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ONE-DIMENSIONAL COMPLEX DYNAMICS

Shaun Bullett
School of Mathematical Sciences
Queen Mary,University of London
Mile End Road
London E1 4NS

CONTENTS

1. Introduction
    1.1 Examples of the behaviour of quadratic maps
    1.2 The Riemann sphere and rational maps: basic essentials from complex analysis
    1.3 Conjugacies, fixed points,and periodic orbits
    1.4 Fatou and Julia sets: equicontinuity, normal families, and Montel's Theorem

2. Properties of Fatou and Julia sets
    2.1 Julia sets: properties and characterizations
    3.2 Counting components of the Fatou set
    3.3 Linearization theorems

3. Hyperbolic geometry and Kleinian groups
    3.1 The hyperbolic plane: half-plane and disc models, isometries
    3.2 Hyperbolic 3-space and its isometries
    3.3 Kleinian groups, ordinary and limit sets, and their properties
    3.4 Fundamental domains for Kleinian groups, Poincaré's polyhedron theorem
    3.5 Examples of Fuchsian and Kleinian groups

4. Quadratic maps and the Mandelbrot Set
    4.1 The Mandelbrot set and its connectivity
    4.2 The geography of the Mandelbrot set
    4.3 Internal and external rays: the 'devil's staircase'
    4.4 The combinatorial Mandelbrot set
    4.5 Kneading sequences and internal addresses in the Mandelbrot set
    4.6 Further topics: polynomial-like maps and their applications (brief introduction)
    4.7 Suggestions for further reading

LECTURE 1. INTRODUCTION

The study of iterated complex maps had its first great flowering with the the work of the French mathematicians Julia and Fatou around 1918-20, though its origins perhaps lie earlier, in the late 19th century, in the more geometric work of Schottky, Poincaré, Fricke and Klein. It has had its second great flowering over the last 25 years, motivated partly by the spectacular computer pictures which started to appear from about 1980 onwards, partly by the explosive growth in the subject of chaotic dynamics which started about the same time, and not least by the revolutionary work in three-dimensional hyperbolic geometry initiated by Thurston in the early 1980's. Some of the names associated with this second wave of activity are Mandelbrot, Douady, Hubbard, Sullivan, Milnor, Thurston, Yoccoz and McMullen. The subject is still very much in a ferment of activity: as we shall see, some of the major conjectures are still waiting to be proved. But the methods are powerful: for example the only conceptual analytic proof of the universality of the Feigenbaum ratios for period doubling in real unimodal maps is that of Sullivan (1992) using techniques from complex iteration theory.

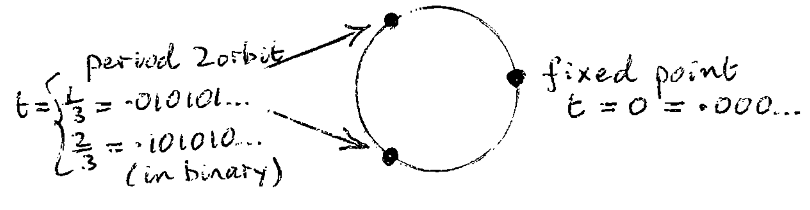
The objective of these lecture notes is to give a brief introduction to the remarkable mixture of complex analysis, hyperbolic geometry and symbolic dynamics that constitutes the subject of complex dynamics. The idea is to give the flavour of the subject, outline some of the main techniques (without detailed proof) and discuss some of the main theorems and open conjectures. As we proceed, we shall also see connections with the symbolic dynamics of maps of the both the real interval and the circle: the complex world is ideal for 'unfolding' problems in the real world.

1.1 Examples of the behaviour of quadratic maps $z \to z^2 + c$

(i) $c = 0$

Here the dynamical behaviour is straightforward. When we iterate $z \to z^2$ any orbit started inside the unit circle heads towards the point 0, any orbit started outside the unit circle heads towards ∞, and any orbit started on the unit circle remains there. The two components of $\{z : |z| \neq 1\}$ are known as the *Fatou set* of the map and the circle $|z| = 1$ is called the *Julia set.*

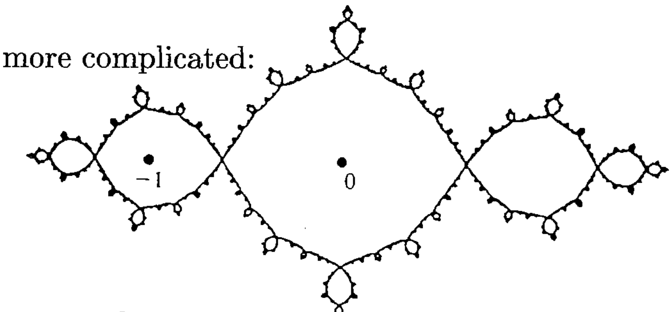
On the unit circle itself the dynamics are those of the *shift.*
Parametrise the circle by $t \in [0, 1) \subset \mathbf{R}$ $\quad (t = arg(z)/2\pi)$: then $z \to z^2$ sends $t \to 2t$ mod 1.



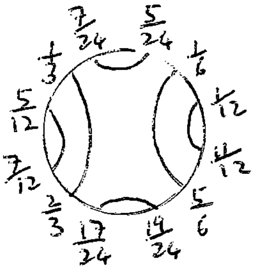
Any $t \in [0, 1)$ of the form $t = p/(2^n - 1)$ (for p integer) is periodic, of period n. Hence the periodic points form a dense set on the unit circle; moreover that the map $z \to z^2$ has *sensitive dependence on initial condition*, since it doubles arguments.

(ii) $c = -1$

Here the dynamical behaviour is much more complicated:



The Fatou set has infinitely many components this time. There is an attracting fixed point at ∞ to which every orbit started in the component of the Fatou set outside the 'filled-in Julia set' is attracted, and a period 2 cycle $0 \to -1 \to 0 \to -1 \to \ldots$ towards which every orbit started in any other component of the Fatou set is attracted. An orbit which starts on the common boundary of the two regions (the 'Julia set', which we shall define formally soon) remains on that boundary. Combinatorially, the Julia set in this example is a *quotient* of the circle, and the dynamics are those of the corresponding *quotient* of the shift. The figure below shows the first few identifications on the unit circle in the construction of this quotient.

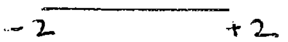


(iii) $c = i$

(For an illustration see the middle picture on the right-hand side on the next page.)

Note that the point 0 is *preperiodic* for this map ($0 \to i \to -1+i \to -i \to -1+i \ldots$). It can be proved that whenever c is such that the critical point 0 of $z \to z^2 + c$ is preperiodic but not periodic, the Julia set is a *dendrite* (that is a connected, simply-connected set with empty interior).

(iv) $c = -2$

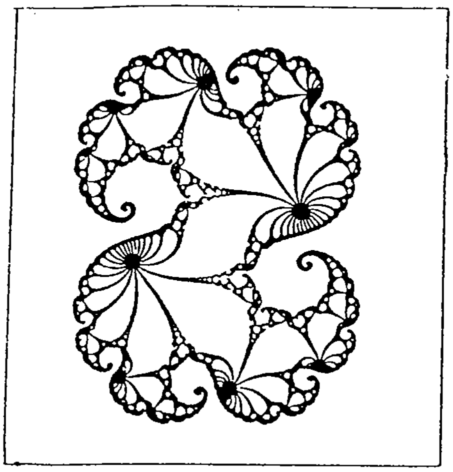


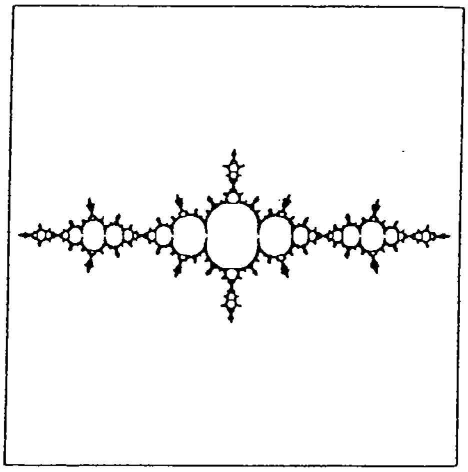
Here again 0 is preperiodic, and the dendrite is a particularly simple one, the real interval $[-2, 2]$.

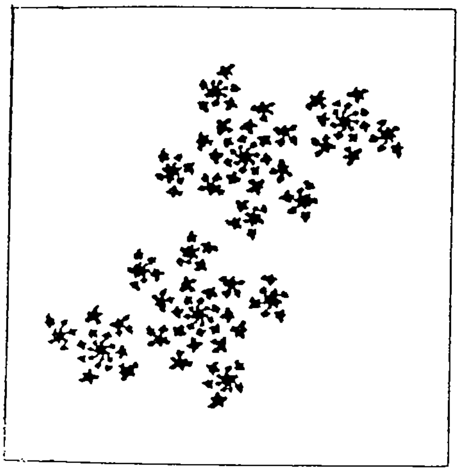
Exercise

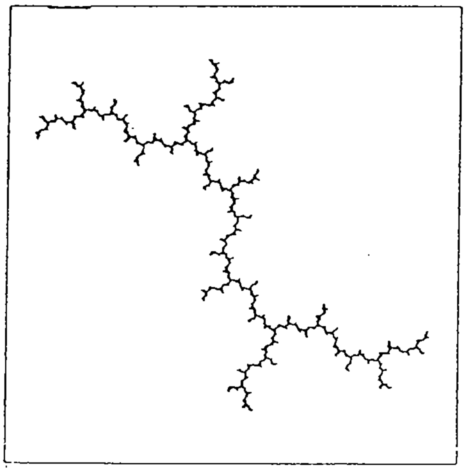
Show that the map $h : z \to z + 1/z$ is a semiconjugacy from $f : z \to z^2$ to $g : z \to z^2 - 2$ (that is, h is a surjection satisfying $hf = gh$) and that h sends the Julia set of f (the unit circle) onto the real interval $[-2, +2]$.

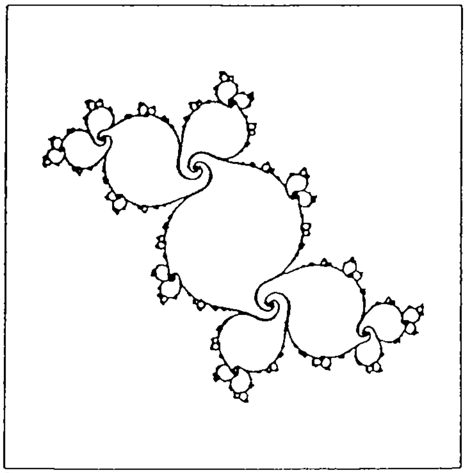
Typical Julia sets of quadratic maps $z \to z^2 + c$

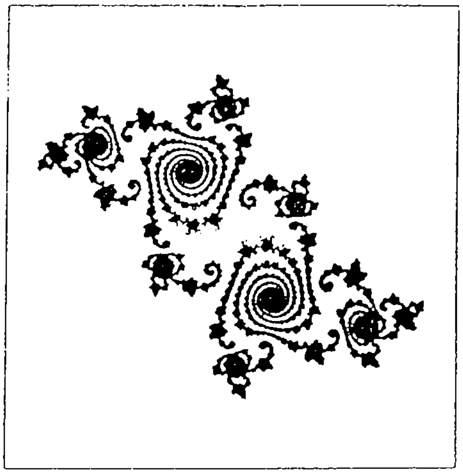












1.2 The Riemann sphere and rational maps: basic essentials from complex analysis

We summarise some basic results of complex analysis before giving the formal definitions of Julia and Fatou sets and beginning the study of their main properties.

The Riemann sphere and rational maps

The *Riemann sphere* is the complex projective line,

$$\mathbf{CP}^1 = \{\mathbf{C}^2 - (0,0)\}/\mathcal{R}$$

where $\mathcal{R}$ is the relation $(z,w) \sim (\lambda z, \lambda w)$ for $\lambda \in \mathbf{C} - 0$. Any equivalence class $[z,w]$ contains $(z/w, 1)$ if $w \neq 0$ or $(1, w/z)$ if $z \neq 0$, so we may think of the Riemann sphere as the union of two copies of the complex plane glued together, $\mathbf{C}_1 \cup \mathbf{C}_2/(z_1 \sim 1/z_2)$, or even more simply as the *extended complex plane* $\hat{\mathbf{C}} = \mathbf{C} \cup \infty$. The bijection

$$\mathbf{CP}^1 \leftrightarrow \hat{\mathbf{C}}$$

is given by $[z,w] \leftrightarrow z/w$ when $w \neq 0$ and $[z,0] \leftrightarrow \infty$. Yet another way to picture the Riemann sphere is as the unit sphere S^2 in $\mathbf{R}^3$: if we remove the north pole $N = (0,0,1)$ the remainder of S^2 maps bijectively onto the $(x,y,0)$-plane under *stereographic projection* from N, and sending $N \to \infty$ completes this to a bijection $S^2 \leftrightarrow \mathbf{C} \cup \infty = \hat{\mathbf{C}}$. With this picture we can define the *spherical metric* on $\hat{\mathbf{C}}$, corresponding to the usual metric on the unit sphere. In what follows it will usually be most convenient to think of the Riemann sphere as $\hat{\mathbf{C}} = \mathbf{C} \cup \infty$, but it might sometimes also be helpful to think in terms of one of the other definitions.

We next want to define what we mean by *differentiable maps* from the Riemann sphere to itself. We approach the definition in stages, recalling some terminolgy from complex analysis.

An open connected set $\Omega \subset \mathbf{C}$ is called a *domain.*

$f : \Omega \to \mathbf{C}$ is called *differentiable* if for each $z_0 \in \Omega$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If f is differentiable then for each $z_0 \in \Omega$ there is a disc neighbourhood of z_0 on which the value of the function $f(z)$ is equal the sum of the Taylor series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ for f at z_0 (Taylor's Theorem). For this reason a differentiable function is often called an *analytic* function.

If $f'(z_0) \neq 0$, then near z_0 we have $f(z) \sim f(z_0) + f'(z_0)(z - z_0)$ so f acts on $z - z_0$ by multiplying it by the scaling factor $|f'(z_0)|$ and turning it through an angle $arg(f'(z_0))$. Thus in particular if $f'(z_0) \neq 0$ the function f is *conformal* (angle-preserving) at z_0.

If $f'(z_0) = 0$, then on a small disc centred at z_0 we have $f(z) \sim f(z_0) + a_n(z - z_o)^n$ for the first coefficient $a_n \neq 0$ and f acts on this disc as an *n to 1 branched covering map* (branched at z_0): note that f is then *not* conformal at z_0.

$f : \Omega \to \hat{\mathbf{C}} = \mathbf{C} \cup \infty$ (Ω still a domain in $\mathbf{C}$) is called *meromorphic* if the only singularities of f on Ω are *poles*, or equivalently if for each $z_0 \in \Omega$ there is a disc neighbourhood of z_0 on which the value of $f(z)$ is equal to the sum of the *Laurent series* $\sum_{n=-m}^{\infty} a_n(z - z_0)^n$ for f at z_0 (where $m = 0$ if $f(z_0) \neq \infty$, and z_0 is a pole of order m if $f(z_0) = \infty$).

Let j denote the function $z \to 1/z$. Note that $f : \Omega \to \hat{\mathbf{C}}$ is meromorphic if and only if f is analytic at those points z_0 where $f(z_0) \neq \infty$ and jf is analytic at those where $f(z_0) \neq 0$.

If Ω is now a domain in $\hat{\mathbf{C}}$ we say that $f : \Omega \to \hat{\mathbf{C}}$ is *meromorphic at* ∞ if fj is meromorphic at 0.

Theorem 1.1 *$f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ is meromorphic if and only if f is a rational function, that is to say there exist polynomials $p(z), q(z)$, with complex coefficients, such that $f(z) = p(z)/q(z)$ for all $z \in \hat{\mathbf{C}}$.*

Proof It is an elementary exercise to show that any rational map is meromorphic. For the converse, let $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be meromorphic. Then f has finitely many poles (else $1/f$ has a convergent sequence of zeros, which, by Taylor's Theorem, is only possible if $1/f$ is identically zero). Let these poles be $\beta_1, ..., \beta_m$, of order $n_1, ...n_m$ respectively. Then $g(z) = (z - \beta_1)^{n_1} ... (z - \beta_m)^{n_m} f(z)$ is analytic $\hat{\mathbf{C}} \to \mathbf{C}$ and equal to its Taylor series $\sum_{n=0}^{\infty} a_n z^n$ everywhere on $\hat{\mathbf{C}}$. In particular g is meromorphic at ∞; that is to say gj is analytic at 0, or in other words $\sum_{n=0}^{\infty} a_n z^{-n}$ has a pole (or a removable singularity) at $z = 0$. Thus only finitely many of the a_n are non-zero and hence g is a polynomial. QED

This is a very powerful result: it tells us that any meromorphic $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ is determined by a *finite* set of data, for example the poles and zeros of f together with the value of f at one other point.

Degree of a rational map

Let $f(z) = p(z)/q(z)$, where p and q are polynomials of degree d_p and d_q respectively, with no common zeros. Then a general point $\zeta \in \hat{\mathbf{C}}$ has $max(d_p, d_q)$ inverse images (just consider the equation $\zeta = p(z)/q(z)$, that is to say $p(z) - \zeta q(z) = 0$: this has $max(d_p, d_q)$ solutions z for any ζ in general position). We define the *degree* of f to be $max(d_p, d_q)$.

Corollary 1.2 *The invertible meromorphic maps $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ are the rational maps of form $f(z) = (az + b)/cz + d)$ having $a, b, c, d \in \mathbf{C}$ and $ad \neq bc$.*

Proof By Theorem 1.1 for f to be meromorphic it must be rational, but to be injective it must have degree 1. Conversely, any f of this form is invertible since it has inverse $f^{-1}(z) = (dz - b)/(-cz + a)$. QED

Critical Points

A *critical point* of a rational map f of degree d is a point z_0 is a point where the degree one term of the Taylor series for f vanishes, in other words the derivative $f'(z_0)$ vanishes. Looked at topologically it is a *branch point* of f, a point where locally f has the form $z \to a_0 + z^n$ for some $n > 1$, and thus in particular where $f^{-1}f(z_0)$ consists of less than d distinct points. (But for $d > 2$ it does not follow that z_0 is a critical point just because $f^{-1}f(z_0)$ consists of less than d distinct points. Why ?) Writing $f(z) = p(z)/q(z)$, we see that $f'(z) = 0 \Leftrightarrow q'(z)p(z) - p'(z)q(z) = 0$ and deduce:

Proposition 1.3 *A degree d rational map has $2d-2$ critical points (counted with multiplicity)*

Möbius transformations

Maps of the form $f(z) = (az+b)/cz+d)$ having $a, b, c, d \in \mathbf{C}$ and $ad \neq bc$ are called *fractional linear* or *Möbius* transformations.

Properties

1. Any invertible linear map $\alpha : \mathbf{C}^2 \to \mathbf{C}^2$ has the form

$$\begin{pmatrix} z \\ w \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} az+bw \\ cz+dw \end{pmatrix}$$

and passes to a map $\mathbf{CP}^1 \to \mathbf{CP}^1$ which in our coordinate z/w on $\hat{\mathbf{C}} = \mathbf{CP}^1$ is

$$z/w \to \frac{az+bw}{cz+dw} = \frac{az/w+b}{cz/w+d} \cdot$$

(where $(a\infty + b)/(c\infty + d)$ is to be interpreted as a/c and so on).

2. Composition of linear maps passes to composition of Möbius transformations. The group of all Möbius transformations is therefore

$$PSL(2, \mathbf{C}) = SL(2, \mathbf{C})/\pm I$$

where $SL(2, \mathbf{C})$ denotes the group of all 2×2 matrices of determinant 1.

3. Given any three distinct points $P, Q, R \in \hat{\mathbf{C}}$, there exists a unique Möbius transformation sending $P \to \infty, Q \to 0, R \to 1$.

4. A Möbius transformation preserves the *cross-ratio*

$$(P, Q; R, S) = \frac{(S-Q)(R-P)}{(S-P)(R-Q)}$$

of any four distinct points P, Q, R, S.

5. Möbius transformations send circles to circles (where a 'circle through ∞' in $\hat{\mathbf{C}}$ is a straight line in $\mathbf{C}$).

1.3 Conjugacies, fixed points,and periodic orbits

Conjugacies

Rational maps f, g are said to be *conjugate* if there exists a Möbius transformation h such that $g = hfh^{-1}$.

Conjugate maps have identical dynamical behaviour (think of h as a 'change of coordinate'). In particular h sends fixed points of f to fixed points of g, periodic points of f to periodic points of g etc. We can often put a rational map into a simpler form by applying a suitable conjugacy.

Examples

1. A rational map f is conjugate to a polynomial if and only if there exists a point $z_0 \in \hat{\mathbf{C}}$ such that $f^{-1}(z_0) = \{z_0\}$. (**Proof**: Move z_0 to ∞ by a Möbius transformation h. Details: exercise.)

2. A rational map f is conjugate to a polynomial of the form $z \to z^n$ (some $n > 0$) if and only if there exist distinct points $z_0, z_1 \in \hat{\mathbf{C}}$ such that $f^{-1}(z_0) = \{z_0\}$ and $f^{-1}(z_1) = \{z_1\}$. (**Proof**: Move z_0 to ∞ and z_1 to 0 by a Möbius transformation h. Details: exercise.)

3. Every degree 2 polynomial $z \to \alpha z^2 + \beta z + \gamma$ ($\alpha \neq 0$) is conjugate to a (unique) one of the form $z \to z^2 + c$. (**Proof**: Exercise. Note that h can be taken of the form $az + b$ since we do not have to move ∞).

Fixed points, periodic points and their types

A *fixed point* of a rational map f is a point $z_0 \in \hat{\mathbf{C}}$ such that $f(z_0) = z_0$. The *multiplier* of f at such a fixed point is the derivative $f'(z_0) = \lambda$. We say that z_0 is

attracting if $|\lambda| < 1$ (if $\lambda = 0$ we say z_0 is *superattracting*);
repelling if $|\lambda| > 1$;
neutral if $|\lambda| = 1$.

The last case is subdivided into *rational* if $\lambda^n = 1$ for some n and *irrational* otherwise.

Exercise Show that multipliers at fixed points of f are preserved when the function f is conjugated by a Möbius transformation (Hint: differentiate hfh^{-1} using the chain rule).

We shall write f^n for the nth iterate of f (not to be confused with the nth derivative of f, which we shall denote $f^{(n)}$ if we ever need it). The *orbit* of a point z under f is the sequence $z, f(z), ..., f^n(z), ...$

When z_0 is an attracting fixed point of f, every point z of $\hat{\mathbf{C}}$ sufficiently close to z has orbit converging to z_0 (as is easily proved using the Taylor expansion of f at z_0, which has the form $f(z) = z_0 + \lambda(z - z_0) +$ *higher order terms*). When z_0 is a repelling fixed point we know that for z sufficiently close to z_0 we have $|f(z) - f(z_0)| > |z - z_0|$, though it may be that the orbit returns to near z_0 at some later stage. When z_0 is a neutral fixed point the behaviour can be much more complicated (as we shall see later).

A point $z_0 \in \hat{\mathbf{C}}$ is said to be a *periodic point* of f if there exists some $n > 0$ such that $f^n(z_0) = z_0$. The least such n is called the *period.* The *multiplier* λ of the orbit $z_0, f(z_0), ..., f^n(z_0) = z_0$ is the derivative of f^n at its fixed point z_0, which, by the chain rule is equal to the product $f'(z_0)f'(z_1)...f'(z_{n-1})$ (where z_m denotes the mth point $f^m(z_0)$ of the orbit of z_0). For periodic orbits we have the same classification into types as for fixed points.

Exercise Let f be the rational map

$$z \to \frac{-2z - 1}{z^2 + 4z + 2}$$

Find the critical points of f and their orbits. Deduce that f is conjugate to $z \to z^2 - 1$.

1.4 Fatou and Julia sets: equicontinuity, normal families, and Montel's Theorem

Definition Let f be a rational map and z_0 be a point of $\hat{\mathbf{C}}$. We say that the family of iterates $\{f^n\}_{n\geq 0}$ is *equicontinuous at* z_0 if given any $\epsilon > 0$ there exists $\delta > 0$ such that for all $n \geq 0$ $d(f^n(z), f^n(z_0)) < \epsilon$ whenever $d(z, z_0) < \delta$. (Here d is the spherical metric on $\hat{\mathbf{C}}$.)

Think of this as saying *'Orbits that start near z_0 remain close to the orbit of z_0'.*

Definitions

The *Fatou set* $F(f)$ of f is the largest open subset of $\hat{\mathbf{C}}$ on which $\{f^n\}_{n\geq 0}$ are equicontinuous at every point.

The *Julia set* $J(f)$ of f is $\hat{\mathbf{C}} - F(f)$.

The Julia set should be thought of as the set of points where we have *'sensitive dependence on initial conditions'.*

Example

$f(z) = z^2$ has Fatou set $F(f) = \{z : |z| \neq 1\}$, and Julia set $J(f) = \{z : |z| = 1\}$.

(Since f doubles length along the unit circle it is clear that $\{z : |z| = 1\} \subset J(f)$. It is not quite so obvious that points not on the unit circle are in $F(f)$. One could try to give a direct formal proof of this, but the details would be messy in practice: the problem is that orbits started close together near (but not on) the unit circle will move apart for a large number of iterations before they start approaching each other again. For a more general method of proof, see the example a few lines below.)

Properties

1. $F(f)$ is open (by definition); hence $J(f)$ is closed and therefore compact (since $\hat{\mathbf{C}}$ is).

2. $F(f)$ is *completely invariant*, that is $f(F(f)) = F(f) = f^{-1}(F(f))$. (This follows from the definition of $F(f)$ and the fact that a rational map is *continuous* and *open.*)

3. $J(f)$ is completely invariant. (This follows at once from 2.)

What kind of families $\mathcal{F}$ of analytic maps $f : \Omega \to \hat{\mathbf{C}}$ are equicontinuous ? Firstly, if all the $f \in \mathcal{F}$ have a common bound on Ω, say $|f(z)| < M$ for all $z \in \hat{\mathbf{C}}$ and all $f \in \mathcal{F}$, then it is an easy exercise using Cauchy's integral formula for $f^{(n)}(z)$ to show that for each n, on each compact subset $K \subset \Omega$ there is a uniform bound on $|f^{(n)}(z)|$ (depending only on n, M and K, not f). By considering Taylor series it follows that in this case the family $\mathcal{F}$ is equicontinuous. In particular

Example Any family of analytic maps of the open unit disc D into itself is equicontinuous. For example $\{z \to z^{2^n}\}_{n\geq 0}$ are equicontinuous on D: thus the Fatou set of $z \to z^2$ contains $\{z : |z| < 1\}$. Conjugating by $j : z \to 1/z$ we see that the Fatou set of $z \to z^2$ also contains $\{z : |z| > 1\}$. Since every point on the unit circle is in the Julia set of $z \to z^2$, we now have a proof that the Fatou and Julia sets of this map are as claimed above.

Definition A family $\mathcal{F}$ of maps $\Omega \to \hat{\mathbf{C}}$ is called *normal* if every infinite set of maps in $\mathcal{F}$ contains a sequence of maps which converges *locally uniformly* to a map $f : \Omega \to \hat{\mathbf{C}}$ (not necessarily in $\mathcal{F}$).

Example $\{z \to z^{2^n}\}_{n\geq 0}$ are a normal family on D, since they converge locally uniformly there to the constant map $z \to 0$.

Theorem 1.4 (Arzelà-Ascoli) *Let* Ω *be a domain in* $\hat{\mathbf{C}}$. *Any family of continuous maps* $\Omega \to \hat{\mathbf{C}}$ *is normal if and only if it is equicontinuous.*
(For a proof see any sufficiently large complex anaylsis textbook.)

This brings us to the key theorem for Fatou-Julia theory:

Theorem 1.5 (Montel, 1911) *let* Ω *be a domain in* $\hat{\mathbf{C}}$. *Every family of analytic maps* $\Omega \to \hat{\mathbf{C}} - \{0, 1, \infty\}$ *is normal (or equivalently, by Arzelà-Ascoli, equicontinuous).*
(For a proof, see, for example, Beardon's book 'Iteration of rational functions'.)

We can replace the points $0, 1, \infty$ in the statement of Montel's Theorem by any other three points of $\hat{\mathbf{C}}$ (just compose with a suitable Möbius transformation). Montel's Theorem is a much more powerful result than our earlier observation that any family of maps with a common bound is equicontinuous. One should perhaps compare it with Picard's Theorem that any analytic function $\mathbf{C} \to \mathbf{C} - \{0, 1\}$ is constant, which is in turn a much more powerful result than Liouville's Theorem that a bounded analytic function on $\mathbf{C}$ is constant.