LECTURE 3. HYPERBOLIC GEOMETRY AND KLEINIAN GROUPS

3.1 The hyperbolic plane: half-plane and disc models, isometries

Around 300BC Euclid of Alexandria wrote a thirteen volume treatise entitled *The Elements*, in which he developed geometry and number theory from a set of *axioms*. His five axioms for geometry in the plane were:

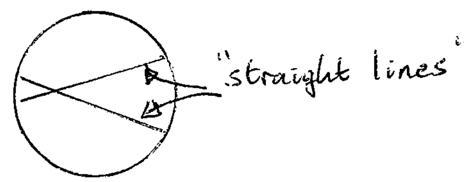
1. A straight line may be drawn from any point to any other point.
2. A finite straight line may be extended.
3. A circle may be drawn with any given centre and radius.
4. All right angles are equal.
5. If a straight line intersects two other straight lines and the sum of the interior angles on one side is less than two right angles, then the two straight lines, if extended indefinitely, meet on the side on which the sum of the angles is less than two right angles.

The fifth postulate is equivalent to:

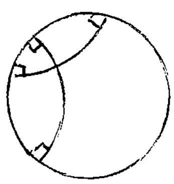
5′. Given any straight line and point not on it, there exists a unique straight line through the point, not meeting the given line.

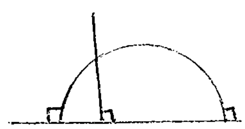
There were many attempts in the following two thousand years to show that the fifth axiom can be deduced from the first four. It appears to be Gauss who was the first to realise that there existed a geometry satisfying axioms 1 to 4 but not 5. He called this a *non-Euclidean geometry*, but though he investigated its properties for ten years in the early 19th century, he did not publish any of his results. It was Lobachevsky (1829) and Bolyai (1832) who first published the discovery of what we now call *hyperbolic geometry*, which has in place of Euclid's axiom that 'there exists a unique parallel', the new axiom that 'there exist infinitely many parallels' to a given line, through a given point not on it.

Beltrami (1868) introduced a Euclidean disc model of the hyperbolic plane



which he used to prove formally that Euclid's fifth axiom is independent of the first four. Klein (1871) gave an interpretation of this model in terms of projective geometry. Beltrami had also introduced conformal disc and upper half-plane models in 1868,





and Poincaré (1882) identified the congruences of the hyperbolic plane with the group $PSL(2, \mathbf{R})$ of the upper half-plane, the key to a host of subsequent developments in mathematics and physics in the subsequent century (from relativity to string theory).

The upper half-plane model

$$\mathcal{H}^2 = \mathcal{H}_+ = \{x + iy : x \in \mathbf{R}, y \in \mathbf{R}^{>0}\} \subset \mathbf{C}$$

Define an infinitesimal metric on $\mathcal{H}^+$ by

$$ds = \frac{1}{y}((dx)^2 + (dy)^2)^{1/2}$$

in other word the 'length' of a path γ in $\mathcal{H}_+$ is defined to be the integral of this quantity ds along γ.

Lemma 3.1 *ds is invariant under $PSL(2, \mathbf{R})$.*
Proof

$$z \to \frac{az+b}{cz+d} = \frac{a}{c} + \frac{r}{cz+d}$$

where $r = b - ad/c$, so it suffices to check invariance under the following three types of transformation: (i) $z \to z + \lambda \quad (\lambda \in \mathbf{R})$; (ii) $z \to \lambda z \quad (\lambda \in \mathbf{R}^{>0})$; (iii) $z \to -1/z$. This is an easy exercise. QED

Definition A path γ is called a *geodesic* from P to Q in $\mathcal{H}_+$ if it is a path of shortest length from P to Q. A proof of the following elementary proposition can be found in any textbook on hyperbolic geometry.

Proposition 3.2 *There is a unique geodesic between any two distinct points P and Q in $\mathcal{H}_+$. It is the segment between P and Q of the unique (Euclidean) semicircle through P and Q which meets $\hat{\mathbf{R}} = \mathbf{R} \cup \infty$ orthogonally. The (hyperbolic) distance from P to Q is $|\ln(P, Q; A, B)|$ where A and B are the points where the semicircle meets $\hat{\mathbf{R}}$.*

Any isometry of the hyperbolic plane $\mathcal{H}^2$ must send geodesics to geodesics, and so in the upper half plane model it must send semicircles orthogonal to $\hat{\mathbf{R}}$ to semicircles orthogonal to $\hat{\mathbf{R}}$. It can be shown that such transformations must have *either* the form

$$z \to \frac{az+b}{cz+d} \quad a, b, c, d \in \mathbf{R}, ad - bc > 0$$

or the form

$$z \to \frac{a\bar{z}+b}{c\bar{z}+d} \quad a, b, c, d \in \mathbf{R}, ad - bc < 0$$

An example of the second type is $z \to -\bar{z}$ (reflection in the imaginary axis).

The disc model

An alternative model to the upper half plane is given by regarding $\mathcal{H}^2$ as the points of the unit disc $D^2 = \{z : |z| < 1\} \subset \mathbf{C}$ and taking as the infinitesimal metric on this disc

$$ds = \frac{((dx)^2 + (dy)^2)^{1/2}}{1 - r^2} \qquad (r^2 = x^2 + y^2)$$

in which case the geodesics are arcs of circles meeting the unit circle (the boundary of D^2) orthogonally. To pass from the upper half plane model to the disc model to the disc model we simply conjugate by any element of $PSL(2, \mathbf{C})$ which sends $\mathcal{H}_+$ to D^2, for example the map

$$z \to \frac{iz + 1}{z + i}$$

which sends $-1, 0, 1$ to $-1, -i, 1$ respectively and hence sends $\hat{\mathbf{R}}$ to the unit circle (and moreover sends the upper half plane to the interior of this circle since it sends i to 0).

The *orientation-preserving isometries* of the hyperbolic plane are the elements of $PSL(2, \mathbf{R})$, acting as fractional linear maps

$$z \to \frac{az + b}{cz + d}$$

in the upper half-plane model. We use their fixed points to classify them into types.

Definition A non-identity element $\alpha \in PSL(2, \mathbf{R})$ is said to be
elliptic if it has a fixed point in $\mathcal{H}_+$;
parabolic if it has precisely one fixed point on $\hat{\mathbf{R}}$;
hyperbolic if has two fixed points on $\hat{\mathbf{R}}$.

If we normalise our matrix representing $\alpha \in PSL(2, \mathbf{R})$ so that $ad - bc = 1$, we can distinguish the three types by the *trace*, $a + d$ of α as follows. The fixed points of α are the solutions of the equation

$$cz^2 + (d + a)z - b = 0$$

This has a complex conjugate pair of roots $\Leftrightarrow (d + a)^2 + 4bc < 0 \Leftrightarrow (d + a)^2 - 4 < 0 \Leftrightarrow |tr(\alpha)| < 2$, it has one (repeated) real root $\Leftrightarrow |tr(\alpha)| = 2$, and it has two (distinct) real roots $\Leftrightarrow |tr(\alpha)| > 2$. Thus

Lemma 3.3 *α is elliptic/parabolic/hyperbolic* $\Leftrightarrow |tr(\alpha)| < 2, = 2, > 2$

(Note that we must normalise α to determinant 1 before we compute the trace.)

The trace of a matrix is a conjugacy invariant, and hence so is the *type* of an isometry of the hyerbolic plane (this is also obvious from the definition of *type* in terms of fixed points). In calculations and proofs it can often be useful to conjugate an isometry to a standard form. The following can easily be verified:

Lemma 3.4 *If $\alpha \in PSL(2, \mathbf{R})$ is parabolic then α is conjugate (in $PSL(2, \mathbf{R})$) to $z \to z+1$ or to $z \to z - 1$.*

Lemma 3.5 *In the disc model, the elliptic elements fixing the origin* 0 *are the Euclidean rotations of the disc.*

Lemma 3.6 *If* $\alpha \in PSL(2, \mathbf{R})$ *is hyperbolic then* α *is conjugate (in* $PSL(2, \mathbf{R})$*) to* $z \to \lambda z$ *for some* $\lambda \in \mathbf{R}^{>0}$.

We may change λ to λ^{-1} by further conjugating our map by $z \to 1/z$ (interchanging 0 and ∞), but since the eigenvalues of α are λ and λ^{-1}, and these are conjugacy invariants of α, the value of λ in Lemma 3.6 is unique up to replacement by λ^{-1}.

3.2 Hyperbolic 3-space and its isometries

Definition $\mathcal{H}^3 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 > 0\}$

Just as in the two-dimensional case we may define an infinitesimal metric:

$$ds = \frac{1}{x_3}((dx_1)^2 + (dx_2)^2 + (dx_3)^2)^{1/2}$$

With this metric $\mathcal{H}^3$ becomes the *upper half-space model of hyperbolic* 3*-space.* The geodesics are the semicircles in $\mathcal{H}^3$ orthogonal to the plane $x_3 = 0$.

Now think of the plane $x_3 = 0$ in $\mathbf{R}^3$ as the complex plane $\mathbf{C}$ ($(x_1, x_2, 0) \leftrightarrow x_1 + ix_2$), add the point '$\infty$', and think of $\hat{\mathbf{C}}$ as the *boundary* of $\mathcal{H}^3$. Every fractional linear map

$$\alpha : z \to \frac{az + b}{cz + d} \quad (a, b, c, d \in \mathbf{C}, ad - bc \neq 0)$$

mapping $\hat{\mathbf{C}}$ to $\hat{\mathbf{C}}$, has an extension to an isometry from $\mathcal{H}^3$ to $\mathcal{H}^3$. One way to see this is to break down α into a composition of maps of the form

(i) $$z \to z + \lambda \quad (\lambda \in \mathbf{C})$$

(ii) $$z \to \lambda z \quad (\lambda \in \mathbf{C})$$

(iii) $$z \to -1/z$$

We extend these as follows on $\mathcal{H}^3$ (where z denotes $x_1 + ix_2$):

(i) $$(z, x_3) \to (z + \lambda, x_3)$$

(ii) $$(z, x_3) \to (\lambda z, |\lambda| x_3)$$

(iii) $$(z, x_3) \to \left(\frac{-\bar{z}}{|z|^2 + x_3^2}, \frac{x_3}{|z|^2 + x_3^2}\right)$$

The expressions above come from decomposing the action on $\hat{\mathbf{C}}$ of each of the elements of $PSL(2, \mathbf{C})$ in question into two *inversions* (reflections) in circles in $\hat{\mathbf{C}}$. Each such inversion has a unique extension to $\mathcal{H}^3$ as an inversion in the hemisphere spanned by the circle and composing appropriate pairs of inversions gives us these formulae. It is now an exercise along the lines of Lemma 3.1 to show that $PSL(2, \mathbf{C})$ preserves the metric ds on $\mathcal{H}^3$ and another exercise, along the lines of Lemma 3.2 to show that the geodesics are the arcs of semicircles as claimed. Moreover every isometry of $\mathcal{H}^3$ can be seen to be the extension of a conformal map of $\hat{\mathbf{C}}$ to itself, since it sends hemispheres orthogonal to $\hat{\mathbf{C}}$ to hemispheres orthogonal to $\hat{\mathbf{C}}$, hence circles in $\hat{\mathbf{C}}$ to circles in $\hat{\mathbf{C}}$. Thus all orientation-preserving isometries of $\mathcal{H}^3$ are given by elements of $PSL(2, \mathbf{C})$ acting as above, and all orientation-reversing isometries are extensions of anti-holomorphic Möbius transformations of $\hat{\mathbf{C}}$.

Comments

1. The fact that the orientation-preserving isometry group of $\mathcal{H}^3$ is $PSL(2, \mathbf{C})$ was first observed by Poincaré.

2. The *disc model* for hyperbolic three-space is the interior D of the unit disc in Euclidean three-space $\mathbf{R}^3$, equipped with the metric

$$ds = \frac{((dx_1)^2 + (dx_2)^2 + (dx_3)^2))^{1/2}}{1 - r^2}$$

(where $r^2 = x_1^2 + x_2^2 + x_3^2$). Geodesics are arcs of circles orthogonal to the boundary sphere.

3. One can construct higher dimensional hyperbolic spaces $\mathcal{H}^n$ in the analagous way. In each case the *conformal* transformations of the boundary extend uniquely to give the *isometries* of the interior.

Types of isometries of hyperbolic 3-space

Non-identity elements $\alpha \in PSL(2, \mathbf{C})$ are of four types.

Definition α is said to be

elliptic $\Leftrightarrow$ α fixes some geodesic in $\mathcal{H}^3$ pointwise;

parabolic $\Leftrightarrow$ α has a single fixed point in $\hat{\mathbf{C}}$;

hyperbolic $\Leftrightarrow$ α has two fixed points in $\hat{\mathbf{C}}$, no fixed points in $\mathcal{H}^3$, and every hyperplane in $\mathcal{H}^3$ which contains the geodesic joining the two fixed points in $\hat{\mathbf{C}}$ is invariant (mapped to itself) under α;

loxodromic $\Leftrightarrow$ α has two fixed points in $\hat{\mathbf{C}}$, no fixed points in $\mathcal{H}^3$, and no invariant hyperplane in $\mathcal{H}^3$.

Note The distinction between *hyperbolic* and *loxodromic* is not always made: some authors use either word for an isometry having two fixed points in $\hat{\mathbf{C}}$ and none in $\mathcal{H}^3$.

Lemma 3.7 *α is elliptic/parabolic/hyperbolic/loxodromic*
$\Leftrightarrow (tr(\alpha))^2 \in [0, 4) \subset \mathbf{R}^{\geq 0}, = 4, \in \mathbf{R}^{\geq 0} - [0, 4), \in \mathbf{C} - \mathbf{R}^{\geq 0}$

Proof

If α has two fixed points in $\hat{\mathbf{C}}$ we may assume (after conjugating α by an appropriate Möbius transformation) they are at 0 and ∞ and that α has the form $z \to \lambda z$ (and $tr(\alpha) = \lambda^{1/2} + \lambda^{-1/2}$).

Case 1: $|\lambda| = 1$, say $\lambda = e^{i\theta}$. Then on $\hat{\mathbf{C}}$ α is a rotation about 0 through an angle θ, and fixes the x_3-axis in $\mathcal{H}^3$ pointwise. As a matrix, normalised to determinant 1,

$$\alpha = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

and so $(tr(\alpha))^2 = 4\cos^2(\theta/2) \in [0, 4)$.

Case 2: $|\lambda| \neq 1$. then α acts on the x_3-axis in $\mathcal{H}^3$ as multiplication by $|\lambda|$. Writing $\lambda = |\lambda|e^{i\theta}$ we have

$$\alpha = \begin{pmatrix} |\lambda|^{1/2}e^{i\theta/2} & 0 \\ 0 & |\lambda|^{-1/2}e^{-i\theta/2} \end{pmatrix}$$

so $(tr(\alpha))^2 \in \mathbf{C} - [0, 4]$. Now if λ is real (i.e. $\theta = 0$ or π) α is hyperbolic and $(tr(\alpha))^2 \in \mathbf{R}^{\geq 0} - [0, 4]$ and if λ is not real, α is loxodromic and $(tr(\alpha))^2 \in \mathbf{C} - \mathbf{R}^{\geq 0}$.

Finally if α has a single fixed point in $\hat{\mathbf{C}}$ then we can place this fixed point at ∞ (by conjugating α if necessary) in which case α has the form $z \to z + \lambda$ (indeed we may even conjugate it to $z \to z + 1$). Then α is parabolic and $(tr(\alpha))^2 = 4$. QED.

Dynamics of Möbius transformations on $\mathcal{H}^3 \cup \hat{\mathbf{C}}$

$z \to e^{2\pi i\theta} z$ (θ real)

Here the fixed points $0, \infty$ on $\hat{\mathbf{C}}$ are *neutral*. For $z \to e^{i\theta}z$ with θ real, all orbits on $\mathcal{H}^3$ have finite period if θ is a rational multiple of π, and densely fill circles around the x_3 axis if not.

$z \to ke^{2\pi i\theta} z$ ($k > 1, \theta$ real)

Here all orbits in $\mathcal{H}^3$ head away from a repelling fixed point 0 and towards an attracting fixed point ∞, spiralling around the x_3 axis as they go. The nature of the spiralling depends on θ: in particular if $\theta = 0$ or π each orbit remains in a hyperplane.

$z \to z + 1$

In this example the (unique) fixed point ∞ is neutral (multiplier 1) and all orbits on $\mathcal{H}^3$ head towards the fixed point under both forward and backward time. Any parabolic map α will have this behaviour.

3.3 Kleinian groups, ordinary and limit sets, and their properties

Definition A *Kleinian group* is a *discrete* subgroup $G < PSL(2, \mathbf{C})$.

Thus for a subgroup $G < PSL(2, \mathbf{C})$ to be called Kleinian we require that there be no sequence $\{g_n\}$ of distinct elements of G tending to a limit $g \in PSL(2, \mathbf{C})$. Here the topology on $PSL(2, \mathbf{C})$ is that induced by the norm

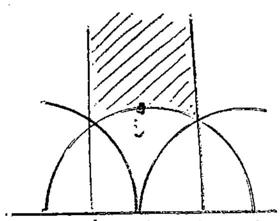
$$\left|\left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|\right| = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}$$

on $SL(2, \mathbf{C})$ (so that two elements of $PSL(2, \mathbf{C})$ are close together if and only if they are representable by $A_1, A_2 \in SL(2, \mathbf{C})$ with $||A_2 - A_1||$ small).

Note If G is discrete then for any $N > 0$ the number of elements of G having norm $\leq N$ is *finite*, since every infinite sequence with bounded norm has a convergent subsequence. Hence every discrete G is *countable.*

Definition The action of G is *discontinuous* at $z \in \hat{\mathbf{C}}$ if there exists a neighbourhood U of z such that $g(U) \cap U = \emptyset$ for all but finitely many $g \in G$.

Example



$G = PSL(2, \mathbf{Z})$ acts discontinuously on $\hat{\mathbf{C}} - \hat{\mathbf{R}}$. For z in the shaded region above, each $z \neq i, \pm 1/2 + i\sqrt{3}/2$ has a neighbourhood U such that $g(U) \cap U = \emptyset$ for all non-identity $g \in G$, the point $z = i$ has a neighbourhood U such that $g(U) \cap U = \emptyset$ for all $g \in G - \{I, S\}$ where $S : z \to -1/z$, and the point $z = -1/2 + i\sqrt{3}/2$ has a neighbourhood U such that $g(U) \cap U = \emptyset$ for all $g \in G - \{I, ST, (ST)^2\}$ where $ST : z \to -1/(z+1)$, etc.

Definition The set of all $z \in \hat{\mathbf{C}}$ at which the action of G is discontinuous is called the *regular* (or *ordinary* or *discontinuity*) set $\Omega(G)$.

Comments

1. It follows at once from the definition that $\Omega(G)$ is *open* and G*-invariant.*
2. In the example above observe that the origin 0 is not in $\Omega(G)$, since any U containing 0 has $g(U) \cap U \neq \emptyset$ for all

$$g = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

with n sufficiently large. In fact in this example $\Omega(G) = \hat{\mathbf{C}} - \hat{\mathbf{R}}$ (as we shall prove later).

A subgroup $G < PSL(2, \mathbf{C})$ acts on $\mathcal{H}^3$ as well as on its boundary $\hat{\mathbf{C}}$. The following theorem establishes an important relationship between these actions.

Theorem 3.8 *A subgroup $G < PSL(2, \mathbf{C})$ is discrete if and only if it acts discontinuously on $\mathcal{H}^3$*

Proof. If G is not discrete there exists $\{g_n\} \in G$ with limit $g \in PSL(2, \mathbf{C})$. So for all $x \in \mathcal{H}^3$, $g_m^{-1} g_n(x) \to x$ as $m, n \to \infty$. Thus for any $x \in \mathcal{H}^3$ and neighbourhood U of x, for m and n sufficiently large $g_m^{-1} g_n(U) \cap U \neq \emptyset$. Hence G does not act discontinuously at x.

Conversely, if G does not act discontinuously at $x \in \mathcal{H}^3$, then for any neighbourhood U of x there exist a sequence $\{x_n\} \in U$ and (distinct) $g_n \in G$ such that each $g_n(x_n) \in U$. Take U compact. Then by passing to subsequences we may assume the x_n tend to a point y and the $g_n(x_n)$ tend to a point z (with both y and z in U). Now let k be an isometry of $\mathcal{H}^3$ having $k(z) = y$ and let $\{h_n\}$, $\{j_n\}$ be sequences of isometries, both tending to the identity, and having $h_n(y) = x_n$ and $j_n g_n(x_n) = z$ respectively. Consider $f_n = k j_n g_n h_n$. For each n this fixes y (by construction). But the isometries of $\mathcal{H}^3$ fixing a common point of $\mathcal{H}^3$ are a compact group (the Euclidean rotations, in the Poincaré disc model with the common point the origin). Hence the $\{f_n\}$ have a convergent subsequence. Hence so do the $\{g_n\}$, in other words G is not discrete. QED

Limit sets of Kleinian groups

One can define the notion of the *limit set* $\Lambda(G)$ of a Kleinian group G, either in terms of its action on $\mathcal{H}^3$, or in terms of the action on the boundary $\hat{\mathbf{C}}$ of $\mathcal{H}^3$. We shall see later that the two definitions are equivalent.

Definition 1. Let x be any point of $\mathcal{H}^3$. Then set

$$\Lambda(x) = \{w \in \hat{\mathbf{C}} : \exists g_n \in G \quad \text{with} \quad g_n(x) \to w \quad \text{as} \quad n \to \infty\}$$

(where convergence is taken in the Euclidean metric on the Poincaré disc model of $\mathcal{H}^3$). Note that the $\{g_n(x)\}$ cannot have accumulation points in $\mathcal{H}^3$, since G acts discontinuously there. Thus an alternative description of $\Lambda(x)$ is as the accumulation set in $\mathcal{H}^3 \cup \hat{\mathbf{C}}$ of the orbit Gx on $\mathcal{H}^3$. This accumulation set is independent of the initial point $x \in \mathcal{H}^3$, since if we choose another initial point y the hyperbolic distance from $g(x)$ to $g(y)$ is constant for all g and therefore the *Euclidean* distance from $g(x)$ to $g(y)$ tends to zero as $g(x)$ and $g(y)$ approach the boundary $\hat{\mathbf{C}}$ of the Poincaré disc. We *define* $\Lambda(G)$ to be $\Lambda(x)$ for any $x \in \mathcal{H}^3$.

Definition 2. Let z be any point of $\hat{\mathbf{C}}$. Set

$$\Lambda(z) = \{w \in \hat{\mathbf{C}} : \exists g_n \in G \quad \text{with} \quad g_n(z) \to w \quad \text{as} \quad n \to \infty\}$$

(where convergence is taken in the spherical metric on $\hat{\mathbf{C}}$). It can be shown that when G is *non-elementary* (see below for definition) $\Lambda(z)$ is independent of $z \in \hat{\mathbf{C}}$. We define $\Lambda(G)$ to be $\Lambda(z)$ for any $z \in \hat{\mathbf{C}}$.

Comments

1. The restriction that G be 'non-elementary' is included in definition 2 in order to exclude just one class of examples where the limit $\Lambda(z)$ depends on z. Consider $G = \{g^n : n \in \mathbf{Z}\}$, where g is loxodromic, with fixed points z_0 and z_1. The limit set by definition 1 is $\Lambda(G) =$

$\{z_0\} \cup \{z_1\}$, but definition 2 gives $\Lambda(z_0) = z_0$, $\Lambda(z_1) = z_1$ (although $\Lambda(z) = \{z_0\} \cup \{z_1\}$ for any other choice of z).

2. We shall adopt definition 2 until we have proved the equivalence of the two notions (later in this section). Meanwhile we remark that the underlying reason that the definitions are equivalent is that to an observer inside $\mathcal{H}^3$ an orbit of G of $\mathcal{H}^3$ is viewed as accumulating at $\Lambda(G)$ on the 'visual sphere' $\hat{\mathbf{C}}$.

3. A third equivalent definition is that $\Lambda(G)$ consists of the points $z \in \hat{\mathbf{C}}$ where the family $g \in G$ fail to be a normal family (with respect, as always, to the spherical metric). We shall prove this too later in the present section.

4. It follows at once from definition 2 (or indeed from definition 1) that $\Lambda(G)$ is both *closed* and *G-invariant.*

It is clear from the definitions of $\Omega(G)$ and $\Lambda(G)$ that $\Omega(G) \cap \Lambda(G) = \emptyset$, but we shall prove the stronger statement that $\Lambda(G)$ is the *complement* of $\Omega(G)$ in $\hat{\mathbf{C}}$. First we deal with some special cases.

Elementary Kleinian groups

Definition A Kleinian group G is called *elementary* if there exists a finite G orbit on either $\mathcal{H}^3$ or $\hat{\mathbf{C}}$.

All elementary Kleinian groups G belong to the following three classes. For a proof see for example Beardon's book 'Geometry of Discrete Groups' or Ratcliffe's book 'Foundations of Hyperbolic Manifolds.'

(i) G is conjugate to a finite subgroup of $SO(3)$ acting on the Poincaré disc by rigid rotations fixing the origin (for example the symmetry group of a regular solid). In this case $\Lambda(G) = \emptyset$.

(ii) G is conjugate to a discrete group of Euclidean motions of $\mathbf{C}$ (i.e. fixing $\infty \in \hat{\mathbf{C}}$) (for example $G = < z \to z+1, z \to z+i >$). Then $|\Lambda(G)| = 1$.

(iii) G is conjugate to a group all of the elements of which are of the form $z \to kz$ or $z \to k/z$ for $k \in \mathbf{C}$. Then $|\Lambda(G)| = 2$.

It is not hard to see that if G is Kleinian then $\Lambda(G) = \emptyset \Rightarrow G$ elementary of type (i), $|\Lambda(G)| = 1 \Rightarrow G$ elementary of type (ii), and $|\Lambda(G)| = 2 \Rightarrow G$ elementary of type (iii), so elementary groups are characterised by the size of their limit sets. Indeed

Proposition 3.9 *A Kleinian group G is elementary if and only $|\Lambda(G)| \leq 2$, and non-elementary if and only if $\Lambda(G)$ is infinite.*

Proof. If $\Lambda(G)$ is finite and non-empty then any G orbit in $\Lambda(G)$ is a finite G orbit on $\hat{\mathbf{C}}$ so G is elementary by definition and has $|\Lambda(G)| = 1$ or 2 by the above classification. QED

We state, without proof, the following properties of ordinary and limit sets of Kleinian groups:

Theorem 3.10 *Any Kleinian group G acts discontinuously on $\hat{\mathbf{C}} - \Lambda(G)$. Hence $\hat{\mathbf{C}}$ is the disjoint union of $\Omega(G)$ and $\Lambda(G)$.*

Proposition 3.11 *Let G be a non-elementary Kleinian group. Then any non-empty closed G-invariant subset S of $\hat{\mathbf{C}}$ contains $\Lambda(G)$*

Corollary 3.12 *Let G be a Kleinian group. Then either $\Lambda(G) = \hat{\mathbf{C}}$ or $\Lambda(G)$ has empty interior.*

Corollary 3.13 *Let G be a non-elementary Kleinian group. Then $\Lambda(G)$ is the closure of the set of all fixed points of loxodromic and hyperbolic elements of G.*

Comment. If G has any parabolic elements their fixed points must lie in $\Lambda(G)$, but elliptic elements may have fixed points in either $\Omega(G)$ or $\Lambda(G)$.

Corollary 3.14 *Let G be a non-elementary Kleinian group. Then $\Lambda(G)$ is perfect (and hence, in particular, uncountable).*

Corollary 3.15 *Definitions 1 and 2 for the limit set $\Lambda(G)$ of a non-elementary Kleinian group G are equivalent.*

Proof. We show that the limit set as defined by definition 1 has exactly the same characterising property as that specified by Proposition 3.11 for $\Lambda(G)$ (where we used definition 2). Let S be any closed G-invariant subset of $\hat{\mathbf{C}}$ (note that S must be infinite, since G is non-elementary). Then $C(S)$, the convex hull of S in $\mathcal{H}^3 \cup \hat{\mathbf{C}}$, is also closed and G-invariant. Take any $x \in C(S) \cap \mathcal{H}^3$. Its orbit Gx is contained in $C(S)$ and the accumulation set of this orbit is contained in $C(S) \cap \hat{\mathbf{C}} = S$. Hence S contains the definition 1 limit set of G. QED

The results stated above for ordinary and limit sets of Kleinian groups exhibit a very close analogy with our earlier results on Fatou and Julia sets for rational maps. This raises the question as to whether we can make the *definitions* analogous too. The answer is yes.

Proposition 3.16 *Let G be a Kleinian group. Then $\Omega(G)$ is the largest open subset of $\hat{\mathbf{C}}$ on which the elements of G form an equicontinuous family.*

Proof. Assume G non-elementary (as usual elementary groups can be dealt with on a case by case basis). Then $\Lambda(G)$ contains at least three points (in fact infinitely many) so $\Omega(G)$ is contained in the equicontinuity set by Montel's Theorem. But given any $z \in \Lambda(G)$, by Corollary 3.13 there must be a repelling fixed point of some $g \in G$ arbitrarily close to z, so the family of maps G cannot be equicontinuous at z. QED

We deduce the following two consequences (useful for plotting $\Lambda(G)$).

Theorem 3.17 *Let G be a non-elementary Kleinian group, and U be any open subset of $\hat{\mathbf{C}}$ meeting $\Lambda(G)$. Then*

$$\bigcup_{g \in G} gU = \hat{\mathbf{C}}$$

Proof. The union $\bigcup_{g \in G} gU$ covers all of $\hat{\mathbf{C}}$ except at most two points (else the family G would be equicontinuous on U by Montel's Theorem). But the complement of this union is a finite G-invariant set and therefore empty (since G is non-elementary). QED

The following corollary is immediate.

Corollary 3.18 *Let G be a non-elementary Kleinian group, and U be any open subset of $\hat{\mathbf{C}}$ meeting $\Lambda(G)$. Then*

$$\bigcup_{g \in G} g(U \cap \Lambda(G)) = \Lambda(G)$$