Queen Mary
University of London

B. Sc. Examination by course unit 2010

MTH6111 Complex Analysis

Duration: 2 hours

Date and time: 17 May 2010, 14.30h–16.30h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best 4 questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): Shaun Bullett

Question 1 In parts (a) and (b) of this question U is an open subset of $\mathbb{C}$ and f is a function from U to $\mathbb{C}$.

(a) What is meant by the statement that f is *(complex) differentiable* at $z_0 \in U$? What does it mean to say that f is *holomorphic* in U? [3]

(b) Let u and v denote the real and imaginary parts of f. Show that if f is differentiable at $z_0 = x_0 + iy_0 \in U$ then the partial derivatives of u and v exist at (x_0, y_0) and satisfy the *Cauchy-Riemann equations* there. [8]

(c) Let $f(z) = z(z + \bar{z})$. Show that f cannot be differentiable at any point $z \neq 0$ of $\mathbb{C}$, prove from first principles that f is differentiable at $z = 0$, and compute the derivative of f there. [7]

(d) Let $u : \mathbb{R}^2 \to \mathbb{R}$ be the function $u(x, y) = x^2 - y^2 - y$. Find a harmonic conjugate $v : \mathbb{R}^2 \to \mathbb{R}$ for u, and express $u + iv$ as a (holomorphic) function of z, where $z = x + iy$. [7]

Question 2 Let U be an open subset of $\mathbb{C}$, f be a continuous function $U \to \mathbb{C}$, and $\gamma : [\alpha, \beta] \to U$ be a C^1 path in U.

(a) Define $\int_\gamma f(z)dz$, the *integral of f along γ*. [3]

(b) Evaluate $\int_\gamma f(z)dz$ when $f(z) = z - \bar{z}$ and γ is the straight line path between $z = 1$ and $z = i$. [7]

(c) Prove that if f has an antiderivative F (that is, $f = F'$), then

$$\int_\gamma F'(z)dz = F(\gamma(\beta)) - F(\gamma(\alpha)).$$

(You may assume the Fundamental Theorem of Calculus for functions $\mathbb{R} \to \mathbb{R}$.) [6]

(d) Let $C(z_0, r)$ denote the path which goes anticlockwise once around the circle which has centre z_0 and radius r. The following integrals are all zero. Give a reason (or reasons) in each case.

$$(i) \int_{C(0,1)} \frac{1}{z+2}dz \qquad (ii) \int_{C(0,2)} \frac{\sin(z-1)}{z-1}dz \qquad (iii) \int_{C(1,1)} \frac{1}{(z-1)^2}dz$$

[9]

Question 3 Define the *length* of a C^1 path γ in $\mathbb{C}$, and state the *Estimation Lemma* for an integral $\int_\gamma f(z)dz$. [5]

EITHER

State *Cauchy's Theorem for a Triangle*, and use it to prove that every holomorphic function on a convex open set has an antiderivative there.

OR

State *Cauchy's Integral Formula* and use it to prove that every bounded entire function is constant *(Liouville's Theorem)*. [20]

Question 4 State the *Residue Theorem.* [5]

Let $a > 0$ be a real number. Prove that the real integral

$$\int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx = \frac{\pi}{4a}$$

(Justify any estimates you make in your proof.) [20]

Question 5 (a) For $z \in \mathbb{C} \setminus \{0\}$, how is the (multivalued) function $\log z$ defined? [2]

(i) Describe the *Riemann surface* for $\log z$. [4]

(ii) If $z \in \mathbb{C} \setminus \{0\}$ and $\alpha \in \mathbb{C}$, how is z^α defined? [2]

(iii) Find all values of $(1+i)^i$. [5]

(b) State the Riemann Mapping Theorem. [2]

(i) Find a Möbius transformation which maps the right-hand half of the complex plane $\mathcal{R} = \{x + iy : x > 0\}$ bijectively onto the unit disc. [5]

(ii) Let $S = \{x + iy : \ x > y > 0\}$. Find a conformal transformation which sends S bijectively onto the unit disc. [5]

Question 6 What is meant by an *isolated singularity* of a complex function f? [2]

(a) Define the three types of isolated singularity *(removable, pole, essential)* and describe (without proof) the behaviour of f in a neighbourhood of a singularity of each type. [7]

(b) What is meant by saying that f is *meromorphic in* $\mathbb{C}$, and by saying that it is *meromorphic at* ∞? Prove that every rational function is meromorphic at ∞. [7]

(c) What is meant by an *elliptic function*, and what is meant by its *order*? Prove that a (non-constant) elliptic function cannot have order 0 or 1. [9]

End of Paper