VI. Further Topics

We introduce (very briefly) two of the many areas of current research in mathematics which involve complex analysis. *(These two topics are not for exam.)*

26. The Zeta function: Riemann's Hypothesis

The end of the 20th century and the start of the 21st century have seen the proofs of many celebrated and long-lasting mathematical questions, including the Four Colour Theorem (Appel and Haken 1976), Fermat's Last Theorem (Wiles 1995) and the Poincaré Conjecture (Perelman 2003). But the *Riemann Hypothesis*, formulated by Riemann in 1859, has so far resisted all efforts and is regarded by many mathematicians as the most important unsolved problem.

For $s \in \mathbb{C}$ with $Re(s) > 1$, the *Riemann zeta function* is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This series converges when $Re(s) > 1$. The zeta function is connected with the prime numbers as can be seen from:

The Euler Product Formula

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{\left(1 - \frac{1}{p^s}\right)}$$

which follows from expanding each of the terms on the right

$$\frac{1}{\left(1 - \frac{1}{p^s}\right)} = 1 + \frac{1}{p^s} + \frac{1}{(p^s)^2} \cdots$$

then observing that their product is therefore a sum of terms of the form

$$\frac{1}{(p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k})^s}$$

and finally using the fact that positive integer n has a unique expression of the form $n = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$.

In order to analytically continue $\zeta(s)$ to other parts of the complex plane, we shall need the *gamma function*, which for real values of $s > 0$ is defined by the real integral

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx$$

When s is a positive integer n, we can evaluate this integral (inductively, using integration by parts) and see that $\Gamma(n) = (n-1)(n-2)\ldots 2.1 = (n-1)!$. But the integral formula can also be used to define $\Gamma(s)$ for any complex value value of s which has $Re(s) > 0$, and it can then be proved that

$$\Gamma(s) = (s-1)\Gamma(s-1) \ \ \forall\ s \in \mathbb{C} \text{ with } Re(s) > 1$$

Finally we can use this relation between $\Gamma(s-1)$ and $\Gamma(s)$ to extend our definition of $\Gamma(s)$ to *all* $s \in \mathbb{C}$ except for the points $s = 0, -1, -2, -3, \ldots$. Indeed one can show that Γ is meromorphic on $\mathbb{C}$ except for simple poles at these points, and that Γ has no zeros on $\mathbb{C}$.

It can be shown that for s in the region in which we have so far defined $\zeta(s)$, that is to say $Re(s) > 1$,

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^s}{e^z - 1} \frac{dz}{z}$$

where C is a path which runs just above the positive real axis from ∞ to the origin, goes round the origin in a small semicircle, and then runs back, just below the positive real axis, to ∞.

This last formula for $\zeta(s)$ can be used to define $\zeta(s)$ for *all* $s \in \mathbb{C}$. It provides a meromorphic continuation of our original function ζ, with a simple pole at $s = 1$ and zeros at $s = -2, -4, -6, \ldots$ (it turns out that $\zeta(-1) = -1/12$, in other words "$1 + 2 + 3 + 4 + ... = -1/12$", and that more generally $\zeta(s) \neq 0$ at each of the negative odd integers). Our original function ζ had no zeros in the region $Re(s) > 1$ and it is easily proved that for this meromorphic continuation the only zeros in the region $Re(s) < 0$ are the 'trivial' zeros at the negative even integers. This leaves the strip $0 \leq Re(s) \leq 1$.

Riemann's Hypothesis *All non-trivial zeros of* $\zeta(s)$ *lie on the vertical line* $Re(s) = 1/2$.

Hardy (1914) proved that $\zeta(s)$ has infinitely many zeros on this line, and modern computer calculations have been used to check that first 10^{13} non-trivial zeros all lie on $Re(s) = 1/2$, but no-one has yet *proved* there are no non-trival zeros off the line. A proof would have important implications for the distribution of the primes. Let $\pi(x)$ denote the number of primes less than x. The *Prime Number Theorem*, proved by Hadamard and de la Vallée Poussin in the late 1890s, states that

$$\pi(x) \sim \frac{x}{\ln x}$$

Riemann's Hypothesis, if true, would give a great deal of information about the error term in this result, and would have important consequences throughout the theory of numbers.

For more on Riemann's Hypothesis see:

H.M.Edwards, 'Riemann's Zeta Function', Dover Paperback 2001 (Academic Press hardback, 1974) - this includes an English translation of Riemann's original 8-page article in 1859.

Or see many other books - or the Wikipedia article.

27. Complex iteration: Julia sets and the Mandelbrot set

The theory of iterated rational maps on the Riemann sphere was initiated by Fatou and Julia in the 1920s. They developed a very elegant theoretical framework and proved many theorems concerning what are now known as 'Julia sets', but it was only with coming of the microcomputer in the 1980s that mathematicians were able to draw pictures displaying the astonishing fractal structure of these sets. The modern theory of holomorphic dynamics has played a central role in the study of nonlinear dynamics ('chaos') over the last thirty years, in particular in the elucidation of universal constants, such as the Feigenbaum constants for period-doubling. In that time two Fields Medals have been awarded for work in holomorphic dynamics: to Jean-Cristophe Yoccoz in 1994 and to Curt McMullen in 1998.

Attracting and repelling orbits

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. The *orbit* of an initial point z_0 is the sequence of points

$$z_0,\ z_1 = f(z_0),\ z_2 = f(z_1), \ldots,\ z_n = f(z_{n-1}), \ldots$$

We say that z_0 is a *fixed point* if $z_1 = z_0$, and that it is a *point of period* n if $z_n = z_0$. A fixed point z_0 is described as *attracting* if the derivative of f at z_0 has modulus < 1 and *repelling* if the derivative of f at z_0 has modulus > 1. Similarly if z_0 has period n we say its orbit is *attracting* if f^n (the composite of n copies of f) has derivative of modulus < 1 at z_0, and *repelling* if this derivative has modulus > 1. Orbits started near an attracting periodic orbit get pulled towards it: orbits started near a repelling periodic orbit get pushed away from it.

The Fatou set and the Julia set of a rational map

The *Julia set* $J(f)$ of f is defined to be the *closure of the set of repelling periodic orbits of* f. For example, if f is the map $f(z) = z^2$ then $J(f)$ consists of all points on the unit circle, since each point of the form

$$z = e^{2\pi i p/(2^q-1)}$$

is easily checked to be periodic of period q.

The *Fatou set* $F(f)$ is defined to be the largest open subset of $\mathbb{C}$ on which the family of maps $\{f^n\}_{n\geq 1}$ form what is called a *normal* family, or equivalently the largest open subset of $\mathbb{C}$ on which this family of maps is *equicontinuous* (with the usual ϵ and δ definition of continuity, 'equicontinuity' means that for a given ϵ there is a single δ that works for *all* the f^n).

Theorem *$J(f)$ and $F(f)$ form a partition of $\hat{\mathbb{C}}$, that is, every point of $\hat{\mathbb{C}}$ is in one or the other but not both. Moreover both $J(f)$ and $F(f)$ are fully invariant under f.*

To say that a set S is *fully invariant under f* is to say that $z \in S \Leftrightarrow f(z) \in f(S)$. So f maps each of $F(f)$ and $J(f)$ to itself. One should think of $F(f)$ as being the subset of $\hat{\mathbb{C}}$ on which the behaviour of f is predictable, in the sense that orbits which start close together remain close together, and $J(f)$ as being the set on which the behaviour of f is 'chaotic' (orbits which start close together move apart, so a small error in specifying the initial point has large consequences later).

Quadratic polynomials and the Mandelbrot set

Fatou-Julia theory shows (among other things) that the behaviour of an iterated rational map is determined by the behaviour of its critical points. Quadratic polynomials $q_c(z) = z^2 + c$ (where the complex number c is a constant) are the easiest examples to study since they have just two critical points: one is the point $z = 0$, and the other is the point $z = \infty$ (which is an attracting fixed point). For the quadratic polynomial q_c, the *filled Julia set* $K(q_c)$ is defined to be the set of all initial points z such that the orbit of z under q_c is bounded. The Julia set, $J(q_c)$, is the *boundary* of $K(q_c)$. It turns out that if the critical point 0 lies in $K(q_c)$, then $J(q_c)$ is a connected set, and if $0 \notin K(q_c)$ then $J(q_c)$ is totally disconnected (it is 'dust'). Thus an important set to study is the *Mandelbrot set:*

$$M = \{c : \ ((q_c)^n(0))_{n\geq 1} \text{ does not tend to } \infty\}$$

The Mandelbrot set M has an intricate self-similar structure on all scales. Although Mandelbrot's first computer pictures (around 1980) appeared to show off-shore islands, Douady and Hubbard proved:

Theorem (Douady and Hubbard 1982) *There is a Riemann mapping (a conformal bijection) between the open unit disc and $\hat{\mathbb{C}} \setminus M$. Hence M is connected.*

At the same time they conjectured:

The MLC Conjecture *M is locally connected.*

If the MLC conjecture is true then the Riemann mapping of the Theorem can be extended to a continuous map from the boundary of the unit disc (the unit circle) onto the boundary ∂M of the Mandelbrot set, and this would allow us to give a complete combinatorial description of M. But for nearly 30 years now MLC has resisted the efforts of some of the best mathematicians in the world.

Plots of Julia sets and M can be found on the web. To find out more about the mathematics see:

Alan Beardon, 'Iteration of Rational Functions', Graduate Texts in Mathematics 132, Springer Verlag 1991

John Milnor, 'Dynamics in One Complex Variable', Annals of Mathematics Studies 160, Princeton University Press 2006