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ONE-DIMENSIONAL COMPLEX DYNAMICS

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LECTURE 1. INTRODUCTION

The study of iterated complex maps had its first great flowering with the the work of the French mathematicians Julia and Fatou around 1918-20, though its origins perhaps lie earlier, in the late 19th century, in the more geometric work of Schottky, Poincaré, Fricke and Klein. It has had its second great flowering over the last 25 years, motivated partly by the spectacular computer pictures which started to appear from about 1980 onwards, partly by the explosive growth in the subject of chaotic dynamics which started about the same time, and not least by the revolutionary work in three-dimensional hyperbolic geometry initiated by Thurston in the early 1980's. Some of the names associated with this second wave of activity are Mandelbrot, Douady, Hubbard, Sullivan, Milnor, Thurston, Yoccoz and McMullen. The subject is still very much in a ferment of activity: as we shall see, some of the major conjectures are still waiting to be proved. But the methods are powerful: for example the only conceptual analytic proof of the universality of the Feigenbaum ratios for period doubling in real unimodal maps is that of Sullivan (1992) using techniques from complex iteration theory.

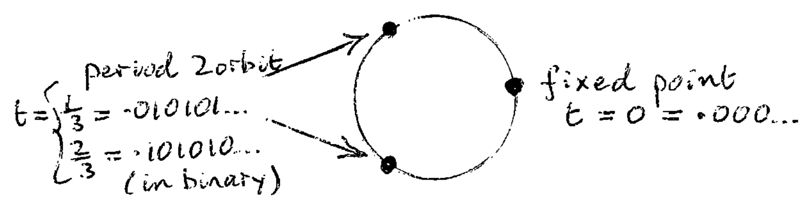
The objective of these lecture notes is to give a brief introduction to the remarkable mixture of complex analysis, hyperbolic geometry and symbolic dynamics that constitutes the subject of complex dynamics. The idea is to give the flavour of the subject, outline some of the main techniques (without detailed proof) and discuss some of the main theorems and open conjectures. As we proceed, we shall also see connections with the symbolic dynamics of maps of the both the real interval and the circle: the complex world is ideal for 'unfolding' problems in the real world.

1.1 Examples of the behaviour of quadratic maps $z \to z^2 + c$

(i) $c = 0$

Here the dynamical behaviour is straightforward. When we iterate $z \to z^2$ any orbit started inside the unit circle heads towards the point 0, any orbit started outside the unit circle heads towards ∞, and any orbit started on the unit circle remains there. The two components of $\{z : |z| \neq 1\}$ are known as the *Fatou set* of the map and the circle $|z| = 1$ is called the *Julia set.*

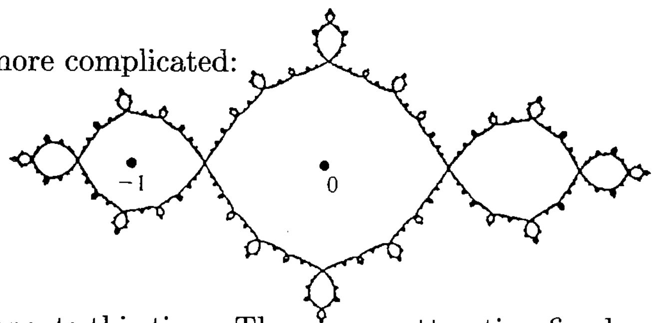
On the unit circle itself the dynamics are those of the *shift.*
Parametrise the circle by $t \in [0, 1) \subset \mathbf{R}$ $(t = arg(z)/2\pi)$: then $z \to z^2$ sends $t \to 2t$ mod 1.



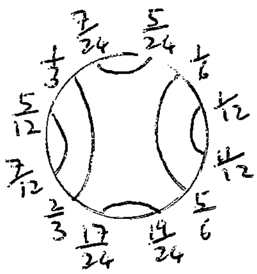
Any $t \in [0, 1)$ of the form $t = p/(2^n - 1)$ (for p integer) is periodic, of period n. Hence the periodic points form a dense set on the unit circle; moreover that the map $z \to z^2$ has *sensitive dependence on initial condition*, since it doubles arguments.

(ii) $c = -1$

Here the dynamical behaviour is much more complicated:



The Fatou set has infinitely many components this time. There is an attracting fixed point at ∞ to which every orbit started in the component of the Fatou set outside the 'filled-in Julia set' is attracted, and a period 2 cycle $0 \to -1 \to 0 \to -1 \to \ldots$ towards which every orbit started in any other component of the Fatou set is attracted. An orbit which starts on the common boundary of the two regions (the 'Julia set', which we shall define formally soon) remains on that boundary. Combinatorially, the Julia set in this example is a *quotient* of the circle, and the dynamics are those of the corresponding *quotient* of the shift. The figure below shows the first few identifications on the unit circle in the construction of this quotient.

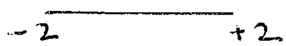


(iii) $c = i$

(For an illustration see the middle picture on the right-hand side on the next page.)

Note that the point 0 is *preperiodic* for this map ($0 \to i \to -1+i \to -i \to -1+i \ldots$). It can be proved that whenever c is such that the critical point 0 of $z \to z^2 + c$ is preperiodic but not periodic, the Julia set is a *dendrite* (that is a connected, simply-connected set with empty interior).

(iv) $c = -2$

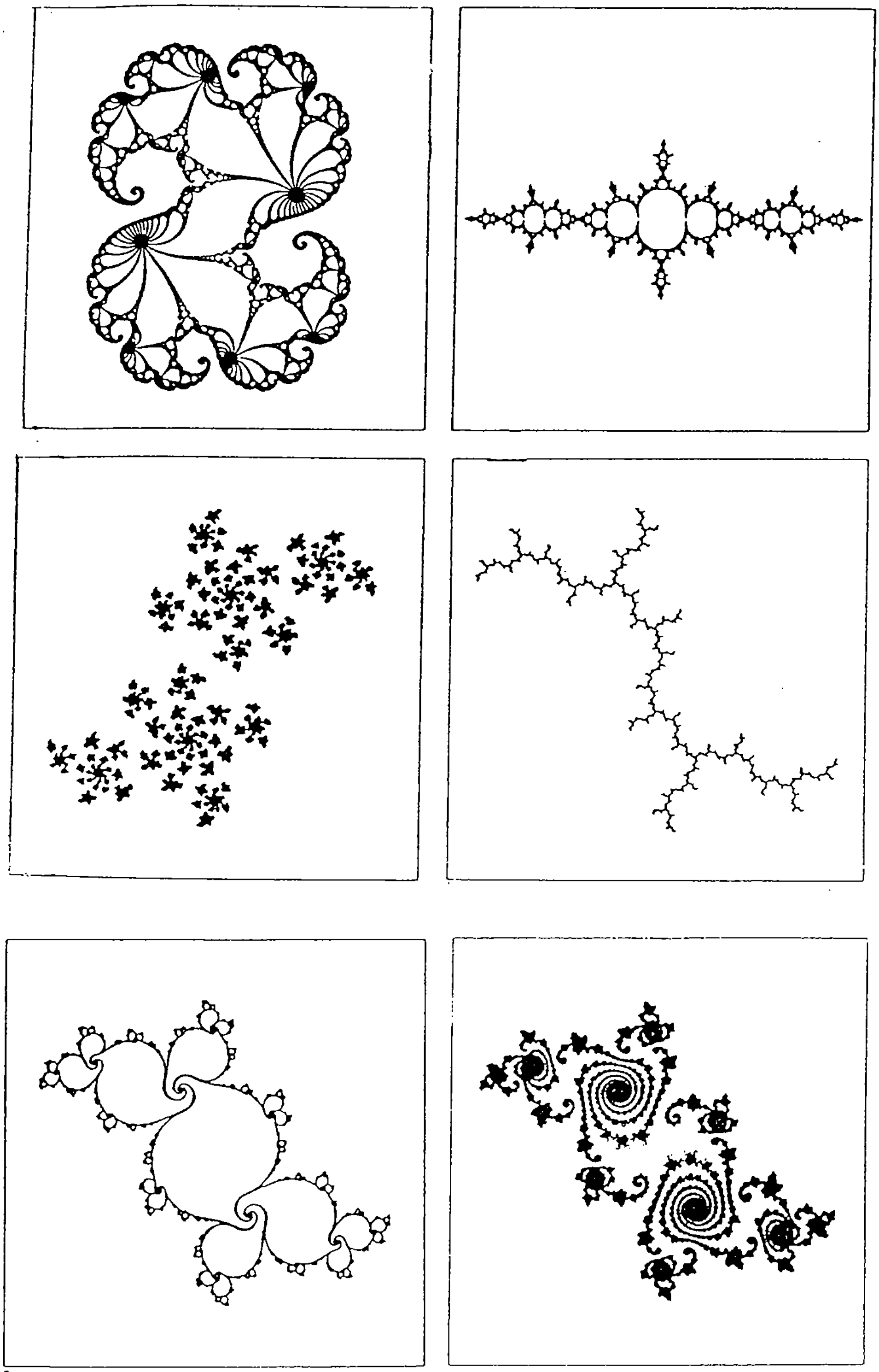


Here again 0 is preperiodic, and the dendrite is a particularly simple one, the real interval $[-2, 2]$.

Exercise

Show that the map $h : z \to z + 1/z$ is a semiconjugacy from $f : z \to z^2$ to $g : z \to z^2 - 2$ (that is, h is a surjection satisfying $hf = gh$) and that h sends the Julia set of f (the unit circle) onto the real interval $[-2, +2]$.

Typical Julia sets of quadratic maps $z \to z^2 + c$



1.2 The Riemann sphere and rational maps: basic essentials from complex analysis

We summarise some basic results of complex analysis before giving the formal definitions of Julia and Fatou sets and beginning the study of their main properties.

The Riemann sphere and rational maps

The *Riemann sphere* is the complex projective line,

$$\mathbf{CP}^1 = \{\mathbf{C}^2 - (0,0)\}/\mathcal{R}$$

where $\mathcal{R}$ is the relation $(z,w) \sim (\lambda z, \lambda w)$ for $\lambda \in \mathbf{C} - 0$. Any equivalence class $[z,w]$ contains $(z/w, 1)$ if $w \neq 0$ or $(1, w/z)$ if $z \neq 0$, so we may think of the Riemann sphere as the union of two copies of the complex plane glued together, $\mathbf{C}_1 \cup \mathbf{C}_2/(z_1 \sim 1/z_2)$, or even more simply as the *extended complex plane* $\hat{\mathbf{C}} = \mathbf{C} \cup \infty$. The bijection

$$\mathbf{CP}^1 \leftrightarrow \hat{\mathbf{C}}$$

is given by $[z,w] \leftrightarrow z/w$ when $w \neq 0$ and $[z,0] \leftrightarrow \infty$. Yet another way to picture the Riemann sphere is as the unit sphere S^2 in $\mathbf{R}^3$: if we remove the north pole $N = (0,0,1)$ the remainder of S^2 maps bijectively onto the $(x,y,0)$-plane under *stereographic projection* from N, and sending $N \to \infty$ completes this to a bijection $S^2 \leftrightarrow \mathbf{C} \cup \infty = \hat{\mathbf{C}}$. With this picture we can define the *spherical metric* on $\hat{\mathbf{C}}$, corresponding to the usual metric on the unit sphere. In what follows it will usually be most convenient to think of the Riemann sphere as $\hat{\mathbf{C}} = \mathbf{C} \cup \infty$, but it might sometimes also be helpful to think in terms of one of the other definitions.

We next want to define what we mean by *differentiable maps* from the Riemann sphere to itself. We approach the definition in stages, recalling some terminolgy from complex analysis.

An open connected set $\Omega \subset \mathbf{C}$ is called a *domain.*

$f : \Omega \to \mathbf{C}$ is called *differentiable* if for each $z_0 \in \Omega$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If f is differentiable then for each $z_0 \in \Omega$ there is a disc neighbourhood of z_0 on which the value of the function $f(z)$ is equal the sum of the Taylor series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ for f at z_0 (Taylor's Theorem). For this reason a differentiable function is often called an *analytic* function.

If $f'(z_0) \neq 0$, then near z_0 we have $f(z) \sim f(z_0) + f'(z_0)(z - z_0)$ so f acts on $z - z_0$ by multiplying it by the scaling factor $|f'(z_0)|$ and turning it through an angle $arg(f'(z_0))$. Thus in particular if $f'(z_0) \neq 0$ the function f is *conformal* (angle-preserving) at z_0.

If $f'(z_0) = 0$, then on a small disc centred at z_0 we have $f(z) \sim f(z_0) + a_n(z - z_o)^n$ for the first coefficient $a_n \neq 0$ and f acts on this disc as an *n to 1 branched covering map* (branched at z_0): note that f is then *not* conformal at z_0.

$f : \Omega \to \hat{\mathbf{C}} = \mathbf{C} \cup \infty$ (Ω still a domain in $\mathbf{C}$) is called *meromorphic* if the only singularities of f on Ω are *poles*, or equivalently if for each $z_0 \in \Omega$ there is a disc neighbourhood of z_0 on which the value of $f(z)$ is equal to the sum of the *Laurent series* $\sum_{n=-m}^{\infty} a_n(z-z_0)^n$ for f at z_0 (where $m = 0$ if $f(z_0) \neq \infty$, and z_0 is a pole of order m if $f(z_0) = \infty$).

Let j denote the function $z \to 1/z$. Note that $f : \Omega \to \hat{\mathbf{C}}$ is meromorphic if and only if f is analytic at those points z_0 where $f(z_0) \neq \infty$ and jf is analytic at those where $f(z_0) \neq 0$.

If Ω is now a domain in $\hat{\mathbf{C}}$ we say that $f : \Omega \to \hat{\mathbf{C}}$ is *meromorphic at* ∞ if fj is meromorphic at 0.

Theorem 1.1 *$f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ is meromorphic if and only if f is a rational function, that is to say there exist polynomials $p(z), q(z)$, with complex coefficients, such that $f(z) = p(z)/q(z)$ for all $z \in \hat{\mathbf{C}}$.*

Proof It is an elementary exercise to show that any rational map is meromorphic. For the converse, let $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be meromorphic. Then f has finitely many poles (else $1/f$ has a convergent sequence of zeros, which, by Taylor's Theorem, is only possible if $1/f$ is identically zero). Let these poles be $\beta_1, ..., \beta_m$, of order $n_1, ...n_m$ respectively. Then $g(z) = (z-\beta_1)^{n_1}...(z-\beta_m)^{n_m} f(z)$ is analytic $\hat{\mathbf{C}} \to \mathbf{C}$ and equal to its Taylor series $\sum_{n=0}^{\infty} a_n z^n$ everywhere on $\hat{\mathbf{C}}$. In particular g is meromorphic at ∞; that is to say gj is analytic at 0, or in other words $\sum_{n=0}^{\infty} a_n z^{-n}$ has a pole (or a removable singularity) at $z = 0$. Thus only finitely many of the a_n are non-zero and hence g is a polynomial. QED

This is a very powerful result: it tells us that any meromorphic $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ is determined by a *finite* set of data, for example the poles and zeros of f together with the value of f at one other point.

Degree of a rational map

Let $f(z) = p(z)/q(z)$, where p and q are polynomials of degree d_p and d_q respectively, with no common zeros. Then a general point $\zeta \in \hat{\mathbf{C}}$ has $max(d_p, d_q)$ inverse images (just consider the equation $\zeta = p(z)/q(z)$, that is to say $p(z) - \zeta q(z) = 0$: this has $max(d_p, d_q)$ solutions z for any ζ in general position). We define the *degree* of f to be $max(d_p, d_q)$.

Corollary 1.2 *The invertible meromorphic maps $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ are the rational maps of form $f(z) = (az+b)/cz+d)$ having $a, b, c, d \in \mathbf{C}$ and $ad \neq bc$.*

Proof By Theorem 1.1 for f to be meromorphic it must be rational, but to be injective it must have degree 1. Conversely, any f of this form is invertible since it has inverse $f^{-1}(z) = (dz-b)/(-cz+a)$. QED

Critical Points

A *critical point* of a rational map f of degree d is a point z_0 is a point where the degree one term of the Taylor series for f vanishes, in other words the derivative $f'(z_0)$ vanishes. Looked at topologically it is a *branch point* of f, a point where locally f has the form $z \to a_0 + z^n$ for some $n > 1$, and thus in particular where $f^{-1}f(z_0)$ consists of less than d distinct points. (But for $d > 2$ it does not follow that z_0 is a critical point just because $f^{-1}f(z_0)$ consists of less than d distinct points. Why ?) Writing $f(z) = p(z)/q(z)$, we see that $f'(z) = 0 \Leftrightarrow q'(z)p(z) - p'(z)q(z) = 0$ and deduce:

Proposition 1.3 *A degree d rational map has $2d-2$ critical points (counted with multiplicity)*

Möbius transformations

Maps of the form $f(z) = (az+b)/cz+d)$ having $a, b, c, d \in \mathbf{C}$ and $ad \neq bc$ are called *fractional linear* or *Möbius* transformations.

Properties

1. Any invertible linear map $\alpha : \mathbf{C}^2 \to \mathbf{C}^2$ has the form

$$\begin{pmatrix} z \\ w \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} az+bw \\ cz+dw \end{pmatrix}$$

and passes to a map $\mathbf{CP}^1 \to \mathbf{CP}^1$ which in our coordinate z/w on $\hat{\mathbf{C}} = \mathbf{CP}^1$ is

$$z/w \to \frac{az+bw}{cz+dw} = \frac{az/w+b}{cz/w+d} \cdot$$

(where $(a\infty+b)/(c\infty+d)$ is to be interpreted as a/c and so on).

2. Composition of linear maps passes to composition of Möbius transformations. The group of all Möbius transformations is therefore

$$PSL(2, \mathbf{C}) = SL(2, \mathbf{C})/\pm I$$

where $SL(2, \mathbf{C})$ denotes the group of all 2×2 matrices of determinant 1.

3. Given any three distinct points $P, Q, R \in \hat{\mathbf{C}}$, there exists a unique Möbius transformation sending $P \to \infty, Q \to 0, R \to 1$.

4. A Möbius transformation preserves the *cross-ratio*

$$(P, Q; R, S) = \frac{(S-Q)(R-P)}{(S-P)(R-Q)}$$

of any four distinct points P, Q, R, S.

5. Möbius transformations send circles to circles (where a 'circle through ∞' in $\hat{\mathbf{C}}$ is a straight line in $\mathbf{C}$).

1.3 Conjugacies, fixed points,and periodic orbits

Conjugacies

Rational maps f, g are said to be *conjugate* if there exists a Möbius transformation h such that $g = hfh^{-1}$.

Conjugate maps have identical dynamical behaviour (think of h as a 'change of coordinate'). In particular h sends fixed points of f to fixed points of g, periodic points of f to periodic points of g etc. We can often put a rational map into a simpler form by applying a suitable conjugacy.

Examples

1. A rational map f is conjugate to a polynomial if and only if there exists a point $z_0 \in \hat{\mathbf{C}}$ such that $f^{-1}(z_0) = \{z_0\}$. (**Proof**: Move z_0 to ∞ by a Möbius transformation h. Details: exercise.)

2. A rational map f is conjugate to a polynomial of the form $z \to z^n$ (some $n > 0$) if and only if there exist distinct points $z_0, z_1 \in \hat{\mathbf{C}}$ such that $f^{-1}(z_0) = \{z_0\}$ and $f^{-1}(z_1) = \{z_1\}$. (**Proof**: Move z_0 to ∞ and z_1 to 0 by a Möbius transformation h. Details: exercise.)

3. Every degree 2 polynomial $z \to \alpha z^2 + \beta z + \gamma$ ($\alpha \neq 0$) is conjugate to a (unique) one of the form $z \to z^2 + c$. (**Proof**: Exercise. Note that h can be taken of the form $az + b$ since we do not have to move ∞).

Fixed points, periodic points and their types

A *fixed point* of a rational map f is a point $z_0 \in \hat{\mathbf{C}}$ such that $f(z_0) = z_0$. The *multiplier* of f at such a fixed point is the derivative $f'(z_0) = \lambda$. We say that z_0 is

attracting if $|\lambda| < 1$ (if $\lambda = 0$ we say z_0 is *superattracting*);
repelling if $|\lambda| > 1$;
neutral if $|\lambda| = 1$.

The last case is subdivided into *rational* if $\lambda^n = 1$ for some n and *irrational* otherwise.

Exercise Show that multipliers at fixed points of f are preserved when the function f is conjugated by a Möbius transformation (Hint: differentiate hfh^{-1} using the chain rule).

We shall write f^n for the nth iterate of f (not to be confused with the nth derivative of f, which we shall denote $f^{(n)}$ if we ever need it). The *orbit* of a point z under f is the sequence $z, f(z), ..., f^n(z), ...$

When z_0 is an attracting fixed point of f, every point z of $\hat{\mathbf{C}}$ sufficiently close to z has orbit converging to z_0 (as is easily proved using the Taylor expansion of f at z_0, which has the form $f(z) = z_0 + \lambda(z - z_0) +$ *higher order terms*). When z_0 is a repelling fixed point we know that for z sufficiently close to z_0 we have $|f(z) - f(z_0)| > |z - z_0|$, though it may be that the orbit returns to near z_0 at some later stage. When z_0 is a neutral fixed point the behaviour can be much more complicated (as we shall see later).

A point $z_0 \in \hat{\mathbf{C}}$ is said to be a *periodic point* of f if there exists some $n > 0$ such that $f^n(z_0) = z_0$. The least such n is called the *period*. The *multiplier* λ of the orbit $z_0, f(z_0), ..., f^n(z_0) = z_0$ is the derivative of f^n at its fixed point z_0, which, by the chain rule is equal to the product $f'(z_0)f'(z_1)...f'(z_{n-1})$ (where z_m denotes the mth point $f^m(z_0)$ of the orbit of z_0). For periodic orbits we have the same classification into types as for fixed points.

Exercise Let f be the rational map

$$z \to \frac{-2z - 1}{z^2 + 4z + 2}$$

Find the critical points of f and their orbits. Deduce that f is conjugate to $z \to z^2 - 1$.

1.4 Fatou and Julia sets: equicontinuity, normal families, and Montel's Theorem

Definition Let f be a rational map and z_0 be a point of $\hat{\mathbf{C}}$. We say that the family of iterates $\{f^n\}_{n\geq 0}$ is *equicontinuous at* z_0 if given any $\epsilon > 0$ there exists $\delta > 0$ such that for all $n \geq 0$ $d(f^n(z), f^n(z_0)) < \epsilon$ whenever $d(z, z_0) < \delta$. (Here d is the spherical metric on $\hat{\mathbf{C}}$.)

Think of this as saying *'Orbits that start near z_0 remain close to the orbit of z_0'*.

Definitions

The *Fatou set* $F(f)$ of f is the largest open subset of $\hat{\mathbf{C}}$ on which $\{f^n\}_{n\geq 0}$ are equicontinuous at every point.

The *Julia set* $J(f)$ of f is $\hat{\mathbf{C}} - F(f)$.

The Julia set should be thought of as the set of points where we have *'sensitive dependence on initial conditions'*.

Example

$f(z) = z^2$ has Fatou set $F(f) = \{z : |z| \neq 1\}$, and Julia set $J(f) = \{z : |z| = 1\}$.

(Since f doubles length along the unit circle it is clear that $\{z : |z| = 1\} \subset J(f)$. It is not quite so obvious that points not on the unit circle are in $F(f)$. One could try to give a direct formal proof of this, but the details would be messy in practice: the problem is that orbits started close together near (but not on) the unit circle will move apart for a large number of iterations before they start approaching each other again. For a more general method of proof, see the example a few lines below.)

Properties

1. $F(f)$ is open (by definition); hence $J(f)$ is closed and therefore compact (since $\hat{\mathbf{C}}$ is).

2. $F(f)$ is *completely invariant*, that is $f(F(f)) = F(f) = f^{-1}(F(f))$. (This follows from the definition of $F(f)$ and the fact that a rational map is *continuous* and *open.*)

3. $J(f)$ is completely invariant. (This follows at once from 2.)

What kind of families $\mathcal{F}$ of analytic maps $f : \Omega \to \hat{\mathbf{C}}$ are equicontinuous ? Firstly, if all the $f \in \mathcal{F}$ have a common bound on Ω, say $|f(z)| < M$ for all $z \in \hat{\mathbf{C}}$ and all $f \in \mathcal{F}$, then it is an easy exercise using Cauchy's integral formula for $f^{(n)}(z)$ to show that for each n, on each compact subset $K \subset \Omega$ there is a uniform bound on $|f^{(n)}(z)|$ (depending only on n, M and K, not f). By considering Taylor series it follows that in this case the family $\mathcal{F}$ is equicontinuous. In particular

Example Any family of analytic maps of the open unit disc D into itself is equicontinuous. For example $\{z \to z^{2^n}\}_{n\geq 0}$ are equicontinuous on D: thus the Fatou set of $z \to z^2$ contains $\{z : |z| < 1\}$. Conjugating by $j : z \to 1/z$ we see that the Fatou set of $z \to z^2$ also contains $\{z : |z| > 1\}$. Since every point on the unit circle is in the Julia set of $z \to z^2$, we now have a proof that the Fatou and Julia sets of this map are as claimed above.

Definition A family $\mathcal{F}$ of maps $\Omega \to \hat{\mathbf{C}}$ is called *normal* if every infinite set of maps in $\mathcal{F}$ contains a sequence of maps which converges *locally uniformly* to a map $f : \Omega \to \hat{\mathbf{C}}$ (not necessarily in $\mathcal{F}$).

Example $\{z \to z^{2^n}\}_{n \geq 0}$ are a normal family on D, since they converge locally uniformly there to the constant map $z \to 0$.

Theorem 1.4 (Arzelà-Ascoli) *Let Ω be a domain in $\hat{\mathbf{C}}$. Any family of continuous maps $\Omega \to \hat{\mathbf{C}}$ is normal if and only if it is equicontinuous.*
(For a proof see any sufficiently large complex anaylsis textbook.)

This brings us to the key theorem for Fatou-Julia theory:

Theorem 1.5 (Montel, 1911) *let Ω be a domain in $\hat{\mathbf{C}}$. Every family of analytic maps $\Omega \to \hat{\mathbf{C}} - \{0, 1, \infty\}$ is normal (or equivalently, by Arzelà-Ascoli, equicontinuous).*
(For a proof, see, for example, Beardon's book 'Iteration of rational functions'.)

We can replace the points $0, 1, \infty$ in the statement of Montel's Theorem by any other three points of $\hat{\mathbf{C}}$ (just compose with a suitable Möbius transformation). Montel's Theorem is a much more powerful result than our earlier observation that any family of maps with a common bound is equicontinuous. One should perhaps compare it with Picard's Theorem that any analytic function $\mathbf{C} \to \mathbf{C} - \{0, 1\}$ is constant, which is in turn a much more powerful result than Liouville's Theorem that a bounded analytic function on $\mathbf{C}$ is constant.

LECTURE 2. PROPERTIES OF FATOU AND JULIA SETS

2.1 Julia sets: properties and characterizations

Before listing a sequence of properties of Julia sets which mostly follow directly from Montel's Theorem, we make a brief excursion into topology to consider *finite* completely invariant sets.

Lemma 2.1 *Let f be a rational map with $deg(f) \geq 2$, and suppose E is a finite completely invariant subset of $\hat{\mathbf{C}}$. Then E contains at most 2 points.*

Proof Suppose E contains k points. Then f must permute these points and hence for some q the iterate $f^q = g$ is the identity on E. Suppose g has degree d. Each point $z \in E$ must be a critical point of g, of *valency* d (i.e. locally g looks like $z \to z^d$), else $g^{-1}(z)$ would contain points other than z. But since the Euler characteristic of $\hat{\mathbf{C}}$ is 2, we know that

$$2d = 2 + \sum_c (\nu_c - 1)$$

(the 'Riemann-Hurwitz formula'). Here the sum is taken over *all* critical points c of g, and ν_c denotes the valency of c. Hence

$$k(d-1) \leq 2d - 2$$

and therefore $k \leq 2$. QED

Definition The *exceptional set* $E(f)$ of a rational map is the union of all finite completely invariant sets. Lemma 2.1 says $|E(f)| \leq 2$. Note that if $|E(f)| = 1$ then f is conjugate to a polynomial (just conjugate by a Möbius transformation sending the exceptional point to ∞), and if $|E(f)| = 2$ then f is conjugate to some $z \to z^d$, with d a positive or negative integer (just send the two exceptional points to ∞ and 0).

Properties of Julia sets of rational maps of degree at least two

1. $J(f) \neq \emptyset$. (See Beardon or Carleson and Gamelin. The basic idea is that if $\{f^n\}_{n \geq 0}$ form a normal family on the whole of $\hat{\mathbf{C}}$ then some subfamily (with degrees tending to infinity) converges locally uniformly to a rational function, which is impossible.)

2. $J(f)$ is infinite. (By Lemma 2.1 the only possibilities for finite completely invariant sets are (up to conjugacy) the set $\{\infty\}$ (for a polynomial) or $\{\infty, 0\}$ (for a map $z \to z^d$). But in both cases these exceptional sets are contained in the Fatou set.)

3. $J(f)$ is the smallest completely invariant closed set containing at least three points. (The complement of a completely invariant closed set containing at least three points is an open completely invariant set omitting at least three points, hence contained in the Fatou set, by Montel's Theorem.)

4. $J(f)$ is perfect, that is, every point of $J(f)$ is an accumulation point of $J(f)$. (For if we let J_0 be the set of accumulation points of J, then J_0 is non-empty, closed and completely invariant - using the facts that f is continuous, open and finite-to-one - and J_0 cannot be finite since it would then be exceptional and hence contained in $F(f)$, so $J_0 = J$ by Property 3.)

5. $J(f)$ is either the whole of $\hat{\mathbf{C}}$ or it has empty interior. (Write $S = \hat{\mathbf{C}} - int(J)$. Then S is the union of the Fatou Set F and the boundary ∂J of J, and either S is empty or it is an infinite closed completely invariant set, so containing J (by Property 3).)

We remark in connection with Property 5 that there exist examples of rational maps f having $J(f) = \hat{\mathbf{C}}$ (e.g. the example of Lattès (1918): $z \to (z^2+1)^2/4z(z^2-1)$) but that for a *polynomial* map the Fatou set always contains the point ∞ and hence is non-empty.

Useful results for plotting $J(f)$

Proposition 2.2 *If f is rational of degree at least 2, and U is any open set meeting $J(f)$, then $\bigcup_{n=0}^{\infty} f^n(U) \supset \hat{\mathbf{C}} - E(f)$.*

Proof If $\bigcup_{n=0}^{\infty} f^n(U)$ misses three or more points of $\hat{\mathbf{C}}$ then f^n are a normal family on U by Montel, contradicting $U \cap J \neq \emptyset$. But if a non-exceptional z lies in $\hat{\mathbf{C}} - \bigcup_{n=0}^{\infty} f^n(U)$ then for some m and n the point $f^{-m}(z)$ must lie in $f^n(U)$ (since $\bigcup_{m\geq 0} f^{-m}(z)$ is infinite). Hence $z \in f^{m+n}(U)$. Contradiction. QED.

Corollary 2.3 *If z_0 is not in $E(f)$, then $J(f) \subset \overline{\bigcup_{n\geq 0} f^{-n}(z_0)}$.*

Proof take any $z \in J(f)$ and neighbourhood U of z. By Proposition 2.2 the given point z_0 lies in some $f^n(U)$. Hence $f^{-n}(z_0) \cap U \neq \emptyset$. QED.

This gives us a very simple algorithm for plotting $J(f)$. One just has to start at any (non-exceptional) z_0 whatever and plot all its images under f^{-1}, then all of their images under f^{-1} etc., or alternatively plot $z_0, z_1, z_2, \ldots$ where each z_{j+1} is a random choice out of the (finite) set of values of $f^{-1}(z_j)$. The resulting set accumulates on the whole of $J(f)$. Even better, if one starts at a point z_0 known to be in $J(f)$ (for example a repelling fixed point) one has $J(f) = \overline{\bigcup_{n\geq 0} f^{-n}(z_0)}$, so that no redundant points are plotted with either algorithm.

Julia sets and repelling periodic points

Obviously every repelling periodic point of f lies in the Julia set. However it is also true that every point of the Julia set has repelling periodic points arbitrarily close to it:

Theorem 2.4 *let f be rational of degree at least two. then $J(f)$ is the closure of the set of all repelling periodic points of f.*

We omit the proof (which can be found in Beardon or in Carleson and Gamelin), but we remark that we had already observed in a simple example, $z \to z^2$, that the Julia set (the unit circle) contains a dense set of repelling periodic points.

The Julia set of $z \to z^2 + c$ for $|c|$ large

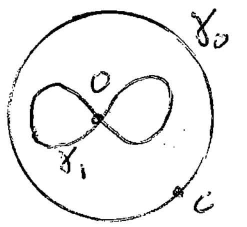
Lemma 2.5 *Let $|c| > 2$. Then for all z with $|z| \geq |c|$, $q_c^n(z) \to \infty$ as $n \to \infty$*

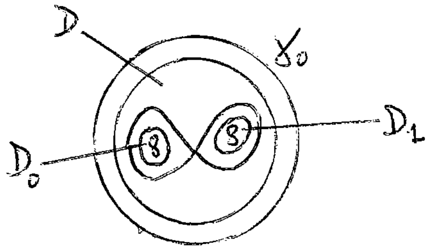
Proof Let $|z| = r$. Then $|q_c(z)| > r^2 - r \geq r(|c|-1) = |z|(|c|-1)$. QED

Definition The *Cantor set C* is the space $\{0,1\}^{\mathbf{N}}$ of all infinite sequences of 0's and 1's, equipped with the product topology (that is, two sequences are close if and only if they have the same initial segments). Recall that every perfect totally disconnected compact subset of $\mathbf{R}^n$ is homeomorphic to C (for example the 'middle thirds' set on the real line).

Proposition 2.6 *For $|c|$ sufficiently large, $J(q_c)$ is homeomorphic to the Cantor set C, and the action of q_c on $J(q_c)$ is conjugate to that of the shift σ on C.*

Proof Let γ_0 be the circle $|z| = |c|$, and let $\gamma_1 = q_c^{-1}(\gamma_0)$. Then γ_1 lies inside γ_0 (by Lemma 2.7) and γ_1 is a *lemniscate* (since 0 is the only critical point of q_c on $\mathbf{C}$). $q_c^{-1}(\gamma_1)$ now consists of a lemniscate inside each lobe of γ_1, and so on:





Let D be any disc containing γ_1 and contained in γ_0. Label the the two discs making up $q_c^{-1}(D)$ as D_0 and D_1, and label the components of $q_c^{-2}(D)$ by

$$D_{00} = D_0 \cap q_c^{-1}(D_0) \quad D_{01} = D_0 \cap q_c^{-1}(D_1) \quad D_{10} = D_1 \cap q_c^{-1}(D_0) \quad D_{11} = D_1 \cap q_c^{-1}(D_1)$$

Continue inductively, setting

$$D_{0s} = D_0 \cap q_c^{-1}(D_s) \quad D_{1s} = D_1 \cap q_c^{-1}(D_s)$$

for any finite sequence s of 0's and 1's. Set

$$\Lambda = \bigcap_1^\infty q_c^{-n}(D)$$

Then Λ is a Cantor set (for large $|c|$ it is easy to show that that q_c^{-1} is contracting, by a definite amount, on both D_0 and D_1), its points are labelled by infinite sequences of 0's and 1's, and the action of q_c on it is conjugate to the action of the shift σ on these sequences. Since Λ is a closed completely invariant set it contains $J(q_c)$; moreover since Λ contains a dense orbit (just write down an infinite sequence of 0's and 1's containing *all* finite sequences) it is a minimal closed completely invariant set and is therefore equal to $J(q_c)$. QED

In fact Proposition 2.6 holds whenever $q_c^n(0) \to \infty$, not just for 'large' $|c|$, but the proof requires a little more work.

2.3 Counting components of the Fatou set

Proposition 2.7 *The Fatou set of a rational map f of degree at least two contains at most two completely invariant simply-connected components*

Proof Any such component is homeomorphic to a disc D, and the restriction of f to D is a branch-covering of degree d. Since D has Euler characteristic 1 we deduce that f has $d-1$ critical points on D (counted with multiplicity). But f has only $2d-2$ critical points. QED.

Example The Fatou set for $z \to z^2$ has exactly two such components.

Dropping the words 'completely invariant' and just counting components, we have:

Proposition 2.8 *If $F(f)$ has more than two components, it has infinitely many components.*

Proof If f has only finitely many components, $D_1, ...D_k$, they must be permuted by f (since each component has image a component and inverse image a union of components). Hence there exists an m such that $g = f^m$ maps each D_j to itself. But $F(g) = F(f)$ (from the definition of a normal family) and the D_j are completely invariant for g. To apply Proposition 2.7 and complete the proof it remains to show that the D_j are simply-connected. But each D_j has boundary ∂D_j closed and completely invariant under g, and hence $\partial D_j = J(f)$. It follows that

$$\hat{\mathbf{C}} - \bar{D}_1 = \hat{\mathbf{C}} - (J(f) \cup D_1) = F(f) - D_1 = D_2 \cup ... \cup D_k$$

Hence $D_2, ...D_k$ are the components of the complement of the connected set D_1 and are therefore simply-connected. Similarly D_1 is simply-connected. QED

Examples

(i) $z \to z^2 - 1$

The basin of infinity is a completely invariant component.

The components containing 0 and -1 form a periodic 2-cycle.

All other components are pre-periodic (fall onto the period two cycle after a finite number of steps).

(ii) $z \to z^2 + c$ with $|c|$ large.

Here $F(f)$ has a single component, the complement in $\hat{\mathbf{C}}$ of a Cantor set (but note that this component is multiply connected).

A key theorem concerning the components of $F(f)$ is

Theorem 2.9 (Sullivan's 'No Wandering Domains Theorem' 1985) *Every component of $F(f)$ is either periodic or preperiodic*

For a proof see Sullivan (Annals 1985). The basic idea is that if there were a wandering domain then it would be possible to construct an infinite-dimensional family of perturbations of f, all of them rational and topologically conjugate to f, but this is impossible since f is determined by a finite set of data (as already remarked earlier). However the technical details take one deep into the theory of Teichmüller spaces and quasiconformal geometry, beyond the scope of these notes. The original conjecture that f could not have wandering domains was due to Fatou.

The *basin* of an attractive fixed point z_0 is the set $\{z : \lim_{n\to\infty} f^n(z) = z_0\}$ and the *immediate basin* is the component of this set containing z_0. There are similar definitions for an attracting period n cycle: here the immediate basin is the set of components of the basin containing points of the cycle.

Theorem 2.10 *The immediate basin of an attractive periodic point (for a rational map f of degree at least two) contains a critical point.*

Proof Without loss of generality we suppose z_0 to be an attracting *fixed point.* If z_0 is superattracting, the result is obvious. if z_0 is attracting but not superattracting then there is a neighbourhood U of z_0 such that $f(U) \subset U$ and $f|_U$ is injective. Let $V = f(U)$ and consider the branch of f^{-1} sending V to U. If f has no *critical value* in U, this branch can be extended to the whole of U and hence f^{-2} has a well-defined branch on V. Repeat. If some $f^{-n}(V)$ contains a critical value then the basin contains a critical point. but if not, then $\{f^{-n}\}_{n>0}$ are all defined on V and have images in the the immediate basin. But then they would form an equicontinuous family (by Montel's Theorem) and this is impossible since z_0 is a *repelling* fixed point for f^{-1}. QED

Corollary 2.11 *If f has degree d then it has at most $2d - 2$ attracting or superattracting cycles.*

Shishikura (1987) improved this bound to 'at most $2d - 2$ non-repelling cycles'.

2.4 Linearisation Theorems

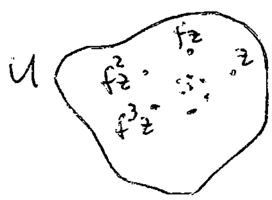
Dynamics of f near a fixed or periodic point

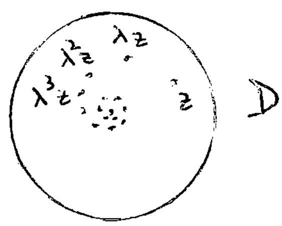
In the neighbourhood of a fixed point, which without loss of generality we take to be $z_0 = 0$, $f(z) = \lambda z + O(z^2)$ (Taylor series), where λ is the multiplier at the fixed point. We say that f is *linearizable* if there is a neighbourhood U on which f is *conjugate* to $z \to \lambda z$ (by a complex analytic conjugacy).

Theorem 2.12 (Koenigs' Linearization Theorem 1884) *If $\lambda \neq 0$ and $|\lambda| \neq 1$ then f is linearizable*

Proof

Assume first that $0 < |\lambda| < 1$. Consider the orbits:





Set

$$h_n(z) = \frac{1}{\lambda^n} f^n(z)$$

Then, by construction $h_n f(z) = \lambda h_{n+1}(z)$, and it suffices to show that the $\{h_n\}$ converge locally uniformly to a function h. Rather than consider the details in general, see the example below.

For the case $1 < |\lambda| < \infty$ one can proceed in exactly the same fashion as before, but with f^{-1} in place of f. QED.

Example

$f(z) = \lambda z + z^2$ (where $|\lambda| < 1$). Here the orbit of z_0 is
$z_1 = f(z_0) = \lambda z_0(1 + z_0/\lambda)$
$z_2 = f(z_1) = \lambda z_1(1 + z_1/\lambda) = \lambda^2 z_0(1 + z_0/\lambda)(1 + z_1/\lambda)$
...
$z_n = \lambda^n z_0(1 + z_0/\lambda)(1 + z_1/\lambda)...(1 + z_{n-1}/\lambda)$

Thus $h_n(z_0) = z_0(1+z_0/\lambda)(1+z_1/\lambda)...(1+z_{n-1}/\lambda)$ where $\{z_n\}$ is the orbit of z_0. As n tends to infinity, z_n tends to 0, and $\{h_n\}$ converge locally uniformly to

$$h(z_0) = z_0 \prod_0^\infty (1 + \frac{z_n}{\lambda})$$

Note how the dynamics have been used to construct an explicit conjugacy. One can also construct the coefficients of h recursively, directly from the functional equation $hf(z) = \lambda h(z)$, but the dynamical motivation is then no longer apparent.

Theorem 2.13 (Böttcher 1904) *If $f(z) = z^k + O(z^{k+1})$ ($k \geq 2$ integer) then f is conjugate to $z \to z^k$ on a neighbourhood of* 0.

Proof Analogously to 2.12, we set $h_n(z) = (f^n(z))^{1/k^n}$. Then $h_n f(z) = (h_{n+1}(z))^k$ and the $\{h_n\}$ converge locally uniformly to a function h conjugating f to $z \to z^k$. QED

The proof above is only a sketch! The right choice of branch of k^nth root in the definition of h_n is important, but rather than fill in the details in general, we consider an example, one that will also be useful later.

Example

Consider $f : z \to z^2 + c$ near the fixed point ∞.
Write this map as $z \to z^2(1 + c/z^2)$.
$z_1 = f(z_0) = z_0^2(1 + c/z_0^2)$
$z_2 = f(z_1) = z_1^2(1 + c/z_1^2) = z_0^4(1 + c/z_0^2)^2(1 + c/z_1^2)$
...
$z_n = z_{n-1}^2(1 + c/z_{n-1}^2) = z_0^{2^n}(1 + c/z_0^2)^{2^{n-1}}(1 + c/z_1^2)^{2^{n-2}}...(1 + c/z_{n-1}^2)$

So $h_n(z_0) = z_0(1 + c/z_0^2)^{1/2}(1 + c/z_1^2)^{1/4}...(1 + c/z_{n-1}^2)^{1/2^n}$ where the choice of each root is the obvious one coming from the binomial expansion. As n tend to∞ the z_n tend to ∞ (since z_0 is outside the filled Julia set). Thus the h_n converge (locally uniformly) to

$$h(z_0) = z_0 \prod_0^\infty (1 + \frac{c}{z_n^2})^{2^{-(n+1)}}$$

Once again one could compute explicit formulae for the coefficients of h using recursion relations based on the functional equation, but they are far less revealing than the dynamical approach above.

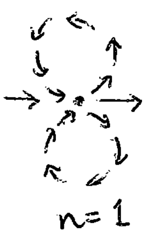
We shall come back to this example when we look at the Mandelbrot set later. Meanwhile, what can be said about linearizability near a *neutral* fixed point ?

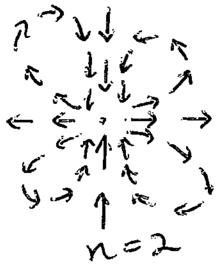
Suppose $f(z) = \lambda z + O(z^2)$, with $|\lambda| = 1$.

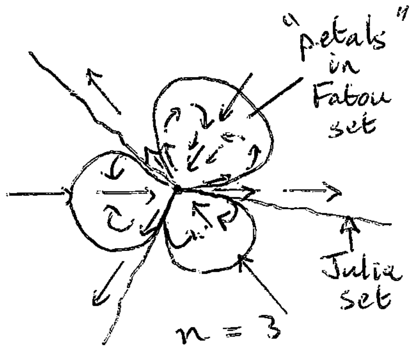
Case 1: $\lambda = e^{2\pi i p/q}$

Then f is not linearizable (unless $f(z) = \lambda z$), since $f^q \neq$ *identity.*

Example $f(z) = z + z^{n+1}$







Theorem 2.14 (Comacho 1978) *If $f(z) = \lambda z + O(z^2)$, with $\lambda = e^{2\pi i p/q}$, then either f^q = identity (in which case $f(z) = \lambda z$) or f is topologically conjugate to a map $z \to \lambda z(1 + z^{kq})$ for some positive integer k*

It follows that the attractive basin contains a 'flower with k petals' as shown. (When $q \neq 1$ the petals rotate through $2\pi p/q$ each time f is applied.)

Notes:

1. The conjugacy in Theorem 2.14 is not in general conformal, indeed it is not in general smooth.

2. A neutral fixed with multiplier $\lambda = e^{2\pi i p/q}$ lies in $J(f)$, but it has a basin of attraction in $F(f)$ and it can easily be shown (using Montel's Theorem) that the immediate basin must contain a crtitical point (see Beardon).

Case 2: $\lambda = e^{2\pi i\alpha}$ with α irrational

Here it all depends on 'how irrational α is'. Write α as a continued fraction

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} = [a_0, a_1, a_2, \ldots]$$

and let p_n/q_n (in lowest terms) be the value of its nth truncation $[a_0, a_1, \ldots, a_n]$.
(For example the golden mean $[0, 1, 1, 1, 1, \ldots]$ has $p_1/q_1 = 1/1, p_2/q_2 = 1/2, p_3/q_3 = 2/3, p_4/q_4 = 3/5, \ldots.$)

Definition α satisfies the *Brjuno condition* if and only if

$$\sum_1^\infty \frac{log(q_{n+1})}{q_n} < \infty$$

We write $\mathcal{B}$ for the set of real numbers satisfing the Brjuno condition.

The combined results of Siegel (1942), Brjuno (1965) and Yoccoz (1987) yield

Theorem 2.15
$\alpha \in \mathcal{B} \Leftrightarrow$ all complex analytic maps $z \to e^{2\pi i\alpha}z + O(z^2)$ are linearizable $\Leftrightarrow z \to e^{2\pi i\alpha}z + z^2$ is linearizable

When a linearisation exists its domain is called a *Siegel disc.*

Notes

1. Yoccoz's proof of the *necessity* of the Brjuno condition is technical and difficult: it is motivated by ideas of *renormalization.*

2. The Siegel disc around a linearizable irrational neutral fixed point is in the Fatou set $F(f)$. It can be shown the Siegel discs 'use up' critical points in the sense that the boundary of a Siegel disc necessarily lies in the accumulation set of the forward orbit of some critical point.

3. The irrational neutral points which are not linearizable are known as *Cremer points* (after Cremer 1928). They lie in $J(f)$. The dynamics around them can be very complicated (Perez-Marco 1992) and has a structure which depends on the continued fraction of α.

Finally in this section we remark that it can be shown (see for example Sullivan 1985) that for a polynomial the only possible components of a Fatou set are components of the basin of

1. a superattracting periodic orbit;
2. an attracting periodic orbit;
3. a rational neutral periodic orbit;
4. a periodic cycle of Siegel discs.

There is one other type that can ocur for rational f (but not polynomial f), components of the basin of

5. a periodic cycle of *Herman rings.*

(A Herman ring is an annulus with dynamics conjugate to an irrational rotation.)

These are the 5 types of 'regular behaviour' of a rational map. To completely understand rational maps we have to understand how they fit together with each other, and with the behaviour on the complement of the regular domain, the Julia set. As we shall see, there are still unanswered questions even in the simplest case, that of quadratic maps $z \to z^2 + c$.