapping Theorem', which we may consider later in this course. The original conjecture that f could not have wandering domains was made by Fatou. Note that Theorem 4.3 is a result about *rational* maps: the Fatou set of a *transcendental* map $\mathbb{C} \to \mathbb{C}$ can have wandering components (these are known as Baker domains).

The *basin* of an attractive fixed point z_0 is the set $\{z : \lim_{n\to\infty} f^n(z) = z_0\}$ and the *immediate basin* is the component of this set containing z_0. There are similar definitions for an attracting period n cycle: here the immediate basin is the set of components of the basin containing points of the cycle.

Theorem 4.4 *The immediate basin of an attractive periodic point (for a rational map f of degree at least two) contains a critical point.*

Proof Without loss of generality we suppose z_0 to be an attracting *fixed point.* If z_0 is superattracting, the result is obvious. if z_0 is attracting but not superattracting then there is a neighbourhood U of z_0 such that

$f(U) \subset U$ and $f|_U$ is injective. Let $V = f(U)$ and consider the branch of f^{-1} sending V to U. If f has no *critical value* in U, this branch can be extended to the whole of U and hence f^{-2} has a well-defined branch on V. Repeat. If some $f^{-n}(V)$ contains a critical value then the basin contains a critical point. but if not, then $\{f^{-n}\}_{n>0}$ are all defined on V and have images in the the immediate basin. But then they would form an equicontinuous family (by Montel) and this is impossible since z_0 is a *repelling* fixed point for f^{-1}. QED

Corollary 4.5 *If f has degree d then it has at most $2d-2$ attracting or superattracting cycles.*

Shishikura (1987) improved this bound to 'at most $2d-2$ non-repelling cycles'.

4.2 Linearisation Theorems

Dynamics of f near a fixed or periodic point

In the neighbourhood of a fixed point, which without loss of generality we take to be 0, $f(z) = \lambda z + O(z^2)$ (Taylor series), where λ is the multiplier at the fixed point. We say that f is *linearisable* if there is a neighbourhood U on which f is *conjugate* to $z \to \lambda z$ (by a complex analytic conjugacy).

Theorem 4.6 (Koenigs' Linearization Theorem 1884) *If $\lambda \neq 0$ and $|\lambda| \neq 1$ then f is linearizable*

Proof Assume first that $0 < |\lambda| < 1$. Set

$$h_n(z) = \frac{1}{\lambda^n} f^n(z)$$

Then, by construction $h_n f(z) = \lambda h_{n+1}(z)$, and it suffices to show that the $\{h_n\}$ converge locally uniformly to a function h, since then $hf = \lambda h$. See the example below for a sketch proof in the case of a particular example, and Milnor (Theorem 8.2) for the general case, which proceeds along the same lines.

For the case $1 < |\lambda| < \infty$ one can proceed in exactly the same fashion, but with f^{-1} in place of f. QED.

Example

$f(z) = \lambda z + z^2$ (where $|\lambda| < 1$). Here the orbit of any initial point z_0 is

$z_1 = f(z_0) = \lambda z_0(1 + z_0/\lambda)$

$z_2 = f(z_1) = \lambda z_1(1 + z_1/\lambda) = \lambda^2 z_0(1 + z_0/\lambda)(1 + z_1/\lambda)$

...

$z_n = \lambda^n z_0(1 + z_0/\lambda)(1 + z_1/\lambda)...(1 + z_{n-1}/\lambda)$

Thus $h_n(z_0) = z_0(1 + z_0/\lambda)(1 + z_1/\lambda)...(1 + z_{n-1}/\lambda)$ where $\{z_n\}$ is the orbit of z_0. As n tends to infinity, z_n tends to 0, and $\{h_n\}$ converge locally uniformly to

$$h(z_0) = z_0 \prod_0^\infty (1 + \frac{z_n}{\lambda})$$

Observe that we have used the dynamics to construct an explicit conjugacy: essentially we have followed an orbit in to very close to the attracting fixed point, and then used the fact that very close the fixed point the map f is very close to $z \to \lambda z$. One can also construct the coefficients of h recursively, directly from the functional equation $hf(z) = \lambda h(z)$, but the dynamical motivation is then no longer so apparent.

Theorem 4.7 (Böttcher 1904) *If $f(z) = z^k + O(z^{k+1})$ ($k \geq 2$ integer) then f is conjugate to $z \to z^k$ on a neighbourhood of 0.*

Proof Analogously to 4.6, we set $h_n(z) = (f^n(z))^{1/k^n}$. Then $h_n f(z) = (h_{n+1}(z))^k$ and the $\{h_n\}$ converge locally uniformly to a function h conjugating f to $z \to z^k$. QED

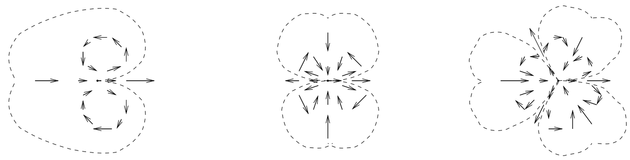

Figure 7: Dynamics of $z \to z + z^{n+1}$ for $n = 1$, $n = 2$ and $n = 3$.

The proof above is only a sketch. See Milnor (Theorem 9.1) for details. The right choice of branch of k^nth root in the definition of h_n is important, but rather than fill in the details in general, we consider an example, one that will also be useful later.

Example Consider $f : z \to z^2 + c$ near the fixed point ∞.

Write this map as $z \to z^2(1 + c/z^2)$.

$z_1 = f(z_0) = z_0^2(1 + c/z_0^2)$

$z_2 = f(z_1) = z_1^2(1 + c/z_1^2) = z_0^4(1 + c/z_0^2)^2(1 + c/z_1^2)$

...

$z_n = z_{n-1}^2(1 + c/z_{n-1}^2) = z_0^{2^n}(1 + c/z_0^2)^{2^{n-1}}(1 + c/z_1^2)^{2^{n-2}}...(1 + c/z_{n-1}^2)$

So $h_n(z_0) = z_0(1 + c/z_0^2)^{1/2}(1 + c/z_1^2)^{1/4}...(1 + c/z_{n-1}^2)^{1/2^n}$ where the choice of each root is the obvious one coming from the binomial expansion. As n tends to ∞ the z_n tend to ∞ (since z_0 is outside the filled Julia set). Thus the h_n converge (locally uniformly) to

$$h(z_0) = z_0 \prod_0^\infty (1 + \frac{c}{z_n^2})^{1/2^{n+1}}$$

Once again one could compute explicit formulae for the coefficients of h using recursion relations based on the functional equation, but they are far less revealing than the dynamical approach above.

We shall come back to this example when we look at the Mandelbrot set later. Meanwhile, what can be said about linearisability near a *neutral* fixed point ?

Suppose $f(z) = \lambda z + O(z^2)$, with $|\lambda| = 1$.

Case 1: $\lambda = e^{2\pi i p/q}$ (in this case $z = 0$ is called a *parabolic* fixed point).

Example $f(z) = z + z^{n+1}$

See Figure 7 for ths example in the cases $n = 1$, $n = 2$ and $n = 3$. The 'attracting petals' bounded by dashed lines are mapped into themselves and each initial point in a petal has orbit which eventually converges to to the fixed point along a direction tangent to the mid-line of the petal. The Julia set (not marked) heads off from the fixed point in directions tangent to the repelling axes (between the petals).

A rational map f is not linearizable around a parabolic fixed point (unless $f(z) = \lambda z$), since $f^q \neq identity$. But by analysing the local power series expansion of $f(z)$ it can be shown that the parabolic point itself lies in the Julia set and its the basin of attraction lies in the Fatou set (See Milnor, Lemma 10.5). It can easily be proved (via Montel) that this basin of attraction must contain a critical point. Similar considerations apply to a parabolic cycle.

The local dynamics around a parabolic fixed point (or cycle) has a very particular topological dynamics, that of a *Leau-Fatou flower*, with 'attracting petals' contained within the Fatou set, as illustrated in the examples above. For $\lambda = e^{2\pi 1 p/q}$ this flower has kq petals, where $k \geq 1$ (see, for example Milnor, Theorem 10.7). The study of holomorphic germs around parabolic points and cycles contains deep and interesting results: Chapter 10 of Milnor's book is an excellent starting point to learn more about this.

Case 2: $\lambda = e^{2\pi i\alpha}$ with α irrational.

Here it all depend on 'how irrational α is'. Write α as a continued fraction

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + ...}} = [a_0, a_1, a_2, ...]$$

and let p_n/q_n (in lowest terms) be the value of its nth truncation $[a_0, a_1, ..., a_n]$.

For example the golden mean $[0, 1, 1, 1, 1, ...]$ has $p_1/q_1 = 1/1, p_2/q_2 = 1/2, p_3/q_3 = 2/3, p_4/q_4 = 3/5, ...$

Definition α satisfies the *Brjuno condition* if and only if

$$\sum_1^\infty \frac{log(q_{n+1})}{q_n} < \infty$$

We write $\mathcal{B}$ for the set of real numbers satisfing the Brjuno condition.

Theorem 4.9 (Brjuno, 1965) $\alpha \in \mathcal{B} \Rightarrow$ *all complex analytic maps* $z \to e^{2\pi i\alpha}z + O(z^2)$ *are linearisable.*

Theorem 4.10 (Yoccoz, 1988) $\alpha \notin \mathcal{B} \Rightarrow z \to e^{2\pi i\alpha}z + z^2$ *is not linearisable.*

When a linearisation exists its domain is known as a *Siegel disc.*

Notes

1. Yoccoz's proof of the *necessity* of the Brjuno condition is motivated by ideas of *renormalization.*

2. The Siegel disc around a linearizable irrational neutral fixed point is in the Fatou set $F(f)$. It can be shown the Siegel discs 'use up' critical points in the sense that the boundary of a Siegel disc necessarily lies in the accumulation set of the forward orbit of some critical point.

3. The irrational neutral points which are not linearizable are known as *Cremer points* (after Cremer 1928). They lie in $J(f)$ and the dynamics around them is complicated. In the 1990s Perez-Marco introduced invariant structures he called 'hedgehogs' and showed they they exist at all Cremer points. These are the subject of continuing research.

4.3 The classification of types of Fatou component

Sullivan's proof of the 'No Wandering Domains Theorem' has the consequence that for a polynomial the only possible components of a Fatou set are components of the basin of:

1. a superattracting periodic orbit;

2. an attracting periodic orbit;

3. a rational neutral periodic orbit;

4. a periodic cycle of Siegel discs.

There is one other type that can ocur for rational f (but not polynomial f), components of the basin of:

5. a periodic cycle of *Herman rings.* (A Herman ring is an annulus with dynamics conjugate to an irrational rotation.)

For a proof of this classification see for example Milnor's book (Chapter 16) or the original paper of Sullivan in 1985.

These are the 5 types of 'regular behaviour' of a rational map. To completely understand rational maps we have to understand how they fit together with each other, and with the behaviour on the complement of the regular domain, the Julia set. As we shall see, there are still unanswered questions even in the simplest case, that of quadratic maps $z \to z^2 + c$.

5 Hyperbolic 3-space and Kleinian groups

Definition $\mathcal{H}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$

Just as in the two-dimensional case we may define an infinitesimal metric:

$$ds = \frac{1}{x_3}((dx_1)^2 + (dx_2)^2 + (dx_3)^2)^{1/2}$$

With this metric $\mathcal{H}^3_+$ becomes the *upper half-space model of hyperbolic* 3*-space*. The geodesics are the semicircles in $\mathcal{H}^3$ orthogonal to the plane $x_3 = 0$.

Now think of the plane $x_3 = 0$ in $\mathbb{R}^3$ as the complex plane $\mathbb{C}$ ($(x_1, x_2, 0) \leftrightarrow x_1 + ix_2$), add the point '$\infty$', and think of $\hat{\mathbb{C}}$ as the *boundary* of $\mathcal{H}^3_+$. Every fractional linear map

$$\alpha : z \to \frac{az + b}{cz + d} \quad (a, b, c, d \in \mathbb{C}, ad - bc \neq 0)$$

mapping $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$, has an extension to an isometry from $\mathcal{H}^3_+$ to $\mathcal{H}^3_+$. One way to see this is to break down α into a composition of maps of the form

$$(i) \;\; z \to z + \lambda \quad (\lambda \in \mathbb{C}) \qquad (ii) \;\; z \to \lambda z \quad (\lambda \in \mathbb{C}) \qquad (iii) \;\; z \to -1/z$$

We extend these as follows on $\mathcal{H}^3_+$ (where z denotes $x_1 + ix_2$):

$$(i) \;\; (z, x_3) \to (z + \lambda, x_3) \qquad (ii) \;\; (z, x_3) \to (\lambda z, |\lambda| x_3) \qquad (iii) \;\; (z, x_3) \to \left(\frac{-\bar{z}}{|z|^2 + x_3^2}, \frac{x_3}{|z|^2 + x_3^2} \right)$$

The expressions above come from decomposing the action on $\hat{\mathbb{C}}$ of each of the elements of $PSL(2, \mathbb{C})$ in question into two *inversions* (reflections) in circles in $\hat{\mathbb{C}}$. Each such inversion has a unique extension to $\mathcal{H}^3_+$ as an inversion in the hemisphere spanned by the circle and composing appropriate pairs of inversions gives us these formulae. It is now an exercise along the lines of Proposition 2.12 to show that $PSL(2, \mathbb{C})$ preserves the metric ds on $\mathcal{H}^3$ and another exercise, along the lines of Proposition 2.13 to show that the geodesics are the arcs of semicircles as claimed. Moreover every isometry of $\mathcal{H}^3$ can be seen to be the extension of a conformal map of $\hat{\mathbb{C}}$ to itself, since it must send hemispheres orthogonal to $\hat{\mathbb{C}}$ to hemispheres orthogonal to $\hat{\mathbb{C}}$, hence circles in $\hat{\mathbb{C}}$ to circles in $\hat{\mathbb{C}}$. Thus all orientation-preserving isometries of $\mathcal{H}^3$ are given by elements of $PSL(2, \mathbb{C})$ acting as above, and all orientation-reversing isometries are extensions of anti-holomorphic Möbius transformations of $\hat{\mathbb{C}}$.

Comments

1. The fact that the orientation-preserving isometry group of $\mathcal{H}^3_+$ is $PSL(2, \mathbb{C})$ was first observed by Poincaré.

2. To verify that the extension of the action of $PSL(2, \mathbb{C})$ from $\hat{\mathbb{C}}$ to $\mathcal{H}^3_+$ is well-defined we should check that when we decompose an element of $PSL(2, \mathbb{C})$ into a product in different ways we get the same extension to $\mathcal{H}^3_+$. We can avoid this problem by writing down a single formula for the action of an element of $PSL(2, \mathbb{C})$ in terms of *quaternions*. (Regard $\mathbb{R}^3_+$ as quaternions of the form $x + yi + tj (+0k)$ with $t > 0$: see Exercise Sheet 3.)

3. In practice we may do many of our computations in $\mathcal{H}^3_+$ by taking a hyperplane 'slice' that looks like $\mathcal{H}^2_+$: given any two points P and Q in $\mathcal{H}^3_+$, the plane through these points orthogonal to the boundary $\hat{\mathbb{C}}$ of upper half-space is a copy of $\mathcal{H}^2$, and so $d(P, Q) = |\ln(P, Q; A, B)|$ where A and B are the endpoints of the semicircle through P and Q orthogonal to $\hat{\mathbb{C}}$.

4. The *disc model* for hyperbolic three-space is the interior $\mathbb{D}^3$ of the unit disc in Euclidean three-space $\mathbb{R}^3$, equipped with the metric

$$ds = \frac{((dx_1)^2 + (dx_2)^2 + (dx_3)^2))^{1/2}}{1 - r^2}$$

(where $r^2 = x_1^2 + x_2^2 + x_3^2$). Geodesics are arcs of circles orthogonal to the boundary sphere S^2 of the disc.

5. One can construct higher dimensional hyperbolic spaces $\mathcal{H}^n_+$ in the analagous way. In each case the *conformal* transformations of the boundary extend uniquely to give the *isometries* of the interior.

5.1 Types of isometries of hyperbolic 3-space

Non-identity elements $\alpha \in PSL(2, \mathbb{C})$ are of four types.

Definition α is said to be

elliptic $\Leftrightarrow$ α fixes some geodesic in $\mathcal{H}^3_+$ pointwise;

parabolic $\Leftrightarrow$ α has a single fixed point in $\hat{\mathbb{C}}$;

hyperbolic $\Leftrightarrow$ α has two fixed points in $\hat{\mathbb{C}}$, no fixed points in $\mathcal{H}^3_+$, and every hyperplane in $\mathcal{H}^3_+$ which contains the geodesic joining the two fixed points in $\hat{\mathbb{C}}$ is invariant (mapped to itself) under α;

loxodromic $\Leftrightarrow$ α has two fixed points in $\hat{\mathbb{C}}$, no fixed points in $\mathcal{H}^3_+$, and no invariant hyperplane in $\mathcal{H}^3_+$.

Note The distinction between *hyperbolic* and *loxodromic* is not always made: some authors use either word for an isometry having two fixed points in $\hat{\mathbb{C}}$ and none in $\mathcal{H}^3_+$.

Lemma 5.1 α *is elliptic/parabolic/hyperbolic/loxodromic*

$\Leftrightarrow (tr(\alpha))^2 \in [0, 4) \subset \mathbb{R}^{\geq 0} \;/\; = 4 \;/\; \in \mathbb{R}^{\geq 0} - [0, 4] \;/\; \in \mathbb{C} - \mathbb{R}^{\geq 0}$ *(where α has been normalised to have det $= 1$).*

Proof

If α has two fixed points in $\hat{\mathbb{C}}$ we may assume (after conjugating α by an appropriate Möbius transformation) they are at 0 and ∞ and that α has the form $z \to \lambda z$ (and $tr(\alpha) = \lambda^{1/2} + \lambda^{-1/2}$).

Case 1: $|\lambda| = 1$, say $\lambda = e^{i\theta}$. Then on $\hat{\mathbb{C}}$ α is a rotation about 0 through an angle θ, and fixes the x_3-axis in $\mathcal{H}^3_+$ pointwise. As a matrix, normalised to determinant 1,

$$\alpha = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

and so $(tr(\alpha))^2 = 4\cos^2(\theta/2) \in [0, 4)$.

Case 2: $|\lambda| \neq 1$. then α acts on the x_3-axis in $\mathcal{H}^3_+$ as multiplication by $|\lambda|$. Writing $\lambda = |\lambda|e^{i\theta}$ we have

$$\alpha = \begin{pmatrix} |\lambda|^{1/2}e^{i\theta/2} & 0 \\ 0 & |\lambda|^{-1/2}e^{-i\theta/2} \end{pmatrix}$$

so $(tr(\alpha))^2 \in \mathbb{C} - [0, 4]$. Now if λ is real (i.e. $\theta = 0$ or π) α is hyperbolic and $(tr(\alpha))^2 \in \mathbb{R}^{\geq 0} - [0, 4]$ and if λ is not real, α is loxodromic and $(tr(\alpha))^2 \in \mathbb{C} - \mathbb{R}^{\geq 0}$.

Finally if α has a single fixed point in $\hat{\mathbb{C}}$ then we can place this fixed point at ∞ (by conjugating α if necessary) in which case α has the form $z \to z + \lambda$ (indeed we may even conjugate it to $z \to z + 1$). Then α is parabolic and $(tr(\alpha))^2 = 4$. QED.

Dynamics of Möbius transformations on $\mathcal{H}^3_+ \cup \hat{\mathbb{C}}$

In the first example in Figure 8 the fixed points $0, \infty$ on $\hat{\mathbb{C}}$ are *neutral*. For $z \to e^{i\theta}z$ with θ real, all orbits on $\mathcal{H}^3_+$ have finite period if θ is a rational multiple of π, and densely fill circles around the x_3 axis if not.

In the second example all orbits in $\mathcal{H}^3_+$ head away from a repelling fixed point 0 and towards an attracting fixed point ∞, spiralling around the x_3 axis as they go. We have this behaviour in general for $z \to ke^{i\theta}z$ (k real > 1) but the nature of the spiralling depends on θ: in particular if $\theta = 0$ or π each orbit remains in a hyperplane.

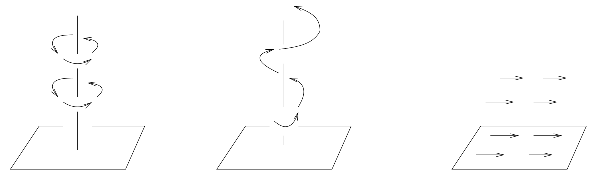

Figure 8: Dynamics of (i) $z \to e^{2\pi i/3}z$ (ii) $z \to 2e^{2\pi i/3}z$ (iii) $z \to z+1$

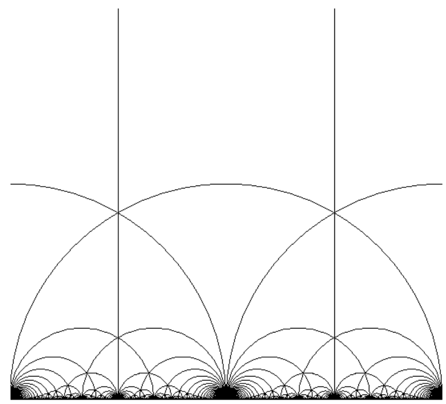

Figure 9: The modular group action on the upper half-plane

In the third example the (unique) fixed point ∞ is neutral (multiplier 1) and all orbits on $\mathcal{H}^3_+$ head towards the fixed point under both forward and backward time. Any parabolic map α will have this behaviour.

5.2 The ordinary set of a Kleinian group

Definition A *Kleinian group* is a *discrete* subgroup $G < PSL(2,\mathbb{C})$.

Thus for a subgroup $G < PSL(2,\mathbb{C})$ to be called Kleinian we require that there be no sequence $\{g_n\}$ of distinct elements of G tending to a limit $g \in PSL(2,\mathbb{C})$. Here the topology on $PSL(2,\mathbb{C})$ is that induced by the norm

$$\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\| = \sqrt{|a|^2+|b|^2+|c|^2+|d|^2}$$

on $SL(2,\mathbb{C})$ (so that two elements of $PSL(2,\mathbb{C})$ are close together if and only if they are representable by $A_1, A_2 \in SL(2,\mathbb{C})$ with $||A_2 - A_1||$ small).

Note If G is discrete then for any $N > 0$ the number of elements of G having norm $\leq N$ is *finite*, since every infinite sequence with bounded norm has a convergent subsequence. Hence every discrete G is *countable*.

Definition The action of G is *discontinuous* at $z \in \hat{\mathbb{C}}$ if there exists a neighbourhood U of z such that $g(U) \cap U = \emptyset$ for all but finitely many $g \in G$.

Example (See Week 3 Exercises) $G = PSL(2,\mathbb{Z})$ acts discontinuously on $\hat{\mathbb{C}} - \hat{\mathbb{R}}$. For z in the region $\Delta = \{z : |z| \leq 1, Re(z) \leq 1/2, Im(z) > 0\}$ (Figure 9) each $z \neq i, \pm 1/2 + i\sqrt{3}/2$ has a neighbourhood U such that $g(U) \cap U = \emptyset$ for all non-identity $g \in G$, the point $z = i$ has a neighbourhood U such that $g(U) \cap U = \emptyset$ for all $g \in G - \{I, S\}$ where $S : z \to -1/z$, and the point $z = -1/2 + i\sqrt{3}/2$ has a neighbourhood U such that $g(U) \cap U = \emptyset$ for all $g \in G - \{I, ST, (ST)^2\}$ where $ST : z \to -1/(z+1)$, etc.

Definition The set of all $z \in \hat{\mathbb{C}}$ at which the action of G is discontinuous is called the *ordinary* (or *discontinuity* or *regular*) set $\Omega(G)$.

Comments

1. It follows at once from the definition that $\Omega(G)$ is *open* and G*-invariant.*

2. In the example in figure 9 observe that the origin 0 is not in $\Omega(G)$, since any U containing 0 has $g(U) \cap U \neq \emptyset$ for all

$$g = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

with n sufficiently large. In fact in this example $\Omega(G) = \hat{\mathbb{C}} - \hat{\mathbb{R}}$.

5.3 The action of a Kleinian group on $\mathcal{H}^3_+$

We next consider the action of a subgroup $G < PSL(2, \mathbb{C})$ on $\mathcal{H}^3_+$ rather than just on its boundary $\hat{\mathbb{C}}$.

Theorem 5.1 *A subgroup $G < PSL(2, \mathbb{C})$ is discrete if and only if it acts discontinuously on $\mathcal{H}^3_+$.*

Proof. If G is not discrete there exists $\{g_n\} \in G$ with limit $g \in PSL(2, \mathbb{C})$. So for all $x \in \mathcal{H}^3_+$, $g_m^{-1} g_n(x) \to x$ as $m, n \to \infty$. Thus for any $x \in \mathcal{H}^3_+$ and neighbourhood U of x, for m and n sufficiently large $g_m^{-1} g_n(U) \cap U \neq \emptyset$. Hence G does not act discontinuously at x.

Conversely, if G does not act discontinuously at $x \in \mathcal{H}^3_+$, then for any neighbourhood U of x there exist a sequence $\{x_n\} \in U$ and (distinct) $g_n \in G$ such that each $g_n(x_n) \in U$. Take U compact. Then by passing to subsequences we may assume the x_n tend to a point y and the $g_n(x_n)$ tend to a point z (with both y and z in U). Now let k be an isometry of $\mathcal{H}^3_+$ having $k(z) = y$ and let $\{h_n\}$, $\{j_n\}$ be sequences of isometries, both tending to the identity, and having $h_n(y) = x_n$ and $j_n g_n(x_n) = z$ respectively. Consider $f_n = k j_n g_n h_n$. For each n this fixes y (by construction). But the isometries of $\mathcal{H}^3_+$ fixing a common point of $\mathcal{H}^3_+$ are a compact group (the Euclidean rotations, in the Poincaré ball model with the common point the origin). Hence the $\{f_n\}$ have a convergent subsequence. Hence so do the $\{g_n\}$, in other words G is not discrete. QED

5.4 Limit sets of Kleinian groups

One can define the notion of the *limit set* $\Lambda(G)$ of a Kleinian group G, either in terms of its action on $\mathcal{H}^3_+$, or in terms of the action on the boundary $\hat{\mathbb{C}}$ of $\mathcal{H}^3_+$. We shall see later that the two definitions are equivalent.

Definition 1. Let x be any point of $\mathcal{H}^3_+$. Then set

$$\Lambda(x) = \{w \in \hat{\mathbb{C}} : \exists g_n \in G \quad \text{with} \quad g_n(x) \to w \quad \text{as} \quad n \to \infty\}$$

(where convergence is taken in the Euclidean metric on the Poincaré disc model of $\mathcal{H}^3_+$). Note that the $\{g_n(x)\}$ cannot have accumulation points in $\mathcal{H}^3_+$, since G acts discontinuously there. Thus an alternative description of $\Lambda(x)$ is as the accumulation set in $\mathcal{H}^3_+ \cup \hat{\mathbb{C}}$ of the orbit Gx on $\mathcal{H}^3_+$. This accumulation set is independent of the initial point $x \in \mathcal{H}^3_+$, since if we choose another initial point y the hyperbolic distance from $g(x)$ to $g(y)$ is constant for all g and therefore the *Euclidean* distance from $g(x)$ to $g(y)$ tends to zero as $g(x)$ and $g(y)$ approach the boundary $\hat{\mathbb{C}}$ of the Poincaré disc. We *define* $\Lambda(G)$ to be $\Lambda(x)$ for any $x \in \mathcal{H}^3_+$.

Definition 2. Let z be any point of $\hat{\mathbb{C}}$. Set

$$\Lambda(z) = \{w \in \hat{\mathbb{C}} : \exists g_n \in G \quad \text{with} \quad g_n(z) \to w \quad \text{as} \quad n \to \infty\}$$

(where convergence is taken in the spherical metric on $\hat{\mathbb{C}}$). It can be shown that when G is *non-elementary* (see below for definition) $\Lambda(z)$ is independent of $z \in \hat{\mathbb{C}}$. We define $\Lambda(G)$ to be $\Lambda(z)$ for any $z \in \hat{\mathbb{C}}$.

Comments

1. The restriction that G be 'non-elementary' is included in definition 2 in order to exclude just one class of examples where the limit $\Lambda(z)$ depends on z. Consider $G = \{g^n : n \in \mathbb{Z}\}$, where g is loxodromic, with fixed points z_0 and z_1. The limit set by definition 1 is $\Lambda(G) = \{z_0\} \cup \{z_1\}$, but definition 2 gives $\Lambda(z_0) = z_0$, $\Lambda(z_1) = z_1$ (although $\Lambda(z) = \{z_0\} \cup \{z_1\}$ for any other choice of z).

2. We shall adopt definition 2 until we have proved the equivalence of the two notions (later in this section). Meanwhile we remark that the underlying reason that the definitions are equivalent is that to an observer inside $\mathcal{H}^3_+$ an orbit of G of $\mathcal{H}^3_+$ is viewed as accumulating at $\Lambda(G)$ on the 'visual sphere' $\hat{\mathbb{C}}$.

3. A third equivalent definition is that $\Lambda(G)$ consists of the points $z \in \hat{\mathbb{C}}$ where the family $g \in G$ fail to be a normal family (with respect, as always, to the spherical metric). We shall prove also later.

4. It follows at once from definition 2 (or indeed from definition 1) that $\Lambda(G)$ is both *closed* and G-*invariant*.

It is clear from the definitions of $\Omega(G)$ and $\Lambda(G)$ that $\Omega(G) \cap \Lambda(G) = \emptyset$, but we shall prove the stronger statement that $\Lambda(G)$ is the *complement* of $\Omega(G)$ in $\hat{\mathbb{C}}$. First we deal with some special cases.

5.5 Elementary Kleinian groups

Definition A Kleinian group G is called *elementary* if there exists a finite G orbit on either $\mathcal{H}^3_+$ or $\hat{\mathbb{C}}$.

All elementary Kleinian groups G belong to the following three classes. For a proof see for example Beardon's book 'Geometry of Discrete Groups' or Ratcliffe's book 'Foundations of Hyperbolic Manifolds.'

(i) G is conjugate to a finite subgroup of $SO(3)$ acting on the Poincaré disc by rigid rotations fixing the origin (for example the symmetry group of a regular solid). In this case $\Lambda(G) = \emptyset$.

(ii) G is conjugate to a discrete group of Euclidean motions of $\mathbb{C}$ (i.e. fixing $\infty \in \hat{\mathbb{C}}$). (For example the group generated by $z \to z + 1$ and $z \to z + i$). Then $|\Lambda(G)| = 1$.

(iii) G is conjugate to a group in which all elements are of the form $z \to kz$ or $z \to k/z$ for $k \in \mathbb{C}$. Then $|\Lambda(G)| = 2$.

It is not hard to see that if G is Kleinian then $\Lambda(G) = \emptyset \Rightarrow G$ elementary of type (i), $|\Lambda(G)| = 1 \Rightarrow G$ elementary of type (ii), and $|\Lambda(G)| = 2 \Rightarrow G$ elementary of type (iii), so elementary groups are characterised by the size of their limit sets. Indeed

Proposition 5.2 *A Kleinian group G is elementary if and only $|\Lambda(G)| \leq 2$, and non-elementary if and only if $\Lambda(G)$ is infinite.*

Proof. If $\Lambda(G)$ is finite and non-empty then any G orbit in $\Lambda(G)$ is a finite G orbit on $\hat{\mathbb{C}}$ so G is elementary by definition and has $|\Lambda(G)| = 1$ or 2 by the above classification. QED

5.6 Properties of ordinary and limit sets

Theorem 5.3 *Every Kleinian group G acts discontinuously on $\hat{\mathbb{C}} - \Lambda(G)$. Hence $\hat{\mathbb{C}}$ is the disjoint union of $\Omega(G)$ and $\Lambda(G)$.*

Proof. *(Outline.)* For groups G with $|\Lambda(G)| = 0, 1$ the result can be verified by checking the corresponding types of elementary Kleinian groups, so we may assume $|\Lambda(G)| \geq 2$. Now let $C(G)$ be the *convex hull* of $\Lambda(G)$ in $\mathcal{H}^3_+ \cup \hat{\mathbb{C}}$, i.e. the space obtained by joining every point of $\Lambda(G)$ to every other point of $\Lambda(G)$ by a geodesic in $\mathcal{H}^3_+$ and then 'filling in the interior' to obtain a convex set in $\mathcal{H}^3_+ \cup \hat{\mathbb{C}}$. (An equivalent definition of $C(G)$ is that it is the space obtained from $\mathcal{H}^3_+$ by 'scooping out' every open half 3-ball bounded by a round 2-disc contained in $\hat{\mathbb{C}} - \Lambda(G)$). The set $C(G)$ is closed and G-invariant, since $\Lambda(G)$ is. There is a uniquely defined retraction map

$$\rho : \mathcal{H}^3_+ \cup (\hat{\mathbb{C}} - \Lambda(G)) \to C(G)$$

sending each point of $\mathcal{H}^3$ to the nearest point of $C(G)$ (in the hyperbolic metric). This map ρ is continuous and commutes with the action of G. Now let z be any point of $\hat{\mathbb{C}} - \Lambda(G)$ and $U \subset \hat{\mathbb{C}} - \Lambda(G)$ be a neighbourhood of z. Then $\rho(U)$ is contained in a neighbourhood V of $\rho(z)$, and by taking U small (in the spherical metric) we can take V as small as we please (in the hyperbolic metric). But now since the action of G is discontinuous (by Theorem 5.1) V meets $g(V)$ for at most finitely many $g \in G$. Hence $g\rho(U)$ meets $\rho(U)$ for at most finitely many $g \in G$, and so $g(U)$ meets U for at most finitely many $g \in G$, in other words $z \in \Omega(G)$. QED

Proposition 5.4 *Let G be a non-elementary Kleinian group. Then any non-empty closed G-invariant subset S of $\hat{\mathbb{C}}$ contains $\Lambda(G)$*

Proof. Let z be any point of S having an infinite orbit under G. Since S is G-invariant it contains the orbit Gz, and since S is closed it contains the accumulation set of Gz. But this accumulation set is $\Lambda(G)$. QED

Corollary 5.5 *Let G be a Kleinian group. Then either $\Lambda(G) = \hat{\mathbb{C}}$ or $\Lambda(G)$ has empty interior.*

Proof. In the elementary case $\Lambda(G)$ has empty interior. In the non-elementary case apply Proposition 5.4 to $\hat{\mathbb{C}} - int\Lambda(G)$. QED

Corollary 5.6 *Let G be a non-elementary Kleinian group. Then $\Lambda(G)$ is the closure of the set of all fixed points of loxodromic and hyperbolic elements of G.*

Proof. If $z \in \hat{\mathbb{C}}$ is a fixed point of a hyperbolic or loxodromic element $g \in G$ then z lies in $\Lambda(G)$ by definition 2. For the converse we remark that the set of fixed points of loxodromic and hyperbolic elements of a non-elementary group is non-empty (by a standard exercise) and is G-invariant since if z is fixed by g, then hz is fixed by hgh^{-1}. The result now follows by Proposition 5.4. QED

Comment. If G has any parabolic elements their fixed points must lie in $\Lambda(G)$, but elliptic elements may have fixed points in either $\Omega(G)$ or $\Lambda(G)$.

Corollary 5.7 *Let G be a non-elementary Kleinian group. Then $\Lambda(G)$ is perfect (and hence, in particular, uncountable).*

Proof. The set of accumulation points of $\Lambda(G)$ is closed and G-invariant. Now apply Proposition 5.4. QED

Corollary 5.8 *Definitions 1 and 2 for the limit set $\Lambda(G)$ of a non-elementary Kleinian group G are equivalent.*

Proof. We show that the limit set as defined by definition 1 has exactly the same characterising property as that specified by Proposition 5.4 for $\Lambda(G)$ (where we used definition 2). Let S be any closed G-invariant subset of $\hat{\mathbb{C}}$ (note that S must be infinite, since G is non-elementary). Then $C(S)$, the convex hull of S in $\mathcal{H}^3_+ \cup \hat{\mathbb{C}}$, is also closed and G-invariant. Take any $x \in C(S) \cap \mathcal{H}^3_+$. Its orbit Gx is contained in $C(S)$ and the accumulation set of this orbit is contained in $C(S) \cap \hat{\mathbb{C}} = S$. Hence S contains the definition 1 limit set of G. QED

5.7 Comparison with Fatou and Julia sets

The results we have proved so far for regular and limit sets for Kleinian groups exhibit a very close analogy with our earlier results on Fatou and Julia sets for rational maps. This raises the question as to whether we can make the *definitions* analogous too. The answer is yes.

Proposition 5.9 *Let G be a Kleinian group. Then $\Omega(G)$ is the largest open subset of $\hat{\mathbb{C}}$ on which the elements of G form an equicontinuous family.*

Proof. Assume G non-elementary (as usual elementary groups can be dealt with on a case by case basis). Then $\Lambda(G)$ contains at least three points (in fact infinitely many) so $\Omega(G)$ is contained in the equicontinuity set by Montel's Theorem. But given any $z \in \Lambda(G)$, by Corollary 5.6 there must be a repelling fixed point of some $g \in G$ arbitrarily close to z, so the family of maps G cannot be equicontinuous at z. QED

We deduce the following two consequences (useful for plotting $\Lambda(G)$).

Theorem 5.10 *Let G be a non-elementary Kleinian group, and U be any open subset of $\hat{\mathbb{C}}$ meeting $\Lambda(G)$. Then*

$$\bigcup_{g\in G} gU = \hat{\mathbb{C}}$$

Proof. The union $\bigcup_{g\in G} gU$ covers all of $\hat{\mathbb{C}}$ except at most two points (else the family G would be equicontinuous on U by Montel's Theorem). But the complement of this union is a finite G-invariant set and therefore empty (since G is non-elementary). QED

The following corollary is immediate.

Corollary 5.11 *Let G be a non-elementary Kleinian group, and U be any open subset of $\hat{\mathbb{C}}$ meeting $\Lambda(G)$. Then*

$$\bigcup_{g\in G} g(U \cap \Lambda(G)) = \Lambda(G)$$

Comments

1. A discrete subgroup of $PSL(2,\mathbb{R})$ is called *Fuchsian*. All our results for Kleinian groups in this chapter have obvious specialisations to the Fuchsian case, with $\mathcal{H}^3_+$ replaced by $\mathcal{H}^2_+$, and $\hat{\mathbb{C}}$ replaced by $\hat{\mathbb{R}}$.

2. 'Sullivan's Dictionary' is a continually evolving correspondence between definitions, conjectures and theorems in the realm of iterated rational maps and definitions, conjectures and theorems in the realm of Kleinian groups. Some entries are obvious, e.g. Julia set $\leftrightarrow$ limit set, but not everything works in exactly the same way in the two areas, for example:

Ahlfors $0-1$ **Conjecture**, formulated by Ahlfors in the 1960s and proved by him for *geometrically finite* Kleinian groups, states in its most general form that for any finitely generated Kleinian group G either $\Lambda(G) = \hat{\mathbb{C}}$ or $\Lambda(G)$ has 2-dimensional Lebesgue measure zero. This was finally proved in 2004 as a consequence of work by many authors (see Marden, Theorem 5.6.6).

Fatou's Question. Can the Julia set of a polynomial have positive 2-dimensional Lebesgue measure? This question was finally answered in 2005 by Xavier Buff and Arnaud Chéritat, who proved that there exist quadratic polynomials, $z \to z^2 + c$, with positive area Julia sets. The proof is very technical, but see their paper at the 2010 Intenational Congress of Mathematics in Hyderabad for an overview of their method.

I don't know that the current contents of this dictionary are all written down in one place, but see Chapter 5 of the book by S.Morosawa, Y.Nishimura, M.Taniguchi and T.Ueda for the situation in 2000. More recently Dick Canary gave a talk about the dictionary at Dennis Sullivan's 70th birthday conference at Stony Brook in 2011 and you can find this on the web.

6 Fundamental domains and examples of Kleinian groups

6.1 Fundamental domains

Let G be a Kleinian group, acting on $\mathcal{H}^3_+$, on $\hat{\mathbb{C}}$, or on $\mathcal{H}^3_+ \cup \hat{\mathbb{C}}$, and let $\Omega(G)$ be the ordinary set for the action.

Definition A *fundamental domain* for the action of G on $\Omega(G)$ is a subset F of $\Omega(G)$ such that

(i)
$$\bigcup_{g\in G} g(\bar{F}) = \Omega(G) \quad \text{and}$$

(ii)
$$g(F) \cap h(F) = \emptyset \quad \text{when} \quad g \neq h \quad (g, h \in G)$$

(where in (i), $\bar{F}$ denotes the closure of F).

Thus the images of F *tesselate* $\Omega(G)$ (they cover it without overlapping).

Example The set $\{x + iy : 0 < x < 1\}$ is a fundamental domain for the action of $z \to z + 1$ on the complex plane $\mathbb{C}$ (as indeed is the set $\{x + iy : 0 \leq x < 1\}$).

Note The precise definition of the term 'fundamental domain' varies from author to author: some require F to be closed - in which case of course one must modify condition (ii) above to require only that $g(F) \cap h(F)$ be contained in the *boundary* of both $g(F)$ and $h(F)$, rather than it be empty.

6.2 Dirichlet domains

The simplest construction of fundamental domains makes use of a metric. So for the time being we consider an action of G on $\mathcal{H}^3_+$ (or, if G is Fuchsian, on $\mathcal{H}^2_+$).

Choose $x \in \mathcal{H}^3_+$ such that for all $g \in G$ except the identity, $gx \neq x$. (Exercise: show that there are at most a discrete set of points $x \in \mathcal{H}^3_+$ which do not have this property.) Now for each $g \in G$ define the *half-space*

$$H_g = \{y \in \mathcal{H}^3_+ : d(y, x) < d(y, gx)\}$$

where $d(y, x)$ denotes the hyperbolic distance from y to x.

Definition The *Dirichlet domain centred at* x is the set

$$D_x = \bigcap_{g\in G-\{I\}} H_g$$

Thus D_x consists of those points of $\mathcal{H}^3$ which are nearer to x than they are to any gx ($g \in G - \{I\}$).

This construction was introduced by Dirichlet in the 1850's for the study of Euclidean groups, and later adapted by Poincaré for the hyperbolic case.

Proposition 6.1 *For any Kleinian group G, a Dirichlet domain D_x is a fundamental domain for the action of G on $\mathcal{H}^3_+$.*

Proof. We must prove that D_x satisfies conditions (i) and (ii) of the definition of a fundamental domain. We first observe that

$$g(D_x) = \{y : d(y, gx) < d(y, hx) \quad \forall h \in G - \{g\}\}$$

since

$$y \in g(D_x) \Leftrightarrow g^{-1}y \in D_x \Leftrightarrow d(g^{-1}y, x) < d(g^{-1}y, kx) \Leftrightarrow d(y, gx) < d(y, gkx) \ \forall k \in G - \{I\}$$

Now take any $y \in \mathcal{H}^3_+$. Take $g \in G$ (not necessarily unique) such that $d(y, gx)$ is minimal. Then $y \in g(\bar{D}_x)$ so property (i) holds. Moreover it is clear that $g(D_x) \cap h(D_x) = \emptyset$ if $g \neq h$ so property (ii) holds too. QED

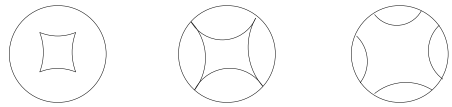

Figure 10: Polygons (in the Poincaré disc model)

Recall that a subset $X \subset \mathcal{H}^3_+$ is said to be *convex* if given any $x, y \in X$ the segment of geodesic joining x to y is entirely contained in X.

Proposition 6.2 *A Dirichlet domain D_x for a Kleinian group G is convex and locally finite (i.e. each compact subset K of $\mathcal{H}^3_+$ meets only finitely many $g(D_x)$).*

Proof. Convexity is obvious since D_x is defined to be an intersection of half-spaces, each of which is convex. For local finiteness, take the Poincaré disc model of $\mathcal{H}^3_+$ and without loss of generality take x to be the origin and K to be the closed ball with centre the origin and (hyperbolic) radius ρ. We claim that if g is any element of G such that $gD_0 \cap K$ is non-empty then $d(0, g0) \leq 2\rho$, which will prove local finiteness since G, being discrete, contains only finitely many elements with $d(0, g0) \leq 2\rho$ (else the orbit of 0 would have an accumulation point in $\mathcal{H}^3_+$, contradicting discontinuity of the action of G there). To prove the claim, take any $y \in gD_0 \cap K$; then $d(0, y) \leq \rho$ (since $y \in K$) and $d(g0, y) \leq d(0, y)$ (since $y \in gD_0$) so $d(0, g0) \leq \rho + \rho = 2\rho$. QED

Definition A convex region P obtained as the intersection of countably many half spaces H_j in $\mathcal{H}^3_+$, with the property that any compact subset of P meets only finitely many of the hyperplanes ∂H_j is called a *polyhedron* (and a subset of $\mathcal{H}^2_+$ with the analogous property is called a *polygon*).

Thus Proposition 6.2 says that a Dirichlet domain is a polyhedron. Note that the proposition does not say that D_x has only *finitely many faces*, at least it only says this when D_x is *compact*. When D_x has finitely many faces (for some x) we say that G is *geometrically finite*.

Now consider any point y on the boundary of D_x, so y is on the boundary of H_g for one of the half-spaces defining D_x, in other words $d(y, x) = d(y, gx)$ for some $g \in G$. Then

$$d(g^{-1}y, g^{-1}x) = d(y, x) = d(y, gx) = d(g^{-1}y, x)$$

so $g^{-1}y$ also lies in the boundary of D_x. Thus each face of D_x is carried to another face of D_x by an appropriate element of G. We call these elements *side-pairing transformations.*

Example Consider the standard action of $PSL(2, \mathbb{Z})$ on the complex upper half-plane. Then for any point iv on the imaginary axis, with $v > 1$, the Dirichlet domain is the region $\{z \in \mathcal{H}^2_+ : |z| > 1,\ |Re(z)| < 1/2\}$ illustrated in Figure 9, and the side-pairing transformations on this domain are $T : z \to z + 1$, $S : z \to -1/z$. (Proof: exercise.)

6.3 Poincaré's Polyhedron Theorem

We have seen that given a Kleinian group G, Dirichlet's construction allows us to find a fundamental domain on which G acts by side-pairing transformations. Poincaré's Polyhedron Theorem takes us in the opposite direction: given a convex polyhedron in $\mathcal{H}^3$ (or polygon in $\mathcal{H}^2$) and a set of side-pairing transformations for that polyhedron it gives us necessary and sufficient conditions for the group generated by those transformations to be discrete (i.e. Kleinian) and for the given polyhedron to be a fundamental domain for the group action. The precise conditions, though conceptually straightforward, are a little cumbersome to state, so we shall restrict ourselves to the two-dimensional case for most of the time. Our main concern (in the next subsection) will be to understand examples.

Let P be a polygon in $\mathcal{H}^2_+$. Note that the definition allows various possibilities. P may be compact (as on the left in Figure 10), it may have *ideal vertices* (vertices on the boundary of $\mathcal{H}^2_+$, as in the middle in Figure 10), or may have infinite area (as on the right in Figure 10).

Definition A *side-pairing* transformation of P is an isometry g_s of $\mathcal{H}^2_+$, sending one side s of P bijectively to another, s', and such that $g_s(P) \cap P = s'$.

Notation

For x_j a vertex of P which lies inside $\mathcal{H}^2_+$ (and so the two edges of P meeting at x_j meet at a non-zero angle), we let N_j denote an ϵ-neighbourhood (in the hyperbolic metric) of x_j intersected with P.

For y_j be an ideal vertex of P (so the two edges of P 'meeting' at y_j have angle zero between them), we let N'_j denote an ϵ-neighbourhood (in the Euclidean metric) of y_j intersected with P.

Theorem 6.4 (Poincaré's Polygon Theorem) *Let P be a polygon in $\mathcal{H}^2_+$, equipped with a set of side-pairing transformations g_s, one for each side of P and with $g_{s'} = g_s^{-1}$ if g_s pairs s with s'. If there exists a real $\epsilon > 0$ such that:*

• *for each vertex $x_0 \in \mathcal{H}^2_+$ of P there are vertices $x_1, ..., x_n$ of P (not necessarily all different) and isometries $f_0 = I, f_1, ...f_n, f_{n+1} = I$ such that*

(i) each $f_{j+1} = f_j g_s$ for some s, and

(ii) $f_j(N_j)$ are non-overlapping and have union the disc centre x_0 radius ϵ

and

• *for each ideal vertex y_0 of P there are ideal vertices $y_1, ..., y_n$ of P and isometries $f_0 = I, f_1, ...f_{n+1}$ with f_{n+1} fixing y_0 and parabolic, and such that*

(i)′ each $f_{j+1} = f_j g_s$ for some s, and

(ii)′ the $f_j(N'_j)$ are contiguous and non-overlapping

then

the group G generated by the side-pairing transformations g_s is discrete, P is a fundamental domain for the action of G on $\mathcal{H}^2_+$, and all relations in G are consequences of cycles $f_{n+1} = I$ corresponding to vertices of P in $\mathcal{H}^2_+$.

For a proof of this theorem see Beardon's book on discrete groups, or Ratcliffe or Maskit.

Comments

1. The condition on ideal vertices does not introduce any new relations, but it does ensure that P/G is *complete* (or equivalently that the translates of P cover the whole of $\mathcal{H}^2_+$).

2. The version for $\mathcal{H}^3_+$ (Poincaré's Polyhedron Theorem) is analogous. Now the 'sides' that are paired by the g_s are two-dimensional *faces* and instead of conditions (i) and (ii) we ask that neighbourhoods of *edges* of P fit together neatly (neighbourhoods of vertices then automatically fit together properly). Around edges which end at ideal vertices we ask that there be parabolic cycles (as in (i)′,(ii)′ above). Each edge which has ends inside $\mathcal{H}^3$ gives rise to a relation between the g_s and all relations in G are consequences of these.

6.4 Examples of Fuchsian and Kleinian groups

Examples in $PSL(2, \mathbb{R})$ (Fuchsian groups)

1. $PSL(2, \mathbb{Z})$ *(the modular group)*

Take our standard fundamental domain with side-pairings given by $S : z \to -1/z$ and $T : z \to z + 1$. Around $x_1 = (1 + i\sqrt{3})/2$ the picture is just that around $x_0 = (-1 + i\sqrt{3})/2$, conjugated by T. The vertex $y_0 = \infty$ is ideal, and T is parabolic ($z \to z + 1$). Poincaré's Polygon Theorem tells us that

$$PSL(2, \mathbb{Z}) = < S, T : S^2 = I, (ST)^3 = I >$$

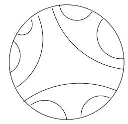

Figure 11: A truncated triangle and its first three reflections (Poincaré disc model)

2. *Surface groups*

Let P be a regular octagon with vertex angles all $\pi/4$. (To find such an octagon in the Poincaré disc model, just take a small regular octagon centred at the origin and blow it up steadily in size until the angles are $\pi/4$: this case must occur, by continuity, since in the limiting case when all vertices are ideal the angles are 0). Mark a pairing of the sides of P by labelling pairsof (oriented) sides such a way that one circuit anticlockwise around the boundary reads $ABA^{-1}B^{-1}CDC^{-1}D^{-1}$. Now think of A as an isometry carrying the first side marked A to the second side marked A etc. Then P is a fundamental domain for the group

$$G =< A, B, C, D : [A, B][C, D] = I >$$

(where $[A, B][C, D] = ABA^{-1}B^{-1}CDC^{-1}D^{-1}$). Note that $\mathcal{H}^2_+/G$ is a surface of genus two. (Higher genus surfaces may be obtained similarly.)

Comment. The octagon need not be regular: all that is really needed is that the angles add up to 2π and that the sides paired be of the same length. This is the beginning of the Teichmüller theory of hyperbolic stuctures on surfaces.

3. *Triangle groups*

Consider a triangle in $\mathcal{H}^2_+$ with angles π/p, π/q, π/r, where p, q, r are positive integers such that $1/p+1/q+1/r < 1$. We can always draw such a triangle in $\mathcal{H}^2_+$ by taking a small Euclidean triangle at the origin in the Poincaré disc model and gradually enlarging it until the angles are those desired. The (hyperbolic) area of such a triangle is π minus the angle sum. Now let G be the group generated by reflections in the sides of the triangles, and let G_0 be its orientation-preserving subgroup (products of even numbers of reflections). G_0 has generators $g_1 = R_2R_3$ and $g_2 = R_3R_1$. By Poincaré's Theorem G_0 is discrete, a quadrilateral made up of the initial triangle and one of its reflections is a fundamental domain for G_0, and a presentation for G_0 is

$$G_0 =< g_1, g_2 : g_1^p = g_2^q = (g_1g_2)^r = I >$$

(Note that if $1/p + 1/q + 1/r > 1$ we can construct a *spherical triangle* and the group G_0 is then *finite.*)

4. *Limit sets of triangle and truncated triangle groups*

When the fundamental polygon for G is compact, the limit set of G is the entire boundary circle S^1 of the Poincaré disc (the translates of P get smaller and smaller in the Euclidean metric as we move towards the boundary circle, so the orbit of any point inside the disc accumulates everywhere on S^1).

When the fundamental domain has ideal vertices the limit set remains the entire circle, but we can go further and take for example a 'truncated triangle' for our polygon P (see Figure 11). As before let G be the group generated by reflections R_1, R_2, R_3, and G_0 be the orientation-preserving subgroup (generated by R_2R_3, R_3R_1). Now R_2R_3 is hyperbolic and the 'gap' between its fixed points is in $\Omega(G_0) \subset S^1$. hence $\Lambda(G_0) \neq S^1$, so $\Lambda(G)$ has empty interior in S^1. Hence $\Lambda(G)$ is totally disconnected, but $\Lambda(G)$ is infinite, perfect, closed and bounded, so $\Lambda(G)$ is a Cantor set. Note that G_0 is freely generated by R_2R_3 and R_3R_1: there are no vertices so no relations.

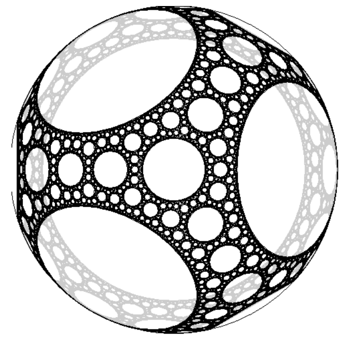

Figure 12: The limit set of a truncated tetrahedron group (picture by McMullen)

Examples in $PSL(2,\mathbb{C})$ (Kleinian groups)

1. *Tetrahedron groups*

Our 'polygon' now becomes a tetrahedron in $\mathcal{H}^3_+$ rather than a triangle in $\mathcal{H}^2_+$, and we consider the group G generated by reflections in its faces, and the orientation preserving subgroup G_0.

A tetrahedron in $\mathcal{H}^3_+$ is determined by its six *dihedral angles* (the angles between adjacent faces). To satisfy the conditions of Poincaré's Theorem we require them all to be of the form π/n with n integer.

A vertex inside $\mathcal{H}^3_+$ must have $1/p_1 + 1/p_2 + 1/p_3 > 1$, an ideal vertex must have $1/p_1 + 1/p_2 + 1/p_3 = 1$, a truncated vertex must have $1/p_1 + 1/p_2 + 1/p_3 < 1$ and where there is a truncated vertex the tetrahedron must meet the boundary of $\mathcal{H}^3_+$ in a $\pi/p_1, \pi/p_2, \pi/p_3$ triangle.

One can show that all combinations of dihedral angles are actually realised by tetrahedra or truncated tetrahedra. If all the vertices are internal or ideal then $\Lambda(G) = \hat{\mathbb{C}}$. If one or more vertices is truncated then $\Lambda(G)$ is a circle-packing (we get a circle as limit set for the triangle group around the truncated vertex, and then other elements of G move this circle around). See Figure 12 for a picture on the Riemann sphere and see Bullett and Mantica (Nonlinearity 1992) for more pictures and explanations.

2. *'Strings of beads'*

Here $C_1, ..., C_n$ are circles in $\hat{\mathbb{C}}$, each of the same size, touching the circle on each side and orthogonal to the unit circle S^1. Let R_m denote inversion in C_m, and extend R_m to a reflection in the hemisphere H_m spanning C_m in $\mathcal{H}^3_+$. Now, by Poincaré's Theorem, the part of $\mathcal{H}^3_+$ remaining after 'scooping out' all the hemispheres is a fundamental domain for the action of $G =< R_1, ..., R_n >$ and the only relations are $R_m^2 = I >$.

Note that the limit set here is S^1, but that if we pull the circles C_m apart the limit set becomes a Cantor set, and that if we perturb the sizes and positions of the circles C_m, but keeping them touching adjacent circles, the limit set becomes a *quasicircle* (a fractal homeomorphic to a circle). Going up in dimension an anologous construction can be used to obtain a group having limit set a wildly embedded circle in S^3.

3. *Schottky groups*

Take $g \geq 1$ pairs of mutually disjoint circles $C_1, C_1', \dots C_g, C_g'$ in $\mathbb{C}$ with mutually disjoint interiors. For each j choose any Mobius transformation A_j that maps C_j to C_j' and the interior of C_j to the exterior of C_j'. The group G generated by $\{A_j\}_{1 \leq j \leq g}$ is called a *Schottky group of genus g*. Writing D_j and D_j' for the interiors of C_j and C_j' in $\hat{\mathbb{C}}$ bounded by C_j, it is easy to see that $\hat{\mathbb{C}} - (\bigcup_j D_j \cup \bigcup_j D'j)$ is a fundamental domain for G (so in particular G is discrete) that $\Lambda(G)$ is a Cantor set, that G is a free group on the generators $\{A_j\}$ and that $\Omega(G)/G$ is a surface S_g of genus g. The quotient $\mathcal{H}^3_+/G = M_g$ is a *handlebody*, a 3-manifold M_g constructed by adding g handles to a sphere. The boundary of M_g is S_g. *(Observe that in this example the fundamental domain on $\hat{\mathbb{C}}$ is not a Dirichlet domain, indeed $PSL(2,\mathbb{C})$ does not preserve any metric on $\hat{\mathbb{C}}$.)*

7 Quadratic maps and the Mandelbrot Set

7.1 The Mandelbrot set and its connectivity

Proposition 7.1 Every quadratic map $f(z) = \alpha z^2 + \beta z + \gamma$ with $\alpha \neq 0$ is conjugate to $q_c(z) = z^2 + c$ for a unique c.

Proof The conjugacy h must send ∞ to itself, and hence have the form $h(z) = kz + l$.

$$hf(z) = k(\alpha z^2 + \beta z + \gamma) + l \qquad q_c h(z) = (kz + l)^2 + c$$

These are equal (for all z) if and only if $k\alpha = k^2, k\beta = 2kl$ and $k\gamma + l = l^2 + c$. Thus we must have $k = \alpha, l = \beta/2$ and $c = \alpha\gamma + \beta/2 - \beta^2/4$. QED

Another useful parametrisation of the quadratic maps is given by the *logistic family*

$$p_\lambda(z) = \lambda z(1 - z)$$

Clearly p_λ is conjugate to q_c if and only if $c = \lambda/2 - \lambda^2/4$ (by Proposition 7.1).

The q_c parametrisation is more convenient when we are dealing with critical points, and the p_λ parametrisation is more convenient when we are dealing with fixed points and their multipliers. Note that q_c has critical points $0, \infty$, the latter a superattracting fixed point, and p_λ has fixed points 0 and $1 - 1/\lambda$, with multipliers λ and $2 - \lambda$ respectively.

Definition The *Mandelbrot set* is the subset of parameter space defined by

$$M = \{c : J(q_c) \text{ connected}\} \subset \mathbb{C}$$

Theorem 7.2 *M is the set of values of the parameter c such that the orbit $q_c^n(0)$ of the critical point 0 does not tend to the point ∞*

Proof If the orbit of 0 does not tend to ∞ then there is no critical value other than ∞ in the basin of attraction, $B(\infty)$, of ∞, and so there is no obstruction to extending the Böttcher coordinate (Section 5 of these notes) from a neighbourhood of ∞ to the whole of this basin. Hence B_∞ is homeomorphic to the open unit disc and its complement $\hat{\mathbb{C}} \setminus B_\infty$ is therefore connected, as is their common boundary ∂B_∞. But ∂B_∞ is closed and completely invariant, and cannot contain any points of the Fatou set (since any point in ∂B_∞ has bounded orbits, yet arbitrarily close to it are points with orbits going to ∞). So ∂B_∞ is the Julia set $J(q_c)$.

Conversely, if the orbit of 0 does go to ∞ then $J(q_c)$ is totally disconnected (a Cantor set) by the argument sketched earlier for the example $|c|$ large. QED

Definition The *filled Julia set* of q_c is $K(q_c) = \{z : q_c^n(z) \not\to \infty\}$

Note that $\partial K(q_c) = J(q_c)$, and that if $c \notin M$ then $K(q_c) = J(q_c) =$ Cantor set.

Theorem 7.3 (Douady and Hubbard 1982) *The Mandelbrot set M is connected*

Proof In fact Douady and Hubbard proved a much stronger result, that there is a conformal bijection between the complement $\hat{\mathbb{C}} - M$ of the Mandelbrot set and the complement $\hat{\mathbb{C}} - \mathbb{D}$ of the open unit disc. It is an immediate consequence of this that M is connected.

When $c \in M$, the Böttcher coordinate defines a conformal bijection

$$\phi_c : \hat{\mathbb{C}} - K(q_c) \to \hat{\mathbb{C}} - \mathbb{D}$$

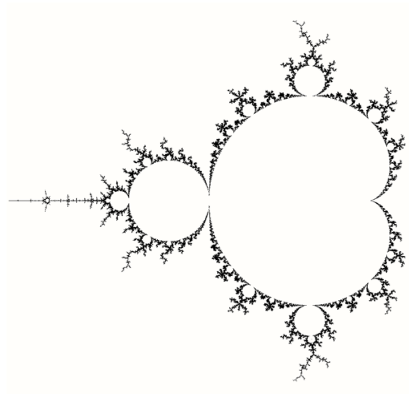

Figure 13: The Mandelbrot set

$$\phi_c(z_0) = z_0(1+\frac{c}{z_0^2})^{1/2}(1+\frac{c}{z_1^2})^{1/4}(1+\frac{c}{z_2^2})^{1/8}...$$

(conjugating q_c to $z \to z^2$). When $c \notin M$ the map ϕ_c, though not defined on the whole of the complement of K_c, is nevertheless defined on a neighbourhood of ∞ and as far as the critical value c of q_c. Define

$$\Psi : \hat{\mathbb{C}} - M \to \hat{\mathbb{C}} - \mathbb{D}$$

$$\Psi(c) = \phi_c(c)$$

This is a conformal bijection (see Douady and Hubbard, Comptes Rendues 1982, for more details). QED

Conjecture ('MLC') *M is locally connected*

A set X is called *locally connected* if every $x \in X$ has arbitrarily small connected open neighbourhoods. If M is locally connected then by a theorem of Carathéodory the map Ψ^{-1} extends to a continuous map from the boundary of $\hat{\mathbb{C}} - \mathbb{D}$ (a circle) onto the boundary ∂M of the Mandelbrot set. This would give us a purely combinatorial description of ∂M and many open questions concerning M would be resolved.

Definition A component of the interior of M is said to be *hyperbolic* if for every c in the component the map q_c has an attracting or superattracting periodic orbit.

Conjecture ('Hyperbolicity is dense') *Every component of the interior of M is hyperbolic*

Douady and Hubbard showed in their 1985 Orsay lecture notes that 'MLC' implies 'Hyperbolicity is dense'.

Both conjectures seem to be very difficult to resolve. Over the past 3 decades there has been a great deal of work on them. The set of points of ∂M at which local connectivity is known to hold has been steadily increased: Yoccoz proved it for 'all but infinitely renormalizable points' and Lyubich extended this to certain of these. Most experts seem to believe that MLC should be true, but it is known that the analogous set for cubics in place of quadratics is *not* locally connected (Lavaurs, Milnor), and that there exist quadratic maps q_c having non-locally-connected Julia sets. As far as 'Hyperbolicity is dense' is concerned, this has been proved for components of M meeting the real axis (Lyubich, McMullen, Swiatek: see McMullen's 1994 book 'Complex Dynamics and Renormalization') but the general question is still unresolved. Shishikura's proved in 1994 that the boundary ∂M of the Mandelbrot set has Hausdorff dimension 2.

7.2 The geography of the Mandelbrot set

We examine some of the more prominent features of M (Figure 13).

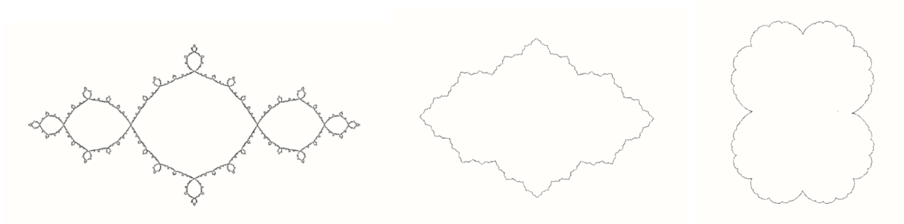

Figure 14: Julia sets for c=-1, c=-0.5 and c=+0.25

Let

$M_0 = \{c : q_c \textit{ has an attracting (or superattracting) fixed point}\}$

$= \{c : J(q_c) \textit{ is a (topological) circle}\}$

Lemma 7.4 $M_0 = \{c : c = \lambda/2 - \lambda^2/4 \textit{ for some } \lambda \textit{ with } |\lambda| < 1\}$

Proof Consider the logistic map p_λ. The multipliers of its fixed points are $\lambda, 2 - \lambda$. Hence

$M_0 = \{c : c = \lambda/2 - \lambda^2/4 \textit{ for some } \lambda \textit{ with } |\lambda| < 1 \textit{ or } |2 - \lambda| < 1\}$

But $\lambda/2 - \lambda^2/4 = (2 - \lambda)/2 - (2 - \lambda)^2/4$. QED

Thus M_0 is a *cardioid* (with a boundary that is smooth except at the cusp $c = 1/4$). Note that there is a bijection between points of M_0 and values of λ such that $|\lambda| < 1$. Thus M_0 is parametrised by the multiplier of the fixed point of q_c. The maps q_c with $c \in M_0 \setminus \{0\}$ are topologically conjugate to one another, indeed they are *quasiconformally* conjugate to one another (as we shall see in Section 8). Note that they cannot be *conformally* conjugate to one another as they have different multipliers at the fixed point. Note also that none of them can be topologically conjugate to $q_0 : z \to z^2$, since for q_0 the critical point 0 is a fixed point, and this is not true for any q_c with $c \neq 0$.

7.2 The intersection of M with the real axis

We consider how the behaviour of q_c varies as we vary the parameter c along the real axis. See Figure 14.

For $c > 1/4$, $J(q_c)$ is a Cantor set (it is an easy exercise to show that the orbit of 0 under q_c tends to ∞).

At $c = 1/4$, there is a neutral fixed point $z = 1/2$, with multiplier 1.

For $-3/4 < c < 1/4$, q_c has an attracting fixed point and $J(q_c)$ is a (topological, indeed *quasi-conformal*) circle, with dynamics conjugate to that of the shift. In particular $J(q_c)$ contains a dense set of repelling periodic orbits.

At $c = -3/4$, both points on the repelling period 2 orbit collide with the attracting fixed point, at a neutral fixed point (which has multiplier -1):

For $-5/4 < c < -3/4$, q_c has an attracting period 2 orbit, and the topology of $J(q_c)$ is the same as that for the (superattractive) case $c = -1$.

We digress briefly to justify the bounds $-5/4 < c < -3/4$:

Lemma 7.5 q_c *has an attracting period* 2 *orbit if and only if* $|1 + c| < 1/4$

Proof The points of period 1 or 2 are the solutions of $q_c^2(z) = z$. Expanding $q_c^2(z) - z$ we have

$$q_c(q_c(z)) - z = ((z^2 + c)^2 + c - z = (z^2 - z + c)(z^2 + z + 1 + c) = (z - \alpha)(z - \beta)(z - u)(z - v)$$

where α, β are the fixed points and u, v is the period 2 cycle. The multiplier of the period 2 cycle is $q_c'(u)q_c'(v) = 4uv = 4(1 + c)$. The period 2 cycle is attracting if and only if this has modulus less than 1. QED.

Returning to our journey in parameter space along the real axis:

For $-2 < c < -5/4$, as c decreases through this range, we have a sequence of period doublings until we reach the Feigenbaum point (the 'period-doubling limit point'. This is followed by the whole Milnor/Thurston sequence of periods for real unimodal maps, familiar to dynamicists (in particular this contains all the natural numbers in the Sarkovskii order). The most prominent component of $int(M)$ along the axis after the period-doubling limit is one corresponding to a period three attracting orbit, and we finish at $c = -2$ where the Julia set is the real interval $[-2, +2]$ (and q_c is semi-conjugate to $z \to z^2$: see the exercise early on in these notes).

For $c < -2$, it is again easily proved that the orbit of the critical point 0 tends to ∞ and hence that the Julia set is again a Cantor set.

The behaviour for c at different points along the real axis is is no surprise to real dynamicists since the quadratic family is conjugate to the logistic family. However with c complex we can now leave the main cardioid M_0 at other points than just $c = -3/4$. When c is on the boundary of M_0 at the point where $\lambda = e^{2\pi i p/q}$, q_c has a neutral periodic point with this as multiplier, and when c passes into the adjoining component q_c has an attracting period q orbit. There are then further bifurcations as we pass along a path through different components of $int(M)$. In the next subsection we shall find that studying the combinatorics of 'external rays' can give us a great deal of information about the overall structure of M

Exercise Compute the values of c where q_c has a superattractive period three orbit (that is, where the point 0 has period three).

7.3 Internal and external rays: the 'devil's staircase'

When $c \in M$, for any $\theta \in [0, 1)$, the radial line $arg(z) = 2\pi\theta$ on $\hat{\mathbb{C}} - \mathbb{D}$ (where $\mathbb{D}$ is the unit disc) maps under the inverse ϕ_c^{-1} of the Böttcher map to the *external ray* $\mathcal{R}_\theta$ of argument $2\pi\theta$ on $\hat{\mathbb{C}} - K(q_c)$.

Similarly, in the parameter plane, the radial line $arg(z) = 2\pi\theta$ on $\hat{\mathbb{C}} - \mathbb{D}$ maps under the inverse Ψ^{-1} of the Douady-Hubbard map to the *external ray* $\mathcal{R}_\theta$ of argument $2\pi\theta$ on $\hat{\mathbb{C}} - M$.

A ray is said to *land*, if it accumulates at a unique point of $J(q_c)$ (in the dynamical case) or ∂M (in the parameter case). If $J(q_c)$ (or ∂M respectively) is locally connected then all external rays land (by Carathéodory's criterion). Unfortunately there are examples where $J(q_c)$ is known not to be locally connected, and where certain external rays do not land; moreover the conjecture 'MLC' is still unproved so we cannot be sure that all external rays in the parameter space land.

An outline proof of the following theorem can be found in Carleson and Gamelin, and a full proof can be found in the Douady-Hubbard Orsay notes.

Theorem 7.6 (Douady and Hubbard) *Every parameter space external ray with rational angle θ lands at a point c of ∂M. If θ is a rational with odd denominator then q_c has a parabolic cycle. If θ is a rational with even denominator then the critical point 0 of q_c is strictly preperiodic.*

We first consider the external rays which land on the boundary of the main cardioid, M_0. Recall that M_0 is itself parametrised by the unit disc and we can therefore define *internal rays* inside M_0. The *internal ray* of argument ν is the set of values of $c \in M_0$ for which the multiplier of the fixed point of q_c has argument $2\pi\nu$. Consider the end point on ∂M_0 of the internal ray of argument $\nu = 1/3$. This is the value of c for which the fixed point α of q_c has multiplier $e^{2\pi i/3}$ (this c lies at the top of the cardioid: it is where the first period-tripling occurs). The external rays $1/7, 2/7, 4/7$ in the dynamical plane landing at α are as shown in Figure 15.

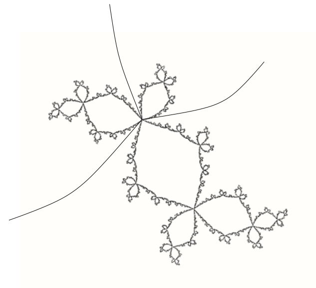

Figure 15: External rays landing at the α-fixed point of Douady's rabbit

Note that we can pick out two particular rays, which together enclose the component of $int(K(q_c))$ containing the critical value. These we have labelled $\theta_-(1/3)$ and $\theta_+(1/3)$. It can be shown that in the parameter space the corresponding external rays with the same arguments, $\theta_-(1/3)$ and $\theta_+(1/3)$, land at c (an example what Douady called 'ploughing in the dynamical plane but harvesting in the parameter plane').

More generally, for c at the end of each internal ray in M_0 of rational argument p/q, the map q_c has a (neutral) fixed point α of rotation number p/q and we can pick out the pair of external rays enclosing the component of $int(K(q_c))$ containing the critical value c. How do we compute the values of $\theta_-(p/q)$ and $\theta_+(p/q)$? Since α is a fixed point of rotation number p/q there are necessarily q external rays landing at α and the effect of q_c on these rays is to permute them in cyclic order. But the action of q_c on arguments of rays is simply that of $t \to 2t \pmod{\mathbb{Z}}$, so our search for candidates for $\theta_\pm(p/q)$ is reduced to a search for finite orbits of $t \to 2t$ on the unit circle $\mathbb{R}/\mathbb{Z}$, arranged in the same order around the circle as an orbit of a rigid rotation through $2\pi p/q$. This is a purely combinatorial question and was answered (though in a slightly different context) by Morse and Hedlund in their pioneering work on symbolic dynamics in the 1930's:

Theorem 7.7 *For each rational p/q there is a unique finite forward invariant orbit $A_{p/q}$ of $t \to 2t$ of rotation number p/q on the circle $\mathbb{R}/\mathbb{Z}$.*

(For a proof of this and other results concerning order-preserving orbits of the shift, see Bullett and Sentenac, Math. Proc. Cam. Phil. Soc. 1994.)

But supposing we have found this orbit $A_{p/q}$, how are we to know which of its points are the special points $\theta_\pm(p/q)$? This turns out to be very straightforward.

Lemma 7.8 *Any ordered orbit of $t \to 2t$ on the circle $\mathbb{R}/\mathbb{Z}$ is contained in a semicircle*

Proof Since $t \to 2t$ doubles distance, any three points on the circle have images in the same order around the circle if and only the three original points lie in a common semi-circle. QED

As a consequence it makes sense to refer to the *least* and *greatest* points of the orbit $A_{p/q}$. We identify the points $\theta_\pm(p/q)$ by observing that the dynamical picure requires that the least point of $A_{p/q}$ be $(\theta_+(p/q))/2$ and the greatest be $(\theta_-(p/q))/2 + 1/2$ (see the picture above for the case $p/q = 1/3$: the inverse image of the component of $int(K(q_c))$ containing the critical value c is that containing the critical point 0).

Algorithm for $\theta_\pm(p/q)$

There is a simple algorithm constructing the binary sequence of each of $\theta_+(p/q)$ and $\theta_-(p/q)$:

Draw a line of slope p/q, through the origin in $\mathbb{R}^2$. To construct $\theta_-(p/q)$, take the integer 'staircase' lying just below this line, but not touching it, and starting at the point $(1,0)$ write 1 for each horizontal step which is followed by a vertical step, 0 for a horizontal step followed by another horizontal one. To construct $\theta_+(p/q)$ do the same with the staircase touching the line.

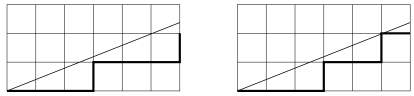

Figure 16: $\theta_-(2/5) = .\overline{01001} = 9/31 \qquad \theta_+(2/5) = .\overline{01010} = 10/31$

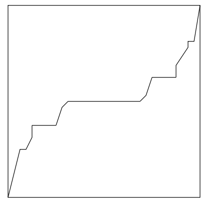

Figure 17: The devil's staircase assigning internal angle to external angles around the cardioid M_0.

Example $p/q = 2/5$: See Figure 16.

Every point on ∂M_0 at the end of an internal ray of *irrational* argument ν corresponds to $\theta_\nu = lim_{p/q \to \nu} \theta_\pm(p/q)$. The assigment of internal angles to external angles as we make a circuit of the boundary of the cardioid M_0 has graph a 'devil's staircase' (see Figure 17). We draw the graph this way round (rather than that assigning external angles to internal angles) in order to have a *continuous* function. It is not difficult to prove that the horizontal steps in the graph above have total length 1. A 'devil's staircase' is the graph of a continuous function that is constant on a set of full measure without being globally constant. This particular devil's staircase another interesting property: it is a theorem due to Douady that every irrational ν corresponds to a *transcendental* θ_ν (see Bullett and Sentenac, Theorem 4).

7.4 External rays landing at points outside the main cardioid

Rational external rays can be used to give us an overall picture of the geography of M. The next step is to consider those landing on the boundary of a component of $int(M)$ immediately adjacent to M_0, say that corresponding to rotation number p/q. This component (which we shall label $M_{p/q}$) has the property that corresponding maps q_c each have an attractive period q orbit. We can parametrise $M_{p/q}$ by the multiplier of this orbit and hence define internal rays inside $M_{p/q}$ in just the same way as we did for M_0. The r/s internal ray in $M_{p/q}$ is the landing point of external rays $\theta_\pm(p/q, r/s)$ obtained from $\theta_\pm(r/s)$ by replacing the digit 0 by the repeating block (of length q) from $\theta_-(p/q)$ and the digit 1 by the repeating block from $\theta_+(p/q)$.

Example

$$\theta_-(1/3, 1/2) = .\overline{001010} \qquad \theta_+(1/3, 1/2) = .\overline{010001}$$

By repeating the same process (which is known as 'tuning') we can compute the arguments of external rays landing on the boundary of any component which is accessible from M_0 by a finite number of boundary crossings. But there are of course components of $int(M)$ which are much further away than this from M_0: for example all components beyond the Feigenbaum point on the real axis are an infinite number of boundary crossings away from M_0. Methods of assigning 'internal addresses' to all hyperbolic components, and algorithms relating these addresses to 'kneading sequences' associated to external rays landing on the components, were developed by Penrose (1990) and by Lau and Schleicher (1994).

7.5 The combinatorial Mandelbrot set

We sketch an algorithm due to Lavaurs (Comptes Rendues 1986) which, if ∂M is locally connected, gives M as the quotient of the unit disc by an equivalence relation defined via a *lamination*.

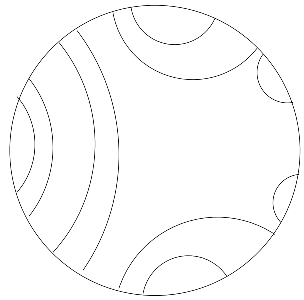

Figure 18: Lavaurs' algorithm

Lavaurs' Algorithm

Write every rational which has odd denominator in the form $p/(2^k-1)$ with k as small as possible.

1. Connect 1/3 to 2/3 (on $\partial\mathbb{D}$) by an arc in $\mathbb{D}$.

2. Assuming all rationals of form $p/(2^{k-1}-1)$ have been connected in pairs, connect pairs of form $p/(2^k-1)$, starting with the smallest number not yet connected, and connecting it to the next smallest one possible without crossing arcs already constructed (see Figure 18 for the construction up to and including $k=4$). The (combinatorial) Mandelbrot set is now obtained by shrinking each of the arcs to a point.

8 Quasiconformal mappings: the Measurable Riemann Mapping Theorem and its applications

8.1 The moduli space and the Teichmüller space of a torus

Given any two Riemann surfaces $\mathcal{S}_1, \mathcal{S}_2$ which are homeomorphic to a sphere, there is *conformal* homeomorphism $\mathcal{S}_1 \to \mathcal{S}_2$. This follows from the Uniformisation Theorem, which tells us that every Riemann surface has universal cover $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$. But Riemann surfaces which are homeomorphic to the torus are another matter. Every such surface $\mathcal{S}$ has universal cover $\mathbb{C}$, and the group Γ of covering transformations of $\mathcal{S}$ is a subgroup of $Aut(\mathbb{C})$ isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Thus Γ is generated by two translations of $\mathbb{C}$ in directions which are linearly independent over $\mathbb{R}$. By conjugating by a scaling and a rotation of $\mathbb{C}$ we may assume that one of the translations is $z \to z+1$ and the other is $z \to z + \lambda$, some $\lambda \in \mathbb{C} - \mathbb{R}$.

Proposition 8.1
Let $\mathcal{S}_1$ be the torus $\mathbb{C}/\Gamma_1$, where Γ_1 is generated by $z \to z+1$ and $z \to z+\lambda_1$, and let $\mathcal{S}_2$ be the torus $\mathbb{C}/\Gamma_2$, where Γ_2 is generated by $z \to z+1$ and $z \to z+\lambda_2$. Then there is a conformal homeomorphism (i.e. an analytic bijection) between $\mathcal{S}_1$ and $\mathcal{S}_2$ if and only if $\lambda_2 = g(\lambda_1)$ for some $g \in PSL(2, \mathbb{Z})$.

Proof First observe that if $\lambda_2 = \lambda_1 + 1$ then $\Gamma_2 = \Gamma_1$ so $\mathcal{S}_1$ and $\mathcal{S}_2$ are the same torus, and if $\lambda_2 = -1/\lambda_1$ then the lattice $\Gamma_2 \subset \mathbb{C}$ is obtained from the lattice Γ_1 by rotating and rescaling $\mathbb{C}$, so $\mathcal{S}_2$ is isomorphic to $\mathcal{S}_1$. Since these two operations generate the action of $PSL(2,\mathbb{Z})$ on λ_1, it follows that if $\lambda_2 = g(\lambda_1)$ for any $g \in PSL(2,\mathbb{Z})$ then $\mathcal{S}_2$ is isomorphic to $\mathcal{S}_1$.

Conversely, if $\mathcal{S}_2$ is isomorphic to $\mathcal{S}_1$ then by the Uniformisation Theorem there must exist an automorphism of $\mathbb{C}$, fixing the origin and conjugating the generators $z \to z+1$ and $z \to z+\lambda_1$ of Γ_1 to a pair of generators of Γ_2, that is to say there must exist $0 \neq \mu \in \mathbb{C}$ such that Γ_2 is the group generated by $z \to \mu$ and $z \to \mu\lambda_1$. Since Γ_2 is also generated by $z \to z+1$ and $z \to z+\lambda_2$ this implies there exist $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ such that

$$\begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$$

and so λ_2 is the image of λ_1 under an element of $PSL(2,\mathbb{Z})$. QED

Thus we get a different complex structure on a topological torus for each different point λ in our fundamental domain Δ for the action of the modular group $PSL(2,\mathbb{Z})$. The complex structures on the torus therefore correspond to the points of the *moduli space*

$$\mathcal{M} = \mathcal{H}^2_+ / PSL(2,\mathbb{Z})$$

which is a sphere with a puncture point (corresponding to ∞), a cone point of angle π (corresponding to i) and a cone point of angle $2\pi/3$ (corresponding to $(-1+i\sqrt{3})/2$). Given a Riemann surface of genus 1, we can *mark* it by choosing two homotopy classes of loops which generate the fundamental group. This corresponds in the universal cover to choosing generators of the covering transformation group $\Gamma = \mathbb{Z} \times \mathbb{Z}$. The *marked* complex structures on the torus correspond to the points on the universal cover of $\mathcal{M}$, the *Teichmüller space* $\mathcal{T} = \mathcal{H}^2_+$.

Remark. For a genus g surface $\mathcal{S}_g$, with $g \geq 2$, the Teichmüller space $\mathcal{T}(\mathcal{S}_g)$ is a copy of $\mathbb{R}^{6g-6}$ (one can give explicit coordinates in terms of lengths of certain loops on $\mathcal{S}_g$), and the moduli space is the qutioent of Teichmüller space by the *mapping class group* of $\mathcal{S}_g$.

However one can construct a homeomorphism from a Riemann surface of genus g to any other Riemann surface of the same genus if we weaken the requirement of conformality to a requirement that the homeomorphism should 'send infinitesimal circles to infinitesimal ellipses having bounded ratios of internal to external diameter'. Such homeomorphisms are called *quasiconformal homeomorphisms.*

Example Figure 19 illustrates a quasiconformal homeomorphism which sends the small circles on the left hand torus to the small ellipses on the right hand torus.

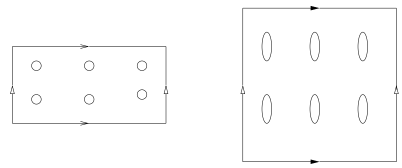

Figure 19: There is a quasiconformal homemorphism between these two tori

8.2 Quasiconformal homeomorphisms and the Measurable Riemann Mapping Theorem

An invertible linear map $\mathbb{R}^2 \to \mathbb{R}^2$ sends a circle centred at the origin to an ellipse centred at the origin, so a C^1-diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ sends an infinitesimal circle at each point $x \in \mathbb{R}^2$ to an infinitesimal ellipse at $f(x)$.

Definition A homeomorphism f between open sets in $\mathbb{C}$ is said to be K-*quasiconformal* if it sends infinitesimally small round circles to infinitesimally small ellipses which have ratio of semi-major axis length to semi-minor axis length less than or equal to K. *(Technical point: we do not require that f be C^1, only that f have "distributional derviatives in L^1". See Milnor, Appendix F.)*

We can write f as a function of z and $\bar{z}$ (if f were conformal it would just be function of z). We can then associate to f the *Beltrami form*:

$$\mu(f) = \frac{\partial f}{\partial \bar{z}} / \frac{\partial f}{\partial z}$$

and it is straightforward to prove that f is K-quasiconformal, with $K = (1 + k)/(1 - k)$, if and only if $\mu(f)$ is defined almost everywhere and has essential supremum $||\mu||_\infty = k < 1$. (See, for example, Carleson and Gamelin.)

Recall that the *Riemann Mapping Theorem* asserts that if U is a bounded open simply-connected subset of $\mathbb{C}$, then there exists a conformal orientation-preserving homeomorphism $\phi : U \to \mathbb{D}$, where $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$. Clearly ϕ is unique up to post-composition by orientation-preserving conformal homeomorphisms of $\mathbb{D}$, that is to say fractional linear maps which send the unit disc $\mathbb{D}$ to itself.

The *Measurable Riemann Mapping Theorem* asserts the analogous result in the case that in addition to U we are given an assigned *complex dilatation* $\mu(z)$ at every point $z \in U$ (except possibly at points in a set of Lebesgue measure zero) and rather than seeking a conformal homeomorphism ϕ from U to $\mathbb{D}$, what we are looking for is a *quasiconformal* homeomorphism $f : U \to \mathbb{D}$ which has the prescribed dilatation $\mu(z)$ at almost every point $z \in U$. We only require that the assignment $z \to \mu(z)$ be measurable, not that it be continuous.

The Measurable Riemann Mapping Theorem is due to Morrey, Bojarski, Ahlfors and Bers. It has various versions appropriate for different applications. The statement below is that of Théorème 5 in Douady's paper in LMS Lecture Notes Volume 274 'The Mandelbrot set. Theme and Variations' (edited by Tan Lei): it is expressed in terms of functions defined on the whole of $\mathbb{C}$, but can be adapted to suit other situations, for example when the domain of μ is a bounded simply-connected open subset U of $\mathbb{C}$ and we seek a quasiconformal homeomorphism $f : U \to \mathbb{D}$ or indeed when the domain of μ is the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and we seek a quasiconformal homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Theorem 8.2 (The Measurable Riemann Mapping Theorem.) *Let μ be any L^∞ function $\mathbb{C} \to \mathbb{C}$ with $||\mu||_\infty = k < 1$. Then there exists an orientation-preserving quasiconformal homeomorphism $f : \mathbb{C} \to \mathbb{C}$ which has complex dilatation $\mu(f)$ equal to μ almost everywhere on $\mathbb{C}$. This homeomorphism is unique if we require that $f(0) = 0$ and $f(1) = 1$. Furthermore if μ depends analytically (respectively continuously) on a parameter λ then the homeomorphism f also depends analytically (respectively continuously) on λ.*

8.3 1st Application: maps in the same hyperbolic component of the interior of the Mandelbrot set are quasiconformally conjugate

For simplicity consider the component consisting of the interior of the main cardioid M_0. We know that for any $c' \neq c$, both in $int(M_0)$, q_c is not conformally conjugate to $q_{c'}$ since they have different multipliers at the attracting fixed point. However, provided both c and c' are non-zero, $q_{c'}$ is quasiconformally conjugate to q_c. To prove this, consider a small circle γ_0 around the attracting fixed point of q_c (sufficiently small that it does not contain the critical value c). The circle γ_0 and its image $\gamma_1 = q_c(\gamma_0)$ bound an annulus A; the map q_c identifies the outer boundary γ_0 of A with its inner boundary γ_1 and the quotient of A under this identification is a torus $\mathcal{S}$. Similarly for q'_c we obtain an annulus A' and a torus $\mathcal{S}'$. Although $\mathcal{S}'$ is not conformally homeomorphic to $\mathcal{S}$ (since the multipliers of the two quadratic maps at their respective attracting fixed points are different), they are quasiconformally homeomorphic, since this is true for any pair of Riemann surfaces of genus 1. Let h be quasiconformal homeomorphism $\mathcal{S} \to \mathcal{S}'$. The derivative of h sends the field of infinitesimal circles on S to a field of infinitesimal ellipses on $\mathcal{S}'$. We can 'spread' this field of ellipses to the whole of the attracting basin $int(K(q_{c'}))$ of the fixed point of $q_{c'}$ by repeatedly applying $q_{c'}$ and $q_{c'}^{-1}$ to the lift A' of $\mathcal{S}'$. This ellipse field, together with the infinitesimal round circle field on $\hat{\mathbb{C}} - int(K(q_{c'}))$, provides us with an ellipse field on $\hat{\mathbb{C}}$ which is preserved by the map $q_{c'}$. Applying the Measurable Riemann Mapping Theorem to this ellipse field on $\hat{\mathbb{C}}$ yields a quasiconformal conjugacy from $q_{c'} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ to a map $q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which is

(i) holomorphic (because q preserves the field of infinitesimal round circles),

(ii) a polynomial (because $q(\infty) = \infty = q^{-1}(\infty)$), and

(iii) necessarily conformally conjugate to q_c (because by construction q has the correct multiplier at its attracting fixed point).

We can apply a similar argument to any hyperbolic component of the interior of the Mandelbrot set M, replacing 'attracting fixed point' by 'attracting periodic cycle' and 'q_c' by '$(q_c)^n$'. But note that each hyperbolic component of $int(M)$ contains one special value c_0 where the attracting cycle is superattracting (i.e. the critical point 0 is periodic), and that q_{c_0} is not even topologically conjugate to the other q_c's: nevertheless the Julia set $J(q_{c_0})$ for this 'postcritically finite' map q_{c_0} is still quasiconformally homeomorphic to the other $J(q_c)$'s, a fact that can be proved by remembering that the Julia set is the closure of the set of all repelling periodic points, and applying the theory of 'holomomorphic motions' to this set.

Remark Notice that when we have a periodic attractor, once we have deformed the complex structure on the 'fundamental torus' A for the attractor, this determines the deformation everywhere in the basin of the attractor. If there were to exist a 'wandering component' of the Fatou set $F(q_c)$ for some c, we would have much more freedom to deform q_c: in fact (as Sullivan proved) we would have an infinite dimensional space of quadratic polynomials none of which would be conformally conjugate to another. This would contradict the fact that up to conformal conjugacy there exists only a one-complex-dimensional family of q_c's. (See Milnor, Appendix F, for the details of Sullivan's proof).

8.4 2nd Application: Bers simultaneous uniformisation: matings between Fuchsian groups

Recall that a *Fuchsian group* is a discrete subgroup of $PSL_2(\mathbb{R})$. Let G_1 be a geometrically finite Fuchsian group (a Fuchsian group which has a fundamental domain with a finite number of sides). Then G_1 acts (by fractional linear maps) on the upper half $\mathcal{U}$ of the complex plane. Suppose the limit set of this action is $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Of course G_1 also acts (by fractional linear maps) on the lower half plane $\mathcal{L}$ and the limit set of this action is also $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Let G_2 be another geometrically finite discrete subgroup of $PSL_2(\mathbb{R})$, such that G_2 is isomorphic to G_1 as an abstract group, and such that the action of G_2 on $\mathcal{U}$ is topologically conjugate to that of G_1 on $\mathcal{U}$.

Theorem 8.3 (Bers' Simultaneous Uniformisation Theorem)

Given subgroups G_1 and G_2 of $PSL_2(\mathbb{R})$ with the properties described above, there exists a discrete subgroup G of $PSL_2(\mathbb{C})$ the action of which on the Riemann sphere $\hat{\mathbb{C}}$ has the following properties:

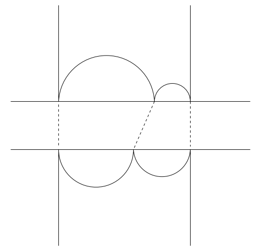

Figure 20: Fundamental domains for G_1 and G_2 on the upper and lower half-planes

(i) The limit set of the action is a quasicircle $\Lambda \subset \hat{\mathbb{C}}$.

(ii) On one component, U, of $\hat{\mathbb{C}} - \Lambda$ *the action of G is conformally conjugate to the action of G_1 on* $\mathcal{U}$.

(iii) On the other component, L, of $\hat{\mathbb{C}} - \Lambda$ *the action of G is conformally conjugate to the action of G_2 on* $\mathcal{L}$.

In the situation described by (i),(ii) and (iii) we might call the *Kleinian* group G (discrete subgroup of $PSL_2(\mathbb{C})$) acting on $\hat{C}$ a *mating* between the Fuchsian group G_1 (discrete subgroup of $PSL_2(\mathbb{R})$) acting on $\mathcal{U}$ and the Fuchsian group G_2 acting on $\mathcal{L}$. The action of G is a *holomorphic realisation* of the dynamical system obtained by gluing together the actions of G_1 on $\mathcal{U}$ and G_2 on $\mathcal{L}$ by means of a (topological) homeomorphism from the boundary $\partial\mathcal{U}$ of $\mathcal{U}$ to the boundary $\partial\mathcal{L}$ of $\mathcal{L}$, which conjugates the action of G_1 on $\partial\mathcal{U}$ to that of G_2 on $\partial\mathcal{L}$.

Bers' Simultaneous Uniformization Theorem may be proved using the Measurable Riemann Mapping Theorem, as we now outline. Since G_1 is geometrically finite, the orbit space $\mathcal{L}/G_1$ is a Riemann surface with a finite number of marked cone points and puncture points. The orbit space $\mathcal{L}/G_2$ is a Riemann surface with a combinatorially identical set of data. It follows by standard Riemann surface theory that there exists a quasiconformal diffeomorphism $h : \mathcal{L}/G_1 \to \mathcal{L}/G_2$, sending marked points to marked points. The complex dilatation μ of h, when composed with the orbit projection, yields an L^∞ function $\mu : \mathcal{L} \to \mathbb{C}$, which we may extend to the whole of $\hat{\mathbb{C}}$ by defining $\mu(z)$ to be zero on $\hat{\mathbb{C}} - \mathcal{L} = \overline{\mathcal{U}}$. Equivalently, if one prefers to think in terms of measurable fields of ellipses, the field of ellipses defined by μ on $\mathcal{L}/G_1$ is pulled back to $\mathcal{L}$ and extended to the rest of $\hat{\mathbb{C}}$ by the standard (round) circle field on $\mathcal{U}$. By the measurable Riemann Mapping Theorem there now exists a quasiconformal diffeomorphism $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ having complex dilatation μ. But, as is easily verified by the chain rule, each element of $G = \phi G_1 \phi^{-1}$ has complex dilatation zero, and so maps infinitesimal round circles to infinitesimal round circles. Thus G is a group of conformal automorphisms of $\hat{\mathbb{C}}$, that is to say a subgroup of $PSL_2(\mathbb{C})$. The limit set of G is the set $\Lambda = \phi(\hat{\mathbb{R}})$, which is a quasicircle by definition, since it is the image of a round circle under a quasiconformal homeomorphism $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Moreover ϕ provides a conformal conjugacy between the actions of G_1 on $\mathcal{U}$ and G on $U = \phi(\mathcal{U})$, and $\phi \circ h^{-1} : \mathcal{L} \to L = \phi(\mathcal{L})$ provides a conformal conjugacy between the actions of G_2 on $\mathcal{L}$ and G on $L = \phi(\mathcal{L})$, where here $h : \mathcal{L} \to \mathcal{L}$ denotes the lift of our quasiconformal diffeomorphism $h : \mathcal{L}/G_1 \to \mathcal{L}/G_2$.

A family of examples: once-punctured torus groups

Consider discrete representations of the free group F_2 on two generators X, Y in $PSL_2(\mathbb{R})$. Let A and B be elements of $PSL_2(\mathbb{R})$ representing X and Y, and restrict attention to the case that A and B are hyperbolic and their commutator $ABA^{-1}B^{-1}$ is parabolic. A generic representation of this kind has fundamental domain a quadrilateral in the upper half-plane, with all four vertices on the (completed) real line and all four sides geodesics in the hyperbolic metric, that is to say arcs of semicircles orthogonal to the real line: the group elements A and B identify pairs of opposite sides of this quadrilateral, and the orbit space is a punctured torus. The cross-ratio of the four vertices of a fundamental domain is a conjugacy invariant of the group (as a subgroup of $PSL_2(\mathbb{R})$). Any two representations G_1 and G_2 of this kind (which in general will have different have different cross-ratios) provide examples of groups to which we may apply Bers' Theorem. We consider G_1 acting on the upper half-plane and G_2 acting on the lower (Figure 20). In general their actions will only match combinatorially on their common limit set, the completed real axis, but if we glue them together combinatorially the conclusion of Bers' theorem tells us that we can realise this topological mating as a *holomorphic* dynamical system, a Kleinian group (discrete subgroup of $PSL_2(\mathbb{C})$) which has as its limit set a quasicircle in place of $\hat{\mathbb{R}}$.

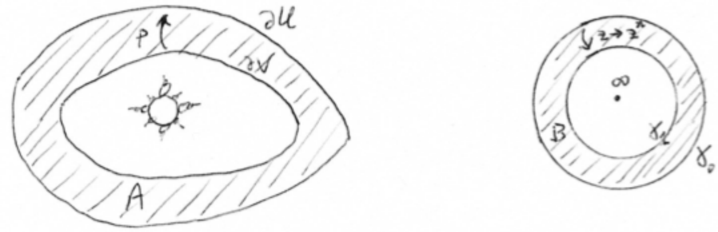

Figure 21: The Straightening Theorem

8.5 3rd Application: Polynomial-like mappings

In their paper *On the dynamics of polynomial-like mappings* (Ann Sci Ec Norm Sup 1985), Douady and Hubbard defined the notion of a *polynomial-like map*. This is a proper holomorphic surjection $p : V \to U$ where U and V are simply-connected open sets in $\mathbb{C}$ with $U \supset \overline{V}$. Such a map has a well-defined *filled Julia set* $K(p) = \bigcap_{n \geq 0} p^{-n}(\overline{V})$.

Theorem 8.4 (The Straightening Theorem of Douady and Hubbard) *For every polynomial-like mapping p there is a genuine polynomial map P which is hybrid equivalent to p.*

Here *hybrid equivalent* means that there is a quasiconformal conjugacy h between p and P on neighbourhoods of $K(p)$ and $K(P)$ such that the Beltrami form of h vanishes on $K(p)$ (so in particular h is conformal on $int(K(p))$. In the case that $K(p)$ is connected, we can think of this as a *mating* between the polynomial-like map p on its filled Julia set and the polynomial $z \to z^n$ on its filled Julia set.

At the heart of the proof of the Straightening Theorem is the Measurable Riemann Mapping Theorem. The idea is as follows. We suppose, for simplicity, that U and V are topological discs with smooth boundaries. A polynomial-like map p has a well-defined degree (the number of times p winds ∂V around ∂U, or equivalently the number of points in a generic $p^{-1}(z)$). Suppose this degree is n. $A = U - V$ is an annulus equipped with a map p of degree n from its inner boundary onto its outer boundary. Let B be the annulus between a circle γ_0 of radius $r > 1$ in $\mathbb{C}$ centred at the origin and its image γ_1 under $z \to z^n$. See Figure 21. It can be shown that it is possible construct a quasiconformal homeomorphism $h : A \to B$ conjugating the boundary map on A to the boundary map on B. Pull back the associated ellipse field on $U - V$ to an ellipse field on $U - K(p)$, by repeatedly applying p^{-1}, and extend it to the whole of U by using the standard field of round circles on $K(p)$. Now add the disc $D = \{z : |z| \geq r\} \cup \{\infty\}$ to U by pasting the annulus A to the annulus B using the quasiconformal homeomorphism h to do the pasting. The maps p on V and $z \to z^n$ on D fit neatly together to give a map a degree n map $Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which preserves the ellipse field. Applying the Measurable Riemann Mapping Theorem to the ellipse field yields a quasiconformal homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ straightening it to the field of round circles and now $fQf^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a genuine polynomial with the desired properties. QED

Tuning, renormalisation and baby Mandelbrot sets

It turns out that there are many parameter values c in the Mandelbrot set M where in some region of the dynamical plane the first return map $(q_c)^n$ is a quadratic-like map, which is then hybrid equivalent to some $q_{c'}$. This process of 'renormalisation' is the dynamical counterpart to the phenomenon of 'tuning' (replacing digits 0 and 1 in external ray addresses by finite strings of digits θ_- and θ_+), discussed briefly in Section 7; it follows from the Straightening Theorem that the Julia set of q_c then contains copies of $J(q_{c'})$. Families of quadratic-like first return maps satisfying certain conditions are called 'Mandelbrot-like families' by Douady and Hubbard in their paper (Ann Sci Ec Norm Sup 1985) : these give rise to 'baby Mandelbrot sets' - small copies of M on finer and finer scales. Indeed every point on the boundary of the Mandelbrot set M is an accumulation point

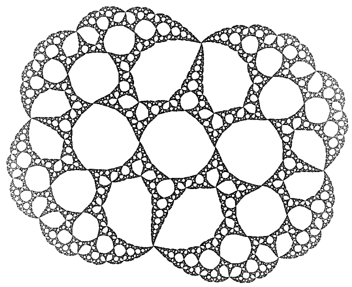

Figure 22: A quadratic rational map mating Douady's rabbit with $z \to z^2 - 1$

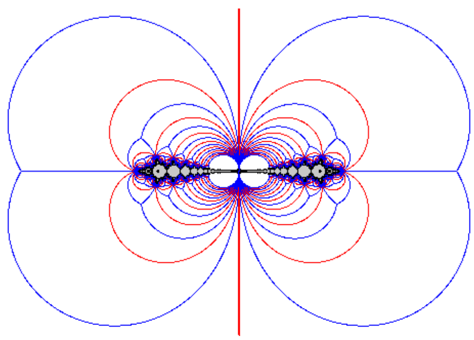

Figure 23: A holomorphic correspondence mating $z \to z^2 - 1$ with the modular group

of baby Mandelbrot sets (see McMullen's paper in the LMS Lecture Notes volume edited by Tan Lei).

Matings of polynomials

The construction described in the outline proof of Theorem 8.4 (the Straightening Theorem) can equally well be used to *mate* q_c with $q_{c'}$ for any pair c, c' in the interior of the cardioid M_0, that is to construct a *rational* map q of degree two which has Julia set $J(q)$ a quasicircle and which is conformally conjugate to q_c on one component of $\hat{\mathbb{C}} - J(q)$ and to $q_{c'}$ on the other component. A more challenging task is to mate q_c with $q_{c'}$ for c and c' elsewhere in the Mandelbrot set, as $J(q_c)$ and $J(q_{c'})$ are now *quotients* of the circle (in the case that they are locally connected) and of course in general they will be different quotients. Mary Rees and Tan Lei proved that *hyperbolic* quadratic polynomials $q_c, q_{c'}$ can be mated if and only if c' is not in the conjugate limb of M to that of c (the proof involves an application of Thurston's criterion for when a topological branched covering of the sphere is 'equivalent' to a rational map). See Figure 22. Finally we remark that if we extend our notions of rational maps and Kleinian groups to include *holomorphic correspondences*, it becomes possible to mate a hyperbolic quadratic polynomial with the modular group (Figure 23), but that is another story.... See the forthcoming book of Bodil Branner and Núria Fagella for the technical details of 'quasiconformal surgery' and many more applications.

The list of 'references' on the next page contains only books. Relevant journal articles have been referred to individually in the course of these Notes.

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