III. Consequences of Cauchy's Theorem

10. Cauchy's formulae

10.1 Cauchy's Integral Formula

Let f be holomorphic on and everywhere inside a simple closed path γ. Then for every point a inside γ,

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz$$

Proof

Since $I(\gamma)$ (the inside of γ) is open, there exists $r > 0$ such that $\bar{D}(a,r) \subset I(\gamma)$. Now, since $f(z)/(z-a)$ is holomorphic everywhere in the annulus between γ^* and $C(a,r)^*$, the deformation principle tells us that

$$\int_\gamma \frac{f(z)}{z-a}dz = \int_{C(a,r)} \frac{f(z)}{z-a}dz$$

We know that

$$\int_{C(a,r)} \frac{f(a)}{z-a}dz = f(a)\int_{C(a,r)} \frac{1}{z-a}dz = f(a)\int_0^{2\pi} \frac{1}{re^{it}}ire^{it}dt = 2\pi i f(a)$$

so it will suffice to show that we can make

$$\int_{C(a,r)} \frac{f(z)-f(a)}{z-a}dz$$

arbitrarily small by taking r sufficiently small.

However given any $\epsilon > 0$, we can take r sufficiently small that $|f(z) - f(a)| < \epsilon$ for all $z \in C(a,r)^*$, by the continuity of $f(z)$ at $z = a$, in which case

$$|\int_{C(a,r)} \frac{f(z)-f(a)}{z-a}dz| \leq \frac{\epsilon}{r}2\pi r = 2\pi\epsilon$$

by the Estimation Lemma. Hence

$$|\int_\gamma \frac{f(z)}{z-a}dz - 2\pi i f(a)| = |\int_{C(a,r)} \frac{f(z)-f(a)}{z-a}dz| \leq 2\pi\epsilon \;\forall \epsilon > 0$$

and so

$$\int_\gamma \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

□

10.2 Liouville's Theorem

If f is entire (that is, holomorphic everywhere on $\mathbb{C}$) and f is bounded on $\mathbb{C}$, then f is constant.

Proof

Let M be a bound for $|f(z)|$ on $\mathbb{C}$. Choose any two points $a, b \in \mathbb{C}$, and take $R \geq 2max\{|a|, |b|\}$. Then for all z with $|z| = R$ we have

$$|z-a| \geq R - R/2 = R/2 \text{ and } |z-b| \geq R - R/2 = R/2$$

Cauchy's Integral Formula applied to $C(0, R)$ gives

$$f(a) = \frac{1}{2\pi i}\int_{C(0,R)} \frac{f(z)}{z-a}dz \text{ and } f(b) = \frac{1}{2\pi i}\int_{C(0,R)} \frac{f(z)}{z-b}dz$$

Thus

$$f(a) - f(b) = \frac{1}{2\pi i}\int_{C(0,R)} f(z)\left(\frac{1}{z-a} - \frac{1}{z-b}\right)dz = \frac{1}{2\pi i}\int_{C(0,R)} f(z)\frac{a-b}{(z-a)(z-b)}dz$$

By the Estimation Lemma we deduce that

$$|f(a) - f(b)| \leq \frac{1}{2\pi}.2\pi R.M\frac{|a-b|}{(R/2)^2} = \frac{4M|a-b|}{R}$$

Since we may take R arbitrarily large, we deduce that $f(a) = f(b)$. Since a and b are arbitrary points of $\mathbb{C}$ we are done. □

Comment It is easy to think of *real* functions which are differentiable everywhere on $\mathbb{R}$, and bounded on $\mathbb{R}$, but not constant - for example $\sin x$. Liouville's Theorem shows that this is not possible on $\mathbb{C}$.

10.3 The Fundamental Theorem of Algebra

Let $p(z)$ be a non-constant polynomial (that is, a polynomial of degree ≥ 1) with coefficients in $\mathbb{C}$. Then there exists $w \in \mathbb{C}$ such that $p(w) = 0$.

Proof

Suppose there is no solution in $\mathbb{C}$ to the equation $p(z) = 0$. Then the function $f(z)$ defined by $f(z) = 1/p(z)$ is entire. We shall show that $f(z)$ is bounded on $\mathbb{C}$, and hence constant by Liouville's Theorem, contradicting our hypothesis that $p(z)$ is not constant. Let

$$p(z) = a_0 + a_1 z + \ldots a_n z^n \text{ where } a_n \neq 0$$

Then

$$p(z) = z^n\left(\left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots + \frac{a_{n-1}}{z}\right) + a_n\right) = z^n(\zeta + a_n)$$

Choose R such that for all z with $|z| > R$ the terms $|a_0/z^n|, \ldots, |a_{n-1}/z|$ are all $< |a_n|/2n$. then for all z with $|z| > R$ we have $|\zeta| < |a_n|/2$ and hence

$$|\zeta + a_n| > |a_n| - \frac{|a_n|}{2} = \frac{|a_n|}{2}$$

Hence for all z with $|z| > R$ we have

$$|p(z)| > \frac{|a_n|R^n}{2} \text{ and so } |f(z)| < \frac{2}{|a_n|R^n}$$

But $\bar{D}(0, R)$ is compact, so $\{|f(z)| : z \in \bar{D}(0, R)\}$ is compact and therefore bounded. thus $\exists M > 0$ such that $|f(z)| \leq M$ for all z with $|z| \leq R$. Now

$$|f(z)| \leq max\{M, \frac{2}{|a_n|R^n}\} \ \forall z \in \mathbb{C}$$

In other words $f(z)$ is bounded on $\mathbb{C}$, and we have our contradiction $\square$.

Comment Most proofs of the Fundamental Theorem of Algebra make use either of complex analysis or algebraic topology. Note that the theorem does not give a constructive method to find a root of a given polynomial. For polynomials of degrees two, three and four there exist formulae for solutions, but it is a famous theorem of Abel that there can be no general solution in radicals (nth roots) for a polynomial of degree five or higher. A general theory of when equations are solvable in radicals was developed by Galois.

10.4 Cauchy's formula for derivatives (extended form of Cauchy's Integral Formula)

Let f be holomorphic on and everywhere inside a simple closed path γ. Then at every point a inside γ, the nth derivative $f^{(n)}(a)$ exists for $n = 0, 1, 2, 3 \ldots$, and

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz$$

Proof This can be proved directly (see Priestley) but we shall see that it emerges immediately as a by-product of our next proof, that of Taylor's Theorem, so we delay the details till then. $\square$

Comment This again is a much stronger property than we have for real functions. It is easy to construct real functions which are differentiable once but not twice.

11. Taylor's Theorem

We first state a Lemma which does most of the work that we need to prove Taylor's Theorem.

11.1 Lemma *Let f be holomorphic on $D(a,R)$. Then there exist constants $c_n \in \mathbb{C}$ such that*

$$f(a+h) = \sum_{n=0}^{\infty} c_n h^n$$

and this series converges absolutely for all h with $|h| < R$. Moreover

$$c_n = \frac{1}{2\pi i} \int_{C(a,r)} \frac{f(z)}{(z-a)^{n+1}} dz$$

for any r with $0 < r < R$.

Proof

Fix $h \in \mathbb{C}$ such that $0 < |h| < R$ and suppose (for the time being) that r satisfies $|h| < r < R$. The Cauchy Integral Formula gives us

$$f(a+h) = \frac{1}{2\pi i} \int_{C(a,r)} \frac{f(z)}{z-(a+h)} dz$$

Expanding $1/(z-(a+h))$ as a power series in h gives

$$\frac{1}{(z-a)-h} = \frac{1}{z-a} . \frac{1}{1-\frac{h}{z-a}} = \sum_{n=0}^{\infty} \frac{h^n}{(z-a)^{n+1}}$$

and this converges absolutely for $z \in C(a,r)$ since then $|z-a| = r > |h|$. Thus

$$f(a+h) = \frac{1}{2\pi i} \int_{C(a,r)} \sum_{n=0}^{\infty} \frac{f(z)}{(z-a)^{n+1}} .h^n dz$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left(\int_{C(a,r)} \frac{f(z)}{(z-a)^{n+1}} .h^n dz \right)$$

because our series of functions converges uniformly on $C(a,r)$ (as we know that $|f(z)|$ is bounded there and $|h|/|z-a| = |h|/r < 1$). Thus

$$f(a+h) = \sum_{n=0}^{\infty} c_n h^n$$

where

$$c_n = \frac{1}{2\pi i} \int_{C(a,r)} \frac{f(z)}{(z-a)^{n+1}} dz$$

Absolute convergence of the series $\sum_{n=0}^{\infty} c_n h^n$ for $|h| < r$ follows from the fact that $|c_n| < M/r^n$ (by the Estimation Lemma, where M is a bound on $|f(z)|$ for $z \in C(a,r)$). Finally we observe that the restriction $|h| < r$ is no longer necessary, since by applying the deformation principle the value of

$$\int_{C(a,r)} \frac{f(z)}{(z-a)^{n+1}} dz$$

is the same for every value of r between 0 and R. □

11.2 Taylor's Theorem

If f is holomorphic on the open set $U \subseteq \mathbb{C}$ then all the higher derivatives of f exist everywhere in U, and for all z in any disc $D(a,R) \subseteq U$, $f(z)$ is equal to the sum of the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

Moreover for any r with $0 < r < R$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_{C(a,r)} \frac{f(z)}{(z-a)^{n+1}}dz$$

Proof

By Lemma 11.1,

$$f(a+h) = \sum_{n=0}^{\infty} c_n h^n \quad \forall h \text{ with } |h| < R$$

so, writing $z = a + h$ we have

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \text{ for } |z-a| < R$$

But we proved in Section I, Theorem 5.6 and Corollary 5.7, that any power series can be differentiated as many times as we wish and that for the power series above,

$$f^{(n)}(a) = n! c_n$$

Finally, by Lemma 11.1 we have already seen that

$$c_n = \frac{1}{2\pi i}\int_{C(a,r)} \frac{f(z)}{(z-a)^{n+1}}dz$$

□

Comments

1. This Theorem is named after Brooke Taylor who in 1715 was the first to publish the approximation of a real function by what is now known as a Taylor series (though the idea was known earlier, for example by Newton). The complex version of Taylor's Theorem is much stronger than the real version, as it says that everywhere inside its radius of convergence the Taylor series is *equal* to the function, not just an approximation to it.

2. Cauchy's Formula for Derivatives (10.4) is an immediate consequence of Taylor's Theorem (11.2), since by the deformation principle, for any simple closed γ such that f is holomorphic on and inside U, and

for any a in the interior of γ, we have

$$\int_\gamma \frac{f(z)}{(z-a)^{n+1}}dz = \int_{C(a,r)} \frac{f(z)}{(z-a)^{n+1}}dz$$

for small r.

3. Taylor coefficients are easy to compute using the formula $c_n = f^{(n)}(a)/n!$. The integral formula for c_n is more useful for estimating the size of c_n than for precise computations - as we shall now see.

11.3 Corollary (Cauchy's Estimate)

If f is differentiable for $|z-a| < R$, if $0 < r < R$ and if $|f(z)| \leq M$ everywhere on the circle $|z-a| = r$, then

$$|f^{(n)}(a)| \leq \frac{M.n!}{r^n} \ \forall n \geq 0$$

Proof

$$|f^{(n)}(a)| = |\frac{n!}{2\pi i}\int_{C(a,r)} \frac{f(z)}{(z-a)^{n+1}}dz| \leq \frac{n!}{2\pi}.\frac{M}{r^{n+1}}.2\pi r = \frac{M.n!}{r^n}$$

□

Comment Cauchy's Estimate for f' can be used to give a quick proof of Liouville's Theorem (see exercise sheet 5).

12. Laurent series

We have seen that any function which is holomorphic on a disc can be written as a power series. We next consider expansions of functions holomorphic on punctured discs, or more generally on annuli.

12.1 Laurent's Theorem

Let $A = \{z \in \mathbb{C} : R_1 < |z-a| < R_2\}$, where R_1 and R_2 are constants such that $0 \leq R_1 < R_2 \leq \infty$, and let $f \in H(A)$ (that is, f is holomorphic on A). Then

$$f(z) = \sum_{-\infty}^{\infty} c_n(z-a)^n \quad \forall z \in A$$

where

$$c_n = \frac{1}{2\pi i}\int_{C(a,r)} \frac{f(w)}{(w-a)^{n+1}}dw \quad \forall n \in \mathbb{Z}$$

for any r such that $R_1 < r < R_2$.

Note: *We say that $\sum_{-\infty}^{\infty} c_n$ converges (to $s = s_1 + s_2$) if and only if $\sum_0^\infty c_n$ converges (to s_1) and $\sum_1^\infty c_{-n}$ converges (to s_2).*

Proof

By a change of coordinates (replacing $z-a$ by z) we may assume that $a=0$. This is just to make it easier to write down all the equations.

Choose any $z \in A$ and let r_1 and r_2 be such that $R_1 < r_1 < |z| < r_2 < R_2$. Now by Cauchy's Integral Formula and the deformation principle

$$f(z) = \frac{1}{2\pi i}\int_{C(0,r_2)} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{C(0,r_1)} \frac{f(w)}{w-z}dz$$

$$= \frac{1}{2\pi i}\int_{C(0,r_2)} \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}} f(w)dw - \frac{1}{2\pi i}\int_{C(0,r_1)} \sum_{m=0}^{\infty} \frac{-w^m}{z^{m+1}} f(w)dw$$

Invoking uniform convergence to interchange $\int$ with $\sum$, we obtain

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C(0,r_2)} \frac{f(w)}{w^{n+1}}\right) z^n + \sum_{m=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C(0,r_1)} f(w).w^m dw\right)\frac{1}{z^{m+1}}$$

Writing $n = -(m+1)$ in the second sum, and observing that by the deformation princple we can replace the integrals around $C(0,r_1)$ and $C(0,r_2)$ by integrals around any $C(0,r)$ with $R_1 < r < R_2$, we are done. $\square$

We do not have as easy a formula for the coefficients of a Laurent series as for those of a Taylor series. However the integral formula for these coefficients can be used to prove that they are *unique*, so any method that we can use to produce a convergent series of the right form will produce the unique Laurent series.

Example To express

$$f(z) = \frac{1}{z(z-1)}$$

as a Laurent series in $A_1 = \{z : 0 < |z| < 1\}$ and in $A_2 = \{z : |z| > 1\}$:

In A_1,

$$f(z) = -\frac{1}{z} + \frac{1}{z-1} = -\frac{1}{z} - \frac{1}{1-z} = -\frac{1}{z} - 1 - z - z^2 - z^3 - \ldots\ldots$$

In A_2,

$$f(z) = -\frac{1}{z} + \frac{\frac{1}{z}}{1-\frac{1}{z}} = -\frac{1}{z} + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ldots$$

Definition We say that a is an *isolated singularity* of f if f is holomorphic on the punctured disc $D'(a,R)$ for some $R > 0$.

In that case f has a Laurent series $\sum_{-\infty}^{\infty} c_n(z-a)^n$ valid on $D'(a,R)$. We define the *residue* of f at a to be the coefficient c_{-1} of $(z-a)^{-1}$ in this series.

12.2 Lemma *If f is holomorphic on $D'(a,R)$ then for any positively oriented simple closed path γ around a in $D'(a,R)$,*

$$\int_\gamma f(z)dz = 2\pi i.res(f,a)$$

Proof By Laurent's Theorem,

$$c_{-1} = \frac{1}{2\pi i}\int_{C(a,r)} f(z)dz$$

The result now follows by the deformation principle. □

Comment An alternative proof is given by the observation that

$$\int_\gamma f(z)dz = \int_\gamma \sum_{-\infty}^{\infty} c_n(z-a)^n = \sum_{-\infty}^{\infty}\int_\gamma c_n(z-a)^n = \int_\gamma \frac{c_{-1}}{z-a} = 2\pi i c_{-1}$$

as all the terms $c_n(z-a)^n$ other than $c_{-1}(z-a)^{-1}$ have antiderivatives on $\mathbb{C}\setminus\{0\}$ and therefore have zero integrals around the closed path γ.

12.3 The Residue Theorem *Let γ be a positively oriented simple closed path in $\mathbb{C}$. If f is holomorphic on γ^* and everywhere inside γ^* except at a finite number of isolated singularities $z_1,\ldots,z_n$ inside γ^*, then*

$$\int_\gamma f(z)dz = 2\pi i\sum_{j=1}^{n} res(f,z_j)$$

Proof Immediate from II, Theorem 9.7 (Cauchy's Theorem for a multiply-connected region) and Lemma 12.2 above. □

13. Evaluating real integrals and series using the Residue Theorem

Computing residues of poles

An isolated singularity of a holomorphic function f is called a *simple pole* if the Laurent series for f in a punctured disc $D'(a,r)$ around a

$$f(z) = \sum_{-\infty}^{\infty} c_n(z-a)^n$$

has $c_{-1}\neq 0$ but $c_n = 0$ for all $n < -1$. So

$$f(z) = \frac{c_{-1}}{z-a} + \sum_{n=0}^{\infty} c_n(z-a)^n$$

An isolated singularity is called a *pole of order* m if $c_{-m} \neq 0$ but $c_n = 0$ for all $n < -m$. So

$$f(z) = \frac{c_{-m}}{(z-a)^m} + \frac{c_{-m+1}}{(z-a)^{m-1}} + \ldots + \frac{c_{-1}}{z-a} + \sum_{n=0}^{\infty} c_n (z-a)^n$$

13.1 Lemma *(i) If* $f(z) = g(z)/(z-a)$ *where* g *is holomorphic at* a *with* $g(a) \neq 0$ *then* a *is a simple pole of* f *and* $Res(f, a) = g(a)$.

(ii) If $f(z) = h(z)/k(z)$ *where* h *and* k *are holomorphic at* a *with* $h(a) \neq 0$, $k(a) = 0$ *and* $k'(a) \neq 0$, *then* a *is a simple pole of* f *and* $Res(f, a) = h(a)/k'(a)$.

Proof

(i) $g(z) = g(a) + (z-a)g'(a) + (z-a)^2 g^{(2)}(a)/2! + \ldots$ (Taylor series for g around $z = a$)
so $g(z)/(z-a) = g(a)/(z-a) + g'(a) + (z-a)g''(a)/2! + \ldots$ is the Laurent series for f around $z = a$.

(ii) $k(z) = k(a) + (z-a)k'(a) + (z-a)^2 k''(a)/2! + \ldots$ (Taylor series for k around $z = a$) So $k(z) = (z-a)(k'(a) + (z-a)k''(a)/2! + \ldots)$, so $h(z)/k(z) = g(z)/(z-a)$ where

$$g(z) = \frac{h(z)}{k'(a) + (z-a)k''(a)/2! + \ldots}$$

But $g(z)$ is holomorphic in some small disc centred at $z = a$ (since the denominator of $g(z)$ is non-zero at $z = a$, and therefore by continuity it is non-zero in a neighbourhood of $z = a$). As $g(a) \neq 0$ the result now follows from (i). $\Box$.

13.2 Lemma *if* $f(z) = g(z)/(z-a)^m$, *where* g *is holomorphic at* a *with* $g(a) \neq 0$, *then* a *is a pole of order* m *of* f *and*

$$Res(f, a) = \frac{g^{(m-1)}(a)}{(m-1)!}$$

Proof

Immediate from dividing the Taylor series for g at $z = a$ by $(z-a)^m$. $\Box$

Evaluating real integrals

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a real function which extends to a complex function f which is holomorphic on the upper half-plane except at a finite number of singularities. Consider a closed path $A_R + B_R$ made up of a semicircular path A_R from $+R$ on the real axis round to $-R$ on the real axis, and a straight line path B_R from $-R$ to $+R$ along the real axis. By the Residue Theorem

$$\int_{A_R+B_R} f(z)dz = 2\pi i(\sum \text{residues inside } A_R + B_R)$$

If we can prove that $\int_{A_R} f$ tends to zero as R tends to infinity, we can deduce the value of

$$\lim_{R\to\infty}\int_{-R}^{+R} f(z)dz$$

Example 1 To find

$$\int_0^\infty \frac{1}{x^6+1}dx$$

The singularities of f are the roots of $z^6=-1$, that is $z^6=e^{i\pi}$, so they are

$$z_k = e^{i\frac{\pi+2k\pi}{6}} \ (k=0,1,\ldots,5)$$

Those in the upper half plane are $z_0=\sqrt{3}/2+i/2$, $z_1=i$, $z_2=-\sqrt{3}/2+i/2$.

So

$$\int_{-R}^{R}\frac{1}{x^6+1}dx+\int_{A_R}\frac{1}{z^6+1}dz = 2\pi i(Res(f,z_0)+Res(f,z_1)+Res(f,z_2))$$

But

$$Res(f,z_k)=\frac{1}{6z_k^5}=-\frac{z_k}{6}$$

So

$$2\pi i(Res(f,z_0)+Res(f,z_1)+Res(f,z_2))=\frac{2\pi}{3}$$

To obtain an upper estimate for $|\int_{A_R} 1/(z^6+1)dz|$ we use the Estimation Lemma and obtain

$$|\int_{A_R}\frac{1}{z^6+1}dz| \leq \frac{\pi R}{R^6-1}\to 0 \text{ as } R\to\infty$$

Hence

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{1}{x^6+1}dx=\frac{2\pi}{3}$$

and as $f(x)$ is an even function (that is $f(-x)=f(x)$) we deduce that

$$\int_0^\infty\frac{1}{x^6+1}dx=\frac{\pi}{3}$$

Example 2 *(We might not do this example in lectures.)* To compute

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+x+1}dx$$

We evaluate this integral by computing the integral of $f(z)=e^{iz}/(z^2+z+1)$ around the same closed path as in Example 1, and then taking the real part.

Briefly, the calculation goes as follows. There is just one singularity of $f(z) = e^{iz}/(z^2 + z + 1)$ in the upper half plane. This is a simple pole at $-1/2 + i\sqrt{3}/2$ and its residue is

$$Res(f, -1/2 + i\sqrt{3}/2) = \frac{e^{i(-1/2+i\sqrt{3}/2)}}{i\sqrt{3}}$$

A bound for e^{iz} on the semicircle A_R is given by observing that if z lies on A_R then e^{iz} has the form $e^{iR(\cos\theta + i\sin\theta)}$ for $0 \le \theta \le \pi$. Thus $|e^{iz}| = e^{-R\sin\theta}$ and since $0 \le \theta \le \pi$ this gives $|e^{iz}| \le 1$ for z on A_R. So

$$|\int_{A_R} \frac{e^{iz}}{z^2 + z + 1} dz| \le \frac{\pi R}{R^2 - R - 1}$$

which goes to zero as R goes to infinity. We deduce that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{x^2 + x + 1} dx = \frac{2\pi e^{i(-1/2+i\sqrt{3}/2)}}{\sqrt{3}}$$

and thus that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\cos x}{x^2 + x + 1} dx = \frac{2\pi e^{-\sqrt{3}/2} \cos 1/2}{\sqrt{3}}$$

There is one more step to verify. The definition of $\int_{-\infty}^{\infty} f(x)dx$ is $\int_0^{\infty} + \int_{-\infty}^{0}$, and we need to check that both exist before we can say that their sum is the same as $\lim_{R\to\infty} \int_{-R}^{R}$. However in the current example we know that $|\cos x/(x^2 + x + 1)| \le 1/(x^2 + x + 1)$ for $x \ge 0$ and that the integral of $1/(x^2 + x + 1)$ exists, so we are OK by the comparison test.

Example 3 *(We might not do this example in lectures.)* To evaluate

$$\int_0^{\infty} \frac{\sin x}{x} dx$$

We follow the method of Example 2 and try to find the imaginary part of $\int_0^{\infty} f(z)dz$, where

$$f(z) = e^{iz}/z$$

In order to estimate the value of the integral efficiently it turns out to be easier in this example to integrate around a large rectangle: from $-X_1$ to X_2 along the real axis, then from X_2 to $X_2 + IY$, then from $X_2 + iY$ to $-X_1 + iY$ and finally from $-X_1 + iY$ to $-X_1$. Using the Estimation Lemma it is not hard to show that the integral of $f(z)$ along the two vertical sides and the top edge of the rectangle all tend to zero as X_1 and X_2 tend to infinity.

This just leaves the straight line path from $-X_1$ to X_2. But the function $f(z)$ has a singularity at $z = 0$, which is on the real axis, so we modify the straight line path from $-X_1$ to X_2 by adding a small semicircle C_ϵ with centre the origin and radius ϵ, so that our closed path now passes just above the origin.

The function $f(z)$ has no singularities inside this new closed path. So we deduce that $\int_{-X_1}^{-\epsilon} f + \int_{C_\epsilon} f + \int_{\epsilon}^{X_2} f$ tends to 0 as X_1 and X_2 tends to infinity.

But for ϵ small we have

$$\frac{e^{iz}}{z} = \frac{1}{z} + \phi(z)$$

where $\phi(z)$ is bounded (since holomorphic). Thus $\int_{C_\epsilon}(f(z) - \frac{1}{z})dz$ tends to zero as ϵ tends to zero.

Finally we can compute

$$\int_{C_\epsilon} \frac{1}{z} dz$$

directly from the definition of this integral and we then have enough information to deduce the value of

$$\int_0^\infty \frac{\sin x}{x} dx$$

(See exercise sheet 7.)

Example 4 To prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(This was first proved by Euler by a different method.)

Let

$$f(z) = \frac{1}{z^2} \cot \pi z = \frac{\cos \pi z}{z^2 \sin \pi z}$$

We integrate this around a large square with vertices $\pm(N + 1/2) \pm (N + 1/2)i$. The function $f(z)$ has a simple pole at each integer value $z = n \neq 0$, with

$$Res(f, n) = \frac{1}{n^2 \pi}$$

and it has a triple pole at $z = 0$, with residue

$$Res(f, 0) = -\frac{\pi}{3}$$

(To compute the residue at $n \neq 0$ we write $f(z)$ as $p(z)/q(z)$, where $p(z) = (\cos \pi z)/z^2$ and $q(z) = \sin \pi z$. The residue is then $p(n)/q'(n)$. To compute the residue at 0 we expand $cos \pi z$ and $\sin \pi z$ as power series in z and thus find the first few terms of the Laurent series around $z = 0$.)

On the right hand edge of the square we have $z = N + 1/2 + iy$ (where $-N - 1/2 \leq y \leq N + 1/2$), and so on this edge we have

$$|\cot \pi z| = |\tan i\pi y| = |\tanh \pi y| \leq 1$$

Similarly on the left hand vertical edge we have $|\cot \pi z| \leq 1$, and on the horizontal edges one can show that

$$|\cot \pi z| \leq \coth (N + 1/2)\pi \leq \coth 3\pi/2$$

It is now a straightforward application of the Estimation Lemma to show that the integral around the square tends to zero as N tends to infinity, and hence (by the Residue Theorem) that

$$-\frac{\pi}{3} + \sum_{n=-\infty}^{\infty} \frac{1}{n^2\pi} = 0$$

where the sum is taken over all integers n except 0.

Hence

$$-\frac{\pi}{3} + 2\sum_{n=1}^{\infty} \frac{1}{n^2\pi} = 0$$

So

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

□

The method of Example 4 can be used to compute $\sum_{n=-\infty}^{\infty} F(n)$ for other functions F (provided $F(n) \sim 1/n^2$ or less). We just replace $(cot\pi z)/z^2$ in the example by $F(z)cot\pi z$.

14. Types of singularity and behaviour near singularities

Let a be an isolated singularity of f, so on some punctured disc $D'(a, R)$ the function $f(z)$ has a Laurent series, which we may write in the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$$

There are three possibilities:

(i) All the b_n are zero. Then we say that a is a *removable* singularity. We remove it by setting $f(a) = a_0$ and now the function $f(z)$ is given by the Taylor series $\sum_0^\infty a_n(z-a)^n$ *everywhere* on the (unpunctured) disc $D(a, R)$.
Example $(\sin z)/z$ has a removable singularity at 0.

(ii) Only finitely many of the b_n are non-zero. In that case there exists an m such that $b_m \neq 0$ but $b_n = 0$ for all $n > m$ and we call a a *pole of order* m.
Example $(\sin z)/z^3$ has a pole of order 2 at 0.

(iii) Infinitely many b_n are non-zero. Then we say that a is an *isolated essential singularity.*
Example $\sin(1/z)$ has an essential singularity at 0.

Behaviour near isolated singularities

It is immediate from considering Laurent series that:

- a is removable if and only if $\lim_{z\to a} f(z)$ exists in $\mathbb{C}$, (i.e. the limit is not allowed to be ∞).
- a is a pole of order m if and only if $lim_{z\to a}(z-a)^m f(z)$ exists in $\mathbb{C}$ and is non-zero.

The following criteria are also useful:

14.1 Proposition *If a holomorphic function $f(z)$ on the punctured disc $D'(a,R)$ is bounded there, then a is a removable singularity of f.*

Proof

Consider the coefficient b_n of $(z-a)^{-n}$ in the Laurent series for f around a. This is given by:

$$b_n = \frac{1}{2\pi i}\int_{C(a,r)} f(z)(z-a)^{n-1}dz$$

If $|f(z)|$ is bounded on $D'(a,R)$ then by taking r arbitrarily small and applying the Estimation Lemma we deduce that $b_n = 0$ (for all $n > 0$). □

14.2 Proposition *$f(z)$ has a pole of order m at a if and only if $1/f(z)$ has a zero of order m at a. (More precisely, if and only if $1/f(z)$ has a removable singularity at a, which when removed gives a zero of order m)*

Proof

Write $g(z) = (z-a)^m f(z)$. Then $f(z)$ has a pole of order m at a

$\Leftrightarrow g(z) = a_0 + a_1(z-a) + \ldots$ (higher powers of $(z-a)$), with $a_0 \neq 0$, on some $D'(a,r)$

$\Leftrightarrow 1/g(z) = 1/a_0 + a_1'(z-a) + \ldots$ (higher powers of $(z-a)$), with $1/a_0 \neq 0$, on some $D'(a,r')$

$\Leftrightarrow 1/f(z)$ $(= (z-a)^m.1/g(z))$ has a zero of order m at $z = a$.

□

14.3 Corollary *If f has a pole of order m at a, then $\lim_{z\to a}|f(z)| = \infty$ (that is, given any $M > 0$ there exists $\delta > 0$ such that $|f(z)| > M$ whenever $|x-a| < \delta$).*

Proof

$$\lim_{z\to a}|f(z)| = \lim_{z\to a}|\frac{g(z)}{(z-a)^m}| = \infty \quad \text{as } g(a) \neq 0$$

□

We now know how a function behaves in the neighbourhood of a removable singularity or a pole. The next result tells us how it behaves near an isolated essential singularity.

14.4 Theorem (Weierstrass-Casorati)

If a is an isolated essential singularity of f, then in every neighbourhood of a the function f takes values arbitraily close to any assigned complex number w (that is, given any $r > 0, \epsilon > 0$ and $w \in \mathbb{C}$, there exists $\zeta \in D'(a,r)$ with $|f(\zeta) - w| < \epsilon|$).

Proof

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

with infinitely many of the b_n non-zero. Given any $w \in \mathbb{C}$, let $\phi(z) = f(z) - w$. Then the Laurent series for ϕ is identical to that for f except that the constant term a_0 is replaced by $a_0 - w$. So ϕ has an essential singularity at a. We are reduced to proving that if ϕ is any function with an essential singularity at a then for every $r > 0$ there exists $\zeta \in D'(a,r)$ with $|\zeta| < \epsilon$.

If a is a limit of zeros of ϕ this is obvious. If not, there exists r' such that $\phi(z) \neq 0$ for $z \in D'(a,r')$. Consider the function $1/\phi(z)$. *Either* there exists $\zeta \in D'(a,r')$ for which $1/|\phi(\zeta)| > 1/\epsilon$ (in which case we are finished) *or* $1/\phi(z)$ is bounded on $D'(a,r')$, in which case $1/\phi(z)$ has a removable singularity at a by 14.1. But then by 14.2 we deduce that $\phi(z)$ has (at worst) a pole at a, contradicting our hypothesis. □

Comment Picard proved the much stronger statement that in every neighbourhood of an isolated essential singularity $f(z)$ *takes every value in* $\mathbb{C}$, with at most one exception. For example for every $r > 0$, however small, $\sin(1/z)$ takes every value in $\mathbb{C}$ in $D'(0,r)$, and $e^{1/z}$ takes every value except 0 in every $D'(0,r)$. This is known as 'Picard's Big Theorem'. Later in this course shall (sketch) prove 'Picard's Little Theorem', which states that every non-constant entire function takes every value in $\mathbb{C}$, except possibly one.

15. Differentiable maps on the Riemann sphere

The extended complex plane (Riemann sphere)

We think of $\mathbb{C}$ as the (x,y)-plane in $\mathbb{R}^3$. Let S^2 denote the unit sphere in $\mathbb{R}^3$, and let $N = (0,0,1)$ denote the 'north pole' of S^2. By stereographic projection from N onto $\mathbb{C}$ we have a bijection from $S^2 \setminus N$ onto $\mathbb{C}$, which sends points on S^2 near to N to points in $\mathbb{C}$ a long way from the origin. This allows to regard the *extended complex plane* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as the *Riemann sphere.*

Differentiability at ∞

Suppose that f is holomorphic on $\{z : |z| > R\}$. Then $g(z) = f(1/z)$ is holomorphic on $D'(0, 1/R)$, so g has an isolated singularity at 0. We say that f has a *removable singularity, pole or essential singularity* *at* ∞ if and only if $g(z)$ has the corresponding type of singularity at 0. If the singularity is removable we set $f(\infty) = g(0)$ and we say that f is *differentiable* at ∞.

Meromorphic functions

A function which is differentiable on $\mathbb{C}$, except for isolated singularities all of which are poles, is called *meromorphic*. If f is meromorphic, we can regard it as a function from $\mathbb{C}$ to $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by setting $f(a) = \infty$ at each pole of f. We can extend a meromorphic f to a function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ if and only if f is *meromorphic at* ∞, that is to say the function $g(z) = f(1/z)$ has a removable singularity or pole at $z = 0$.

15.1 Theorem *A function $f(z)$ is meromorphic on the extended complex plane $\mathbb{C} \cup \{\infty\}$ if and only if $f(z)$ is a rational function (that is $f(z) = p(z)/q(z)$ for some polynomials $p(z)$ and $q(z)$).*

Proof

It is easy to see that a rational function is meromorphic on $\mathbb{C} \cup \{\infty\}$: the only singularities of $p(z)/q(z)$ in $\mathbb{C}$ are the zeros of $q(z)$ and they are all poles; moreover $z = \infty$ is a pole or removable singularity of $p(z)/q(z)$ since $z = 0$ is a pole or removable singularity of $p(1/z)/q(1/z)$.

To prove the converse, suppose f is meromorphic on $\mathbb{C} \cup \{\infty\}$. From the definition of 'meromorphic at ∞' there exists some $R > 0$ such that f has no singularities with $|z| > R$ (except for possibly a singularity at ∞ itself). So all the singularities of f in $\mathbb{C}$ are in the closed disc $\overline{D}(0, R)$. It follows that f has only finitely many singularities in $\mathbb{C}$ (else the singularities would have a limit point, which would be a non-isolated singularity). These are all poles by hypothesis. Suppose they are $z_1, \ldots, z_k$ and have orders $n_1, \ldots, n_k$.

Now let $\psi(z) = (z - z_1)^{n_1} \ldots (z - z_k)^{n_k}$ and set

$$g(z) = \psi(z)f(z) = (z - z_1)^{n_1} \ldots (z - z_k)^{n_k} f(z)$$

The function $g(z)$ is holomorphic on $\mathbb{C}$, and if $f(z)$ has a pole of order N at ∞ then $g(z)$ has a pole of order $M = n_1 + \ldots + n_k + N$ at ∞, since $g(1/z) = \psi(1/z)f(1/z)$ has a pole of order M at $z = 0$.

Consider the Taylor expansion of g (valid on the whole of $\mathbb{C}$):

$$g(z) = \sum a_n z^n$$

Since

$$g(1/z) = \sum a_n z^{-n}$$

the fact that g has a pole of order M at ∞ tells us that $g(1/z)$ has a pole of order M at 0, and hence that $a_n = 0$ for all $n > M$.

Thus $g(z)$ is a polynomial, and $f(z) = g(z)/\psi(z)$ is a rational function. □