IV. Conformal Maps

16. Geometric interpretation of differentiability

We saw from the definition of complex differentiability that if f is holomorphic at $z = a$ and $f'(a) = \zeta \neq 0$, then 'to a first approximation' f maps a small disc centred at a to a disc centred at $f(a)$ by expanding it by the factor $|\zeta|$ and rotating it through the angle $arg(\zeta)$. Thus f 'preserves angles' at $z = a$. We now make this idea more precise.

Definition Let $\gamma_1 : [0, 1] \to \mathbb{C}$ and $\gamma_2 : [0, 1] \to \mathbb{C}$ be C^1 paths through the point $z = a$, so there exist t_1 and $t_2 \in [0, 1]$ with $\gamma_1(t_1) = \gamma_2(t_2) = a$. Suppose $\gamma_1'(t_1) \neq 0$ and $\gamma_2'(t_2) \neq 0$ so the paths have well-defined tangents at a.

Then we say that the *angle between* γ_1^* *and* γ_2^* *at* $z = a$ is

$$arg(\gamma_2'(t_2)) - arg(\gamma_1'(t_1)) = arg\left(\frac{\gamma_2'(t_2)}{\gamma_1'(t_1)}\right)$$

Now let $f : U \to \mathbb{C}$, where U is an open set in $\mathbb{C}$ containing a. We say that f is *conformal* at $z = a$ if the angle between $(f \circ \gamma_1)^*$ and $(f \circ \gamma_2)^*$ at $f(a)$ is equal to that between γ_1^* and γ_2^* at a.

16.1 Lemma *If f is holomorphic at a and $f'(a) \neq 0$ then f is conformal at a.*

Proof By the chain rule

$$\frac{(f \circ \gamma_2)'(t_2)}{(f \circ \gamma_1)'(t_1)} = \frac{f'(\gamma_2(t_2))\gamma_2'(t_2)}{f'(\gamma_1(t_1))\gamma_1'(t_1)} = \frac{f'(a)\gamma_2'(t_2)}{f'(a)\gamma_1'(t_1)} = \frac{\gamma_2'(t_2)}{\gamma_1'(t_1)}$$

The result follows (since $\arg(z_2/z_1) = arg(z_2) - arg(z_1)$ for any non-zero complex numbers z_1 and z_2).
□

Note that the restriction $f'(a) \neq 0$ is necessary, since (for example) the map $z \to z^2$ *doubles* angles at zero. More generally if a holomorphic map f at a has $f'(a) = 0, f''(a) = 0, \ldots, f^{(n-1)}(a) = 0$ but $f^{(n)}(a) \neq 0$ then f will multiply angles at a by n.

17. Automorphisms of the Riemann sphere: Möbius transformations

We call a complex differentible map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ an *automorphism* of the Riemann sphere $\hat{\mathbb{C}}$ if it is a bijection. We already know that the differentiable maps $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ are rational maps. A rational map $z \to p(z)/q(z)$ is said to have *degree* d if the maximum of the degrees of the polynomials p and q is d

(where of course any factor common to $p(z)$ and $q(z)$ has been cancelled). If f is a rational map of degree d then most points $w \in \mathbb{C}$ have inverse image $f^{-1}(w)$ consisting of d distinct points: the only exceptions are when w is a *critical value* of f (the image of a *critical point*, that is, a point where $f'(z) = 0$). As there are only a finite number of critical points (in fact $2d - 2$, counted with multiplicity) there are infinitely many non-critical points and so f cannot be a bijection unless $d = 1$. We deduce:

17.1 Lemma *The automorphisms of $\hat{\mathbb{C}}$ are the maps of the form:*

$$\phi(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ (else $az + b$ would be a scalar times $cz + d$).

Such a map is called a *fractional linear* or *Möbius* transformation.

17.2 Proposition *The composition of two Möbius transformations is a Möbius transformation. The inverse of a Möbius transformation is a Möbius transformation.*

Proof Exercise. $\square$

17.3 Proposition *A Mobius transformation is conformal at every point of $\hat{\mathbb{C}}$.*

Proof The derivative of

$$\phi(z) = \frac{az + b}{cz + d}$$

is

$$\phi'(z) = \frac{ad - bc}{(cz + d)^2}$$

so for $z \neq -d/c$ or ∞ we have $\phi'(z) \neq 0$. For $z = -d/c$ (in which case $\phi(z) = \infty$) it is easily checked that $\psi'(z) \neq 0$, where $\psi(z) = 1/\phi(z)$, and for $z = \infty$ it is easily checked that $\phi'(1/z) \neq 0$. $\square$

Comment It can be shown that stereographic projection is conformal (angle-preserving) so we may think of Möbius transformations as conformal maps of the unit sphere $S^2 \subset \mathbb{R}^3$ to itself.

Types of Möbius transformation

1. $z \to az$ $a \neq 0$ $(b = c = 0, d = 1)$ (for $a = Re^{i\theta}$ this is expansion by R followed by rotation though θ);
2. $z \to z + b$ $(a = d = 1, c = 0)$ (translation by b);
3. $z \to 1/z$ $(a = d = 0, b = c = 1)$ (inversion).

17.4 Proposition *Every Möbius transformation ϕ is a composition of transformations of these three types.*

Proof If $c \neq 0$ we decompose ϕ into

$$z \to Z \to W \to w$$

where $Z = cz + d$, $W = 1/Z$, and $w = \alpha W + \beta$, with

$$\alpha . \frac{1}{cz + d} + \beta = \frac{az + b}{cz + d}$$

that is,

$$\beta = \frac{a}{c} \quad \alpha = b - \frac{ad}{c}$$

If $c = 0$ then ϕ can be written $\phi(z) = az + b$ and this is obviously multiplication by a followed by translation by b. □

17.5 Corollary *Möbius transformations send circles to circles (where a 'circle through ∞' is a straight line in $\mathbb{C}$).*

Proof Each of the three types sends circles to circles. □

17.6 Proposition *Let z_1, z_2, z_3 be any triple of distinct points in $\hat{\mathbb{C}}$ and w_1, w_2, w_3 also be a triple of distinct points in $\hat{\mathbb{C}}$. Then there exists a unique Möbius transformation ϕ such that $\phi(z_j) = w_j$ for $j = 1, 2, 3$.*

Proof

$$\phi : z \to \left(\frac{z_1 - z_3}{z_2 - z_3}\right)\left(\frac{z_2 - z}{z_1 - z}\right)$$

maps z_1, z_2, z_3 to $\infty, 0, 1$ respectively. Similarly

$$\psi : w \to \left(\frac{w_1 - w_3}{w_2 - w_3}\right)\left(\frac{w_2 - w}{w_1 - w}\right)$$

maps w_1, w_2, w_3 to $\infty, 0, 1$. Thus $\psi^{-1} \circ \phi$ has the required property. To show it is unique it suffices to show that the only Möbius transformation which sends $0 \to 0$, $1 \to 1$ and ∞ to ∞ is the identity. This is a straightforward exercise. □.

The complex number

$$[z_1, z_2; z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$$

is known as the *cross-ratio* of the 4-tuple of distinct complex numbers z_1, z_2, z_3, z_4. It follows from Proposition 17.6 that every Möbius transformation ϕ preserves cross-ratios (exercise).

Warning There are various conventions for the order in which to take the points z_1, z_2, z_3, z_4 when defining the cross-ratio. Changing the order changes the actual value of the cross-ratio λ into $1/\lambda$, $1 - \lambda$, $1/(1 - \lambda)$, $1/(1 - 1/\lambda)$ or $1 - 1/\lambda$.

18. Conformal maps between subsets of the plane: the Riemann Mapping Theorem

18.1 Proposition *Every automorphism of the plane* $\mathbb{C}$ *(that is to say holomorphic bijection* $\mathbb{C} \to \mathbb{C}$*) has the form*

$$\phi(z) = az + b$$

for some $a \in \mathbb{C}$ *with* $a \neq 0$*.*

Proof (sketch) It can be shown that if ϕ is an automorphism of $\mathbb{C}$ then for any sequence (z_n) tending to infinity the sequence $\phi(z_n)$ also tends to infinity (this follows from the fact that the image of a compact set under a continuous map is a compact set, but we omit the details). Thus ϕ extends to an automorphism of $\hat{\mathbb{C}}$ and therefore has the form $z \to (az + b)/(cz + d)$. But $\phi(\infty) = \infty$ and hence $c = 0$. □

Examples of conformal bijections between particular subsets of the complex plane

1. If $H = \{z = x + iy : y > 0\}$ (the upper half-plane) and $D = \{z : |z| < 1\}$ (the unit disc) then the Möbius transformation which sends $0 \to -1, 1 \to -i$ and $\infty \to 1$ will send the real axis traversed in a positive direction to the unit circle traversed anticlockwise, and hence it will send H to D.

2. If $S = \{z = x + iy : x > 0, y > 0\}$ then the map $z \to z^2$ will send S bijectively onto the upper half-plane H.

3. Any Möbius transformation which sends the real axis (union ∞) to itself and preserves the orientation of $\mathbb{R}$ will map the upper half-plane to itself conformally. Similarly any Möbius transformation which sends the (oriented) unit circle to itself will map the unit disc conformally to itself.

The proof of the following celebrated theorem is beyond the scope of this course (but not *too* far beyond it).

18.2 The Riemann Mapping Theorem *Let* U *be any simply-connected open proper subset of* $\mathbb{C}$*. Then there exists a bijective conformal mapping from* U *to the open unit disc* $D(0, 1)$*.*

Comment 1 Unfortunately the proof does not tell us how to construct an explicit mapping in every particular case.

Comment 2 There is no conformal bijection from the *whole* of $\mathbb{C}$ to $D(0, 1)$. It is a theorem of Poincaré that for each simply-connected one-dimensional complex manifold M there is a holomorphic bijection between M and exactly one of $\mathbb{C}, \hat{\mathbb{C}}$ and $D(0, 1)$.

19. Harmonic maps

One reason there is so much interest in conformal maps is the connection with *harmonic* maps, which are important in applied mathematic and physics.

Definition Let U be an open subset of $\mathbb{R}^2$. A function $u : U \to \mathbb{R}$ is said to be *harmonic* if

(i) u has continuous second order partial derivatives on U;

(ii) u satisfies *Laplace's equation*:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

19.1 Theorem *Let f be holomorphic in an open set U, with real and imaginary parts u and v. Then both u and v are harmonic in U.*

Proof

We are supposing that $f(x+iy) = u(x,y) + iv(x,y)$. Since f is (complex) differentiable arbitrarily many times (by Taylor's Theorem) we know that u and v have partial derivatives of all orders, and by the Cauchy-Riemann equations we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$$

and similarly

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2}$$

□

There is a converse, which allows us to convert problems about harmonic functions into problems about holomorphic maps:

19.2 Theorem *Let $D \subset \mathbb{R}^2$ be an open disc and suppose that $u : D \to \mathbb{R}$ is harmonic. Then there exists a complex function f, holomorphic in D, such that $u = Re(f)$.*

Proof (omitted to save time, and therefore non-examinable: but there may be examples involving harmonic maps and harmonic conjugates in the examination.)

Let

$$g = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

Then Laplace's equation for u tells us that $\frac{\partial u}{\partial x}$ and $-\frac{\partial u}{\partial y}$, the real and imaginary parts of g, satisfy the Cauchy-Riemann equations. Hence g is holomorphic in D and so it has an antiderivative G there. Writing $Re(G) = H$ and $Im(G) = K$ we see that

$$G' = \frac{\partial H}{\partial x} + i\frac{\partial K}{\partial x} = \frac{\partial H}{\partial x} - i\frac{\partial H}{\partial y}$$

But

$$G' = g = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

So

$$\frac{\partial(H-u)}{\partial x} = 0 \text{ and } \frac{\partial(H-u)}{\partial y} = 0$$

so $H - u$ is constant on D, say $H - u = k$. Now let $f(z) = G(z) - k$. Then $Re(f) = H - k = u$. □

The function $Im(f)$, which is also harmonic, is known as the *harmonic conjugate* of f.

20. Holomorphic maps of the unit disc: Schwarz's Lemma

In this section we shall address the question of finding all automorphisms of the unit disc $D = D(0,1)$. A priori we have no reason to suppose these automorphisms are all Möbius transformations, so first we investigate holomorphic maps $f : D \to \mathbb{C}$ which have $f(D) \subseteq D$. Our main tool will be:

20.1 The Maximum Modulus Principle *If f is holomorphic on an open disc $D = D(a,r)$ and $b \in D$ is a maximum point for $|f|$, that is $|f(b)| \geq |f(z)|\ \forall z \in D$, then f is constant on D.*

This was proved in Exercise 2(ii) on Sheet 5 (as a consequence of Gauss' Mean Value Theorem, which in turn followed from Cauchy's integral formula). A more general form of the Maximum Modulus Principle is that for f holomorphic a maximum for $|f|$ on a set S can only be achieved at a *boundary* point of S.

20.2 Schwarz's Lemma *Let $f : D \to D$ be a holomorphic function of the unit disc to itself such that $f(0) = 0$. Then*
(i) $|f(z)| \leq |z|$ for all $z \in D$.
(ii) If for some $z_0 \neq 0$ we have $|f(z_0)| = |z_0|$ then there is some complex number μ of modulus 1 such that $f(z) = \mu z$.

Proof Let $f(z) = a_1 z + \dots$ be the Taylor series for f, valid on the whole of D by Taylor's Theorem (the constant term is 0 as $f(0) = 0$). Thus $f(z)/z$ is holomorphic for all $z \in D$. If z is a point with $|z| = r < 1$ we have

$$\left|\frac{f(z)}{z}\right| < \frac{1}{r}$$

(since $|f(z)| < 1$). So by the Maximum Modulus Principle we have

$$\left|\frac{f(z)}{z}\right| < \frac{1}{r}$$

for all z with $|z| \leq r$. Letting r tend to 1 proves (i). If in addition we have

$$\left|\frac{f(z_0)}{z_0}\right| = 1$$

for some z_0 in the open unit disc, then again by the Maximum Modulus Principle $f(z)/z$ must be constant and so there is a complex number μ of modulus 1 such that $f(z)/z = \mu$ for all $z \in D$. □

20.3 Corollary *Let $f : D \to D$ be an automorphism of the unit disc and suppose $f(a) = 0$. Then f is a Möbius transformation. Indeed there exists a real number θ such that*

$$f(z) = e^{i\theta} \frac{a - z}{1 - \bar{a}z}$$

Proof The Möbius transformation

$$g(z) = \frac{a - z}{1 - \bar{a}z}$$

sends a to 0 and also sends the unit disc to itself (since it sends any point e^{it} of the unit circle to the point $-e^{-it}(a - e^{it})/(\bar{a} - e^{-it})$, which also lies on the unit circle). Thus $f \circ g^{-1}$ is an automorphism h of the unit disc which has $h(0) = 0$. It will suffice to show that $h(z) = e^{i\theta} z$ for some θ.

Part (i) of the Schwarz Lemma tells us that $|h(z)| \leq |z|$ for all $z \in D$. But applied to h^{-1} it also tells us that $|z| \leq |h(z)|$ for all $z \in D$. So $|h(z)| = |z|$ for all $z \in D$ and part(ii) now tells us that $h(z) = e^{i\theta} z$, completing the proof. □

20.4 Corollary The automorphisms of the complex upper half-plane H are the Möbius transformations of the form

$$z \to \frac{az + b}{cz + d}$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$

Proof We have already seen that there is a Möbius transformation ϕ which sends H bijectively onto the unit disc D. If f is an automorphism of H then $\phi \circ f \circ \phi^{-1}$ is an automorphism of D, and by 5.3 it is a Möbius transformation ψ. Hence f, being equal to $\phi^{-1} \circ \psi \circ \phi$, is also a Möbius transformation. But the only Möbius transformations which send $\mathbb{R} \cup \infty$ to itself and preserve its orientation have the form:

$$f(z) = \frac{az + b}{cz + d}$$

for some $a, b, c, d \in \mathbb{R}$ with $ad - bc > 0$. □

Comment Given any two points z_1 and z_2 in H, there is a unique semicircle orthogonal to the real axis which passes through z_1 and z_2. This semicircle meets the real axis at α and β say. The cross-ratio λ of the four points z_1, z_2, α, β is real, since sending the first three points to $\infty, 0, 1$ will send the fourth one to a point on the real axis. We define the *hyberbolic distance* between z_1 and z_2 to be $|\ln \lambda|$. With this metric H becomes *Poincaré's half-plane model of the hyperbolic plane*, the automorphisms of H become *hyperbolic isometries* and the semicircles orthogonal to $\mathbb{R}$ become *geodesics*.