MTH6111 Complex Analysis 2010-11

I. Holomorphic Functions

1. The Complex Numbers $\mathbb{C}$

A complex number z is a pair of real numbers $(x, y) \in \mathbb{R}^2$. We write z as '$x + iy$'.

Notation: $x = Re(z)$; $y = Im(z)$; $\bar{z} = x - iy$ (conjugate); $|z| = \sqrt{x^2 + y^2}$ (modulus).

Polar notation: $z = re^{i\theta}$ (where $r = |z|$, $\cos\theta = x/|z|$ and $\sin\theta = y/|z|$).

Addition of complex numbers is the usual vector addition in $\mathbb{R}^2$.

Multiplication of complex numbers is defined by:
$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$
(which in polar coordinates is $r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$).

$z \neq 0$ has inverse $z^{-1} = \bar{z}/|z|^2$ (in polar coordinates $(1/r)e^{-i\theta}$).

1.1 Elementary properties

- $\mathbb{C}$ is a field;
- $\overline{z + w} = \bar{z} + \bar{w}$;
- $\overline{zw} = \bar{z}\bar{w}$;
- $|zw| = |z||w|$;
- $|z| = \sqrt{z\bar{z}}$;
- $|Re(z)| \leq |z|$; $|Im(z)| \leq |z|$;
- $|z + w| \leq |z| + |w|$;
- $|z + w| \geq ||z| - |w||$.

The first six are immediate from the definitions; for the last two see your 'Complex Variables' lecture notes, Priestley Chapter 1, or Stewart and Tall Chapter 1.

2. Open sets and closed sets

Let $a \in \mathbb{C}$.

The *open disc* with centre a and radius r is $D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$.
The *closed disc* with centre a and radius r is $\overline{D}(a, r) = \{z \in \mathbb{C} : |z - a| \leq r\}$.
The *punctured disc* with centre a and radius r is $D'(a, r) = \{z \in \mathbb{C} : 0 < |z - a| < r\}$.

Definitions

- $S \subseteq \mathbb{C}$ is *open* if $\forall z \in S\ \exists r > 0$ such that $D(z,r) \subseteq S$;
- $S \subseteq \mathbb{C}$ is *closed* if its complement $\mathbb{C} \setminus S$ is open;
- $z \in \mathbb{C}$ is a *limit point* of S if $\forall r > 0\ D'(z,r) \cap S \neq \emptyset$;
- the *closure* $\overline{S}$ of $S \subseteq \mathbb{C}$ is the union of S and its limit points.

2.1 Proposition *$S \subseteq \mathbb{C}$ is closed if and only if S contains all its limit points (i.e. $\overline{S} = S$).*

Proof

Suppose S is closed. Consider any $z \in \mathbb{C} \setminus S$. As $\mathbb{C} \setminus S$ is open $\exists r > 0$ such that $D(z,r) \subseteq \mathbb{C} \setminus S$. Hence z is not a limit point of S. Thus S contains all its limit points.

Conversely, suppose S contains all its limit points. Again consider any $z \in \mathbb{C} \setminus S$. As z is not a limit point of S, there exists $r > 0$ such that $D'(z,r) \subseteq \mathbb{C} \setminus S$. But since $z \in \mathbb{C} \setminus S$ it follows that $D(z,r) \subseteq \mathbb{C} \setminus S$. So $\mathbb{C} \setminus S$ is open. Hence S is closed. $\square$

Definition $S \subseteq \mathbb{C}$ is *bounded* if $\exists M \in \mathbb{R}$ such that $\forall z \in S$, $|z| \leq M$ (i.e. $S \subseteq \bar{D}(0,M)$).

A subset of $\mathbb{C}$ which is both closed and bounded will be called *compact*.

Comment The standard definition of 'compact' for a topological space X is that a subset $S \subseteq X$ is compact if and only if 'every open covering of S has a finite subcover'. The Heine-Borel Theorem states that in the case that $X = \mathbb{C}$ (indeed more generally in the case $X = \mathbb{R}^n$) this property is equivalent to 'S is closed and bounded'. For this course we will take 'closed and bounded' as the *definition* of 'compact' (as do Priestley, and Stewart and Tall).

3. Limits, continuity and compactness

The results in this section follow from $\mathbb{C}$ being viewed as $\mathbb{R}^2$. They do not use algebraic properties of $\mathbb{C}$.

Definitions

(i) The sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ *converges* to $a \in \mathbb{C}$, written $\lim_{n\to\infty} z_n = a$, if and only if $\forall \epsilon > 0\ \exists N$ such that $z_n \in D(a,\epsilon)\ \forall n \geq N$.

(ii) Let $S \subseteq \mathbb{C}$, let $f : S \to \mathbb{C}$ be a function, and let $a \in \mathbb{C}$. We say that $\lim_{z\to a} f(z) = w$ if and only if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $z \in D'(a,\delta) \cap S \Rightarrow f(z) \in D(w,\epsilon)$.

(iii) Let $S \subseteq \mathbb{C}$, let $f : S \to \mathbb{C}$ be a function, and let $a \in \mathbb{C}$. We say that f is *continuous* at a if and only if $f(a) = \lim_{z\to a} f(z)$ (i.e. $\forall \epsilon > 0\ \exists \delta > 0$ such that $z \in D(a,\delta) \cap S \Rightarrow f(z) \in D(f(a),\epsilon)$).

3.1 Lemma

(i) Let $(z_n)_{n\in\mathbb{N}}$ *be a sequence of complex numbers, let* $x_n = Re(z_n), y_n = Im(z_n)$*, and let* $a = b + ic$*. Then* $(z_n)_{n\in\mathbb{N}}$ *converges to* a *if and only if* $(x_n)_{n\in\mathbb{N}}$ *converges to* b *and* $(y_n)_{n\in\mathbb{N}}$ *converges to* c*.*

(ii) Let $f(z) = u(z) + iv(z)$ *(i.e.* $u(z) = Re(f(z))$ *and* $v(z) = Im(f(z))$*). Then* $\lim_{z\to a} f(z) = s + it$ *if and only if* $\lim_{z\to a} u(z) = s$ *and* $\lim_{z\to a} v(z) = t$*.*

(iii) Let $f(z) = u(z) + iv(z)$ *(as in (ii)). Then* f *is continuous at* $a \in \mathbb{C}$ *if and only if* u *and* v *are continuous at* a*.*

Proof

To save writing we shall use the notation (z_n) in place of $(z_n)_{n\in\mathbb{N}}$.

(i) *If* (z_n) *converges to* a then $\forall\epsilon > 0,\ \exists N$ such that $\forall n \geq N,\ |z_n - a| < \epsilon$.
Thus $\forall n \geq N,\ |x_n + iy_n - (b + ic)| < \epsilon$, in other words $|(x_n - b) + i(y_n - c)| < \epsilon$.
So $\forall n \geq N,\ |x_n - b| < \epsilon$ and $|y_n - c| < \epsilon$.
Hence (x_n) converges to b and (y_n) converges to c.

Conversely, if (x_n) *converges to* b *and* (y_n) *converges to* c then $\forall\epsilon > 0$
$\exists N_1$ such that $\forall n \geq N_1$, $|x_n - b| < \epsilon/2$ and $\exists N_2$ such that $\forall n \geq N_2$, $|y_n - c| < \epsilon/2$.
Let $N = max(N_1, N_2)$. Then $\forall n \geq N$ we have:
$|z_n - a| = |(x_n - b) + i(y_n - c)| \leq |x_n - b| + |y_n - c| < \epsilon/2 + \epsilon/2 = \epsilon$.
So (z_n) converges to c.

(ii) has a similar proof to (i) (and is an exercise on Sheet 1).
(iii) is an immediate consequence of (ii). □

3.2 Theorem *Every bounded sequence in* $\mathbb{C}$ *has a convergent subsequence.*

Proof

Suppose (z_n) is bounded by M. Then (x_n) is bounded by M and so is (y_n).

Let S_0 be the square $|x| \leq M$, $|y| \leq M$.

Divide S_0 into 4 equal squares, each of side length M. At least one of these squares contains z_n for infinitely many n. Choose one such square and call it S_1.

Subdivide S_1 into 4 equal squares, each of side length $M/2$. Choose one of these containing z_n for infinitely many n, and call it S_2.

Repeat, to obtain a sequence of squares $S_0 \supset S_1 \supset S_2 \ldots \supset S_k \supset \ldots$, with each S_k a square which contains infinitely many members of the sequence and has side length $M/2^{k-1}$. In each S_k choose a member z_{n_k} of the sequence, with $n_0 < n_1 < n_2 < \ldots < n_k < \ldots$.

Since S_k has side length $M/2^{k-1}$, the sequences $(x_{n_k})_{k\geq 0}$ and $(y_{n_k})_{k\geq 0}$ are Cauchy sequences in $\mathbb{R}$. (Recall that a sequence (a_n) in $\mathbb{R}$ is called a Cauchy sequence if $\forall\epsilon > 0\ \exists N$ such that $|a_n - a_m| < \epsilon$ $\forall m, n > N$, and recall also that every Cauchy sequence in $\mathbb{R}$ has a limit).

Hence $\lim_{k\to\infty} x_{n_k}$ exists, as does $\lim_{k\to\infty} y_{n_k}$. Writing b and c for these limits we deduce by (3.1) that $\lim_{k\to\infty} z_{n_k}$ exists and is equal to $b + ic$. □

We deduce the following useful characterisation of compactness.

3.3 Corollary *$S \subseteq \mathbb{C}$ is compact if and only if every sequence in S has a subsequence which converges to a limit in S. ("S is compact if and only if S is sequentially compact.")*

Proof

First suppose that S compact. Then (from our definition of compact) S is bounded, so every sequence in S has a subsequence which converges to a limit in $\mathbb{C}$, by (3.2). But (also from our definition of compact) S is closed, so this limit is in S.

For the converse, suppose that *every sequence in S has a subsequence which converges to a limit in S.* We must show that it follows that S is bounded and closed.

But if S were unbounded, we could choose a sequence $z_1, z_2, z_3 \ldots \in S$ with z_1 having $|z_1| > 1$, z_2 having $|z_2| > 2$, z_3 having $|z_3| > 3$ and so on, and this sequence would then have no convergent subsequence. So S is bounded.

To prove that S is closed, note that given any limit point a of S, we can choose a sequence (z_n) with $z_n \in D'(a, 1/n) \cap S$. But now (z_n), and every subsequence of (z_n), converges to a. Hence (by our hypothesis) $a \in S$. But a was an arbitrary limit point of S. Hence S is closed. □

3.4 Theorem *let $S \subseteq \mathbb{C}$ be compact and $f : S \to \mathbb{C}$ be continuous. Then $f(S)$ is compact and $|f|$ attains its bound on S (i.e. if $M = \sup_{z\in S} |f(z)|$ then $\exists a \in S$ such that $|f(a)| = M$).*

Proof Let (w_n) be any sequence in $f(S)$. Then each $w_n = f(z_n)$ for some $z_n \in S$. By (3.3), since S is compact there is a subsequence (z_{n_k}) converging to some $a \in S$.

Since f is continuous, $f(a) = lim_{k\to\infty} f(z_{n_k}) = \lim_{k\to\infty} w_{n_k}$. So $lim_{k\to\infty} w_{n_k} \in f(S)$. Thus (w_n) has a subsequence which converges to a limit in $f(S)$. Hence, again by (3.3), $f(S)$ is compact.

We next use a similar method to show that $|f|$ attains its bound on S. Let (w_n) be a sequence in $f(S)$ which has $\lim_{n\to\infty} |w_n| = M$ $(= \sup_{z\in S} |f(z)|)$. As in the first part of the proof, each $w_n = f(z_n)$ for some $z_n \in S$, and the sequence (z_n) has some subsequence (z_{n_k}) converging to some $a \in S$ (by (3.3)). Now $f(a) = \lim_{k\to\infty} f(z_{n_k}) = \lim_{k\to\infty} w_{n_k}$ (by continuity of f at a). So $|f(a)| = lim_{k\to\infty}|w_{n_k}| = M$. □

Remark Theorem 3.4 is a complex analogue of the well-known theorem of real analysis that 'every continuous real function on a closed interval is bounded and attains its bounds'.

Examples of how we use 3.4 later:

1. If γ is a path in $\mathbb{C}$ (i.e. a continuous function from a closed interval $[\alpha, \beta] \subset \mathbb{R}$ to $\mathbb{C}$) then its image $\gamma([\alpha, \beta])$ is a closed bounded subset of $\mathbb{C}$, so in particular $\gamma([\alpha, \beta]) \subset D(0, R)$ for some finite R.
Proof Think of $[\alpha, \beta]$ as a subset of $\mathbb{C}$ and apply the first part of 3.4.

2. If $\gamma([\alpha, \beta]) \subset D(0, R)$ then $\gamma([\alpha, \beta])$ is contained in the closed disc $\overline{D}(0, r)$ for some $r < R$.
Proof Let $r = sup_{z \in \gamma([\alpha,\beta)])}|z|$. Then (by the definition of 'sup') $\gamma([\alpha, \beta)] \subset \overline{D}(0, r)$. But by the final part of 3.4 there exists a point $z \in \gamma([\alpha, \beta)]$ with $|z| = r$. As $z \in D(0, R)$, we deduce that $r < R$.

4. Differentiation

Definitions

Let U be an open subset of $\mathbb{C}$. We say that $f : U \to \mathbb{C}$ is *differentiable* at $z \in U$ if

$$\lim_{h \in \mathbb{C}, h \to 0} \frac{f(z + h) - f(z)}{h}$$

exists. When it exists, this limit is denoted $f'(z)$ (or df/dz).

If f is differentiable at every point of U we say that f is *holomorphic in* U, and we write $f \in H(U)$.

4.1 Theorem (Cauchy-Riemann equations). *Let $f : U \to \mathbb{C}$ be differentiable at a point $z = x + iy \in U$, and let $u(x, y) = Re(f(z))$ and $v(x, y) = Im(f(z))$.*
Then the partial derivatives $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x, \partial v/\partial y$ exist at (x, y) and satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

there.

Proof

(*)
$$f'(z) = \lim_{h \in \mathbb{C}, h \to 0} \frac{f(z + h) - f(z)}{h}$$

Taking the limit through real values of h we deduce that

$$f'(z) = \lim_{h \in \mathbb{R}, h \to 0} \frac{u(x + h, y) - u(x, y) + i(v(x + h, y) - v(x, y))}{h}.$$

Now, taking real and imaginary parts and applying 3.1(ii), we deduce that $\partial u/\partial x$ and $\partial v/\partial x$ exist, and that

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Taking the limit in (*) through imaginary values $h = ik$ we deduce that

$$f'(z) = \lim_{k \in \mathbb{R}, k \to 0} \frac{u(x, y + k) - u(x, y) + i(v(x, y + k) - v(x, y))}{ik}$$

and now, taking real and imaginary parts and applying 3.1(ii), we deduce that $\partial u/\partial y$ and $\partial v/\partial y$ exist, and that

$$f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

□

Comment 1 The C-R equations on their own at a single point z do not provide a *sufficient* condition for f to be differentiable at z. But if the four partial derivatives exist everywhere in some open disc around z, and are continuous at z and satisfy the C-R equations at z, then f is indeed differentiable at z (see for example Priestley 10.30 or Stewart and Tall 4.6).

Comment 2 Although the C-R equations are named after Cauchy (1789-1852) and Riemann (1826-1866) they had already been noted by D'Alembert (1717-1783) in 1752.

Comment 3 See the exercise sheets for examples using the C-R equations to prove non-differentiability (a topic already covered in the module 'Complex Variables').

Geometrical interpretation of derivatives

If f is differentiable at z, and $f'(z) = \zeta$, then

$$\frac{f(z+h) - f(z)}{h} = \zeta + \epsilon(h) \quad \text{where } \epsilon(h) \to 0 \quad \text{as } h \to 0.$$

So

$$f(z+h) - f(z) = (\zeta + \epsilon(h)).h$$

and hence if $\zeta \neq 0$ then to a first approximation f maps a small disc with centre at z to a disc with centre at $f(z)$ by scaling the radius by a factor $|\zeta|$ and turning the disc through an angle $arg(\zeta)$. If $\zeta \neq 0$ it follows that f is *conformal* (angle-preserving) at z. We shall say more about conformal maps later.

As one might guess, a complex function which has derivative zero everywhere is necessarily constant. But this is not so easy to prove as in the real case (where one uses the Mean Value Theorem). In part (i) of the following proposition we use the result for real functions to prove the result for complex functions.

4.2 Proposition *Let $f \in H(D(0,R))$. Then*
(i) if $f'(z) = 0$ for all $z \in D(0,R)$ then f is constant on $D(0,R)$;
(ii) if $|f(z)|$ is constant on $D(0,R)$ then f is constant on $D(0,R)$.

Proof

(i) Since $f'(z) = 0$, and (as we saw in the proof of the C-R equations)

$$Re(f'(z)) = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad Im(f'(z)) = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

the partial derivatives of u and v with respect to x and y are zero everywhere on $D(0,R)$.

Any pair of points $p, q \in D(0,R)$ can be joined by a horizontal path segment γ_1 followed by a vertical path segment γ_2 or vice versa.

But u and v are constant on γ_1, by the real Mean Value Theorem, since $\partial u/\partial x$ and $\partial v/\partial x$ are zero on $D(0,R)$. And similarly u and v are constant on γ_2, by the real Mean Value Theorem, since $\partial u/\partial y$ and $\partial v/\partial y$ are zero on $D(0,R)$.

Hence $u(q) + iv(q) = u(p) + iv(p)$. But p and q were arbitrary points in $D(0,R)$, so f is constant on $D(0,R)$.

(ii) Suppose $|f| = c$, so $u^2 + v^2 = c^2$, and hence at every point $z \in D(0,R)$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0.$$

Using the Cauchy-Riemann equations we deduce:

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \quad \text{and} \quad u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0.$$

Eliminating $\partial u/\partial y$ between these last two equations yields

$$(u^2 + v^2)\frac{\partial u}{\partial x} = 0, \quad \text{that is,} \quad c^2\frac{\partial u}{\partial x} = 0.$$

Now if $c = 0$, then $|f| = 0$ so $f = 0$ and in particular f is constant.
And if $c \neq 0$ then $\partial u/\partial x = 0$, and similarly $\partial v/\partial x = 0$, so $f'(z) = 0$ at every point $z \in D(0,R)$ (since $f'(z) = \partial u/\partial x + i\partial v/\partial x$). It now follows from (i) that f is constant. □

4.3 Proposition

(i) If $f \in H(U)$ and $g \in H(U)$ and $\lambda \in \mathbb{C}$, then λf, $f + g$ and $f.g$ are all in $H(U)$, and the usual rules for evaluating their derivatives apply.

(ii) If $f \in H(U)$ and $g \in H(f(U))$, then their composition $g \circ f$ is in $H(U)$, and $(g \circ f)'(z) = g'(f(z))f'(z)$ (the chain rule).

(iii) If $f \in H(U)$ and $\forall z \in U$ $f(z) \neq 0$ then $1/f \in H(U)$ and $(1/f)'(z) = -f'(z)/(f(z))^2$.

Proofs We omit these as they look exactly like the proofs for real functions, but note that the chain rule is self-evident from the geometrical interpretation of derivatives. □

Comment Since the function $z \to z$ is holomorphic in $\mathbb{C}$ (with derivative equal to 1 everywhere) 4.3(i) allows us to deduce that every *polynomial* function

$$p(z) = a_0 + a_1 z + \ldots + a_n z^z \quad (\text{where } a_j \in \mathbb{C}\ \forall j)$$

is holomorphic in $\mathbb{C}$, and 4.3(iii) allows us to deduce that every *rational* function

$$\frac{p(z)}{q(z)} = \frac{a_o + \ldots + a_n z^n}{b_0 + \ldots + b_m z^m}$$

is holomorphic except at the roots of $q(z)$. In fact we shall see later that every rational function can be regarded as a holomorphic function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the *Riemann sphere*, and indeed that *every* holomorphic function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map.

5. Power Series

Definition We say that the power series $\sum_{n=0}^{\infty} z_n$ *converges* to $a \in \mathbb{C}$ if and only if its sequence of partial sums (s_n) converges to a (where $s_n = z_0 + \ldots + z_n$). If the series does not converge, we say it *diverges*.

5.1 Proposition

(i) If $z_n = x_n + iy_n$ $\forall n$ and $a = b + ic$ (i.e. x_n, y_n are the real and imaginary parts of z_n, and b and c are the real and imaginary parts of a) then

$$\sum_{n=0}^{\infty} z_n = a \Leftrightarrow \sum_{n=0}^{\infty} x_n = b \text{ and } \sum_{n=0}^{\infty} y_n = c.$$

(ii) If $\sum_{n=0}^{\infty} |z_n|$ converges, then so does $\sum_{n=0}^{\infty} z_n$.

Proof (i) is just 3.1 applied to the sequence of partial sums.
(ii) If $\sum_{n=0}^{\infty} |z_n|$ converges then so do $\sum_{n=0}^{\infty} |x_n|$ and $\sum_{n=0}^{\infty} |y_n|$ by the comparison test (for real series) and hence so do $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ by the well-known result for real series that absolute convergence implies convergence. The result now follows by (3.1). □

5.2 Proposition *For $|z| < 1$, $\sum_0^{\infty} z^n$ converges to $1/(1-z)$.*

Proof

$$\left|\frac{1}{1-z} - s_n\right| = \left|\frac{1}{1-z} - (1 + z + z^2 + \ldots + z^n)\right| = \left|\frac{1}{1-z} - \frac{1-z^{n+1}}{1-z}\right| = \left|\frac{z^{n+1}}{1-z}\right|$$

But if $|z| < 1$ then $\lim_{n\to\infty} |z|^{n+1} = 0$, so

$$\lim_{n\to\infty} s_n = \frac{1}{1-z} \quad \square$$

Definition A *power series* in z is a series of the form $\sum_0^{\infty} a_n z^n$ where $a_n \in \mathbb{C}$ for all n. We say that $\sum_0^{\infty} a_n z^n$ is *absolutely convergent* if $\sum_0^{\infty} |a_n z^n|$ converges.

5.3 Ratio Test *If $\lim_{n\to\infty} |a_{n+1} z^{n+1}|/|a_n z^n|$ exists and is strictly less than 1 then $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent. If the limit exists and is strictly greater than 1 then the series diverges.*

Proof This is a result about the real series $\sum_0^{\infty} |a_n z^n|$. It is proved by comparing $\sum_0^{\infty} |a_n z^n|$ with a suitable geometric series. We omit the details here, but note that the last part is obvious, since if the ratio is eventually greater than 1 the individual terms in the series cannot tend to zero. □

5.4 Proposition *Every power series* $\sum_0^\infty a_n z^n$ *has a radius of convergence, i.e.* $\exists R$*, possibly* 0 *or* ∞*, such that the series converges absolutely for all* z *with* $|z| < R$ *and diverges for all* z *with* $|z| > R$*.*

Proof It will suffice to show that if $\sum_0^\infty a_n w^n$ converges, and $|z| < |w|$, then $\sum_0^\infty a_n z^n$ converges absolutely.

So suppose $\sum_0^\infty a_n w^n$ converges and $|z| < |w|$. Thus $|z|/|w| = b < 1$.

As $\lim_{n\to\infty} a_n w^n = 0$ (else the series could not converge) there exists M such that $|a_n w^n| < M$ for all n. Now

$$|a_n z^n| = |a_n w^n| \frac{|z|^n}{|w|^n} < Mb^n.$$

But the geometric series $\sum_0^\infty Mb^n$ converges and so $\sum_0^\infty |a_n z^n|$ converges, by the comparison test for real series. □

Comments

1. R exists, whether or not we can find it using the ratio test.
2. The series may converge for some z with $|z| = R$ and diverge for others.

5.5 Lemma $\sum_{n=0}^\infty a_n z^n$ *and* $\sum_1^\infty n a_n z^{n-1}$ *have the same radius of convergence.*

Proof We show that if $\sum_0^\infty a_n z^n$ converges absolutely for $|z| < R \neq 0$, then so does $\sum_1^\infty n a_n z^{n-1}$. We leave the converse, which is easier, as an exercise.

Fix z with $0 < |z| < R$. Now choose r with $|z| < r < R$. Now

$$|n a_n z^{n-1}| = \frac{n}{r}\left(\frac{|z|}{r}\right)^{n-1} |a_n r^n|.$$

But $lim_{n\to\infty} \frac{n}{r}(|z|/r)^{n-1} = 0$ (by the ratio test, since $|z|/r < 1$ and $lim_{n\to\infty}(n+1)/n = 1$).
So $\exists M$ with $\frac{n}{r}(|z|/r)^{n-1} < M$ for all n. Hence

$$|n a_n z^{n-1}| < M|a_n r^n|$$

so $\sum_1^\infty |n a_n z^{n-1}|$ converges (by the comparison test, as $\sum_0^\infty |a_n r^n|$ converges since $r < R$). □

5.6 Theorem *If* $\sum_0^\infty a_n z^n$ *has radius of convergence* $R \neq 0$*, and* $f(z)$ *is the sum of this series, then* f *is a holomorphic function on* $D(0,R)$*, and its derivative is* $f'(z) = \sum_1^\infty n a_n z^{n-1}$ *for all* $z \in D(0,R)$*.*

Proof *(Not examinable, as it is hard to get the details right - in fact I won't even write these details on the board in lectures!)*
Write $g(z) = \sum_1^\infty n a_n z^{n-1}$. This function is well-defined for all $z \in D(0,R)$ by (5.5). Now

$$\left|\frac{f(z+h) - f(z)}{h} - g(z)\right| = \left|\sum_{n=1}^\infty a_n \left(\frac{(z+h)^n - z^n}{h} - n z^{n-1}\right)\right|$$

$$= |h|.\left|\sum_{n=1}^{\infty} a_n \left(\sum_{k=2}^{n} \binom{n}{k} z^{n-k}h^{k-2}\right)\right|$$

$$\leq |h| \sum_{n=1}^{\infty} |a_n| \left(\sum_{k=2}^{n} \binom{n}{k} |z|^{n-k}|h|^{k-2}\right)$$

$$= |h| \sum_{n=1}^{\infty} |a_n| \left(\sum_{m=0}^{n-2} \binom{n}{m+2} |z|^{n-m-2}|h|^{m}\right)$$

$$\leq |h| \sum_{n=1}^{\infty} |a_n| \frac{n(n-1)}{2} \left(\sum_{m=0}^{n-2} \binom{n-2}{m} |z|^{n-m-2}|h|^{m}\right) \quad \text{(since } (m+2)(m+1) \geq 2)$$

$$= |h| \sum_{n=1}^{\infty} \frac{n(n-1)}{2} |a_n|(|z| + |h|)^{n-2}$$

Now fix z and choose r with $|z| < r < R$.

$\sum_{n=1}^{\infty} n(n-1)|a_n|r^{n-2}$ converges, say to K, by 5.5 (applied twice).

Thus when $|h| < r - |z|$ (so $|z| + |h| < r$) we have proved that

$$|\frac{f(z_h) - f(z)}{h} - g(z)| \leq \frac{K}{2}|h|$$

which goes to zero as h goes to 0. □

5.7 Corollary *If $f(z) = \sum_0^{\infty} a_n z^n$ has radius of convergence $R \neq 0$ then f has derivatives of all orders on $D(0, R)$ and $f^{(n)}(0) = n!a_n$ for all $n \geq 0$ (where $0!$ is defined to be 1).*

Proof By repeatedly applying 5.6 to $f(z) = \sum_0^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + ...$ we see that all the derivatives of f exist for all $z \in D(0, R)$ and that they are given by:

$$f'(z) = 1.a_1 + 2a_2 z + ... + na_n z^{n-1} + ...$$

$$f^{(2)}(z) = 2.1a_2 + 3.2a_3 z + ... + n(n-1)a_n z^{n-2} + ...$$

$$f^{(3)}(z) = 3.2.1a_3 + ... + n(n-1)(n-2)a_n z^{n-3} + ...$$

.

$$\cdots$$

$$f^{(n)}(z) = n!a_n + \text{higher degree terms}$$

By substituting $z = 0$ into the expression for $f^{(n)}(z)$ we have $f^{(n)}(0) = n!a_n$. □

Exponential, trigonometric and hyperbolic functions

Definition

$$exp(z) = e^z = \sum_0^\infty \frac{z^n}{n!}$$

5.8 Proposition

(i) e^z is entire (that is, differentiable for all $z \in \mathbb{C}$) and the derivative of e^z is e^z.

(ii) $\forall z, w \in \mathbb{C}$, $e^{z+w} = e^z e^w$.

Proof

(i) The radius of convergence of the series if ∞, by the ratio test. Hence the result follows by 5.6.

(ii) For fixed $\zeta \in \mathbb{C}$, consider $f(z) = e^z e^{\zeta - z}$.

By the rule for differentiating a product,

$$f'(z) = e^z e^{\zeta - z} - e^z e^{\zeta - z} = 0$$

So, by 4.2(i), f is constant, and hence $f(z) = f(0)$ for all $z \in \mathbb{C}$, that is to say $f(z) = e^\zeta \ \forall z \in \mathbb{C}$.

In other words $e^z e^{\zeta - z} = e^\zeta \ \forall z \in \mathbb{C}$. Writing w for $\zeta - z$, this gives us

$$e^z e^w = e^{z+w} \ \forall z \in \mathbb{C}$$

Finally, since we can choose whatever value for ζ we please, we deduce

$$e^z e^w = e^{z+w} \ \forall z, w \in \mathbb{C} \quad \square$$

Comment One can also prove (ii) by using results about the products of absolutely convergent complex series, but the proof above is quicker!

Definitions

$$\cos z = \sum_{n=0}^\infty (-1)^n \frac{z^{2n}}{(2n)!} \quad \sin z = \sum_{n=0}^\infty (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

Both these series have infinite radius of convergence (by the ratio test). Hence both functions are entire.

By 5.5 their derivatives are $\cos' z = -\sin z$ and $\sin' z = \cos z$.

Note that $e^{iz} = \cos z + i \sin z$ so

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Definitions

$$\cosh z = \frac{e^z + e^{-z}}{2} = \cos iz \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2} = -i \sin iz$$

$$\tan z = \frac{\sin z}{\cos z} \quad \tanh z = \frac{\sinh z}{\cosh z}$$

Of course $\tan z$ and $\tanh z$ are not entire functions: they are only defined at points where $\cos z \neq 0$ and $\cosh z \neq 0$ respectively.

Logarithms, and powers

We define $\log z$ by $\log z = w \Leftrightarrow e^w = z$.
Writing $w = u + iv$ (with u and v real) this tells us

$$logz = w = u + iv \Leftrightarrow e^{u+iv} = z \Leftrightarrow e^u(\cos v + i \sin v) = z$$

$$\Leftrightarrow e^u = |z| \text{ and } v = \arg z \Leftrightarrow w = \ln |z| + i \arg z$$

Note that $\arg z$ is multivalued, since if θ is an argument for z then so is $\theta + 2k\pi$ for any $k \in \mathbb{Z}$. For example if we regard 3 as a complex number, then its complex logarithm is the set

$$\log 3 = \{\ln 3 + 2k\pi i : k \in \mathbb{Z}\}$$

For real numbers x and a, one way to define the power x^a is by $x^a = e^{a \ln x}$. So for complex z and α we define

$$z^\alpha = e^{\alpha \log z} = e^{\alpha(\ln |z| + i \arg z)}$$

Once again this is multivalued in general. But if α is an integer, say $\alpha = n \in \mathbb{Z}$, the formula gives

$$z^n = e^{n(ln|z| + i \arg z)}$$

which has just one value, the usual value of of z^n, since the different values of $\arg z$ differ by $2k\pi$ for $k \in \mathbb{Z}$, and we know that $e^{2nk\pi i} = 1$.

When $\alpha = 1/n$ the definition of z^α using the formula above gives us a set consisting of the nth roots of the complex number z. More generally, when α is a rational number p/q (in lowest terms) the set of values of z^α contains q elements.

Because of the multivalued nature of logarithms and powers one has to be careful interpreting computations. But it is the fact that $\log z$ has more than one value that will later give us the calculus of residues, enabling us to compute real integrals and sum real series that we cannot compute any other way.