MTH6111 Complex Analysis 2010-2011: Summary of Main Results

WARNING: Just because a result is not in this list does not mean it will not be in the exam. The results listed below are the key ones to understanding the whole course. Results which follow fairly directly from definitions have been omitted. When you revise for the examination you should look through ALL results and ALL proofs, except for those which are labelled "not examinable".

SECTION I: Holomorphic Functions

3.2 Theorem *Every bounded sequence in $\mathbb{C}$ has a convergent subsequence.*

Big idea in proof of 3.2. Repeatedly divide a square into four squares of half the side length. At each step one of the four must contain infinitely many points of the sequence.

3.3 Corollary *$S \subseteq \mathbb{C}$ is compact (i.e. closed and bounded) if and only if every sequence in S has a subsequence which converges to a limit in S. ("S is compact if and only if S is sequentially compact.")*

3.4 Theorem *let $S \subseteq \mathbb{C}$ be compact and $f : S \to \mathbb{C}$ be continuous. Then $f(S)$ is compact and $|f|$ attains its bound on S (i.e. if $M = \sup_{z \in S} |f(z)|$ then $\exists a \in S$ such that $|f(a)| = M$).*

Big ideas in proof of 3.4. Use 3.3 and the fact that since f is continuous $f(\lim z_n) = \lim f(z_n)$ for any convergent sequence (z_n).

4.1 Theorem (Cauchy-Riemann equations). *Let $f : U \to \mathbb{C}$ be differentiable at a point $z = x + iy \in U$, and let $u(x, y) = Re(f(z))$ and $v(x, y) = Im(f(z))$.*
Then the partial derivatives $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x, \partial v/\partial y$ exist at (x, y) and satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

there.

Big idea in proof of 4.1. $\lim_{h \to 0}(f(z + h) - f(z))/h$ has to be the same whatever the direction in which we let h go to zero. Comparing the limits with h real and h imaginary gives the Cauchy-Riemann equations.

4.2 Proposition *Let $f \in H(D(0, R))$ (i.e. f is holomorphic on $D(0, R)$). Then*
(i) if $f'(z) = 0$ for all $z \in D(0, R)$ then f is constant on $D(0, R)$;
(ii) if $|f(z)|$ is constant on $D(0, R)$ then f is constant on $D(0, R)$.

Big idea in proof of 4.2. Let $f = u + iv$.
For (i), between any two points in $D(0, R)$ there is a path made up of a horizontal segment and a vertical segment. Apply the real mean value theorem to prove that u and v are constant.

For (ii), partially differentiate $u^2 + v^2 = c^2$ with respect to x and y and apply the Cauchy-Riemann equations. Deduce that the partial derivatives of u and v with respect to x and y are all zero. Now apply (i).

5.4 Proposition *Every power series* $\sum_0^\infty a_n z^n$ *has a radius of convergence, i.e.* $\exists R$*, possibly* 0 *or* ∞*, such that the series converges absolutely for all* z *with* $|z| < R$ *and diverges for all* z *with* $|z| > R$*.*

Big idea in proof of 5.4. It suffices to show that if $\sum_0^\infty a_n w^n$ converges, and $|z| < |w|$, then $\sum_0^\infty a_n z^n$ converges absolutely. Let $b = \frac{|z|}{|w|}$. Using the fact that $\sum b^n$ converges show that $\sum |a_n z^n|$ converges.

5.5 Lemma $\sum_{n=0}^\infty a_n z^n$ *and* $\sum_1^\infty n a_n z^{n-1}$ *have the same radius of convergence.*

Big idea in proof of 5.5. Show that if $\sum_0^\infty a_n z^n$ converges absolutely for $|z| < R \neq 0$, then so does $\sum_1^\infty n a_n z^{n-1}$. To do this, choose r with $|z| < r < R$. Now

$$|n a_n z^{n-1}| = \frac{n}{r} \left(\frac{|z|}{r} \right)^{n-1} |a_n r^n|.$$

But there exists M with $\frac{n}{r} \left(\frac{|z|}{r} \right)^{n-1} < M$ for all n since $\lim_{n\infty} \frac{n}{r} \left(\frac{|z|}{r} \right)^{n-1} = 0$ (by the ratio test). Finally show $\sum_1^\infty |n a_n z^{n-1}|$ converges, by the comparison test, since $\sum_0^\infty |a_n r^n|$ converges.

5.6 Theorem *If* $\sum_0^\infty a_n z^n$ *has radius of convergence* $R \neq 0$*, and* $f(z)$ *is the sum of this series, then* f *is a holomorphic function on* $D(0, R)$*, and its derivative is* $f'(z) = \sum_1^\infty n a_n z^{n-1}$ *for all* $z \in D(0, R)$*.*
Proof of 5.6 not examinable.

5.7 Corollary *If* $f(z) = \sum_0^\infty a_n z^n$ *has radius of convergence* $R \neq 0$ *then* f *has derivatives of all orders on* $D(0, R)$ *and* $f^{(n)}(0) = n! a_n$ *for all* $n \geq 0$ *(where* $0!$ *is defined to be* 1*).*

5.8 Proposition
(i) e^z *is entire (that is, differentiable for all* $z \in \mathbb{C}$*) and the derivative of* e^z *is* e^z*.*
(ii) $\forall z, w \in \mathbb{C}$, $e^{z+w} = e^z e^w$*.*

Big idea in proof of 5.8. (i) is obvious using 5.6. For (ii), for any $\zeta \in \mathbb{C}$, differentiate the product $f(z) = e^z e^{\zeta - z}$ and deduce that f is constant using 4.2(i).

SECTION II: Integration and Cauchy's Theorem

7.1 Fundamental theorem of calculus *Let* γ *be a path* $[\alpha,\beta]\to\mathbb{C}$*, and let* $F:U\to\mathbb{C}$ *be holomorphic, where* U *is an open set containing* γ^**. Then*

$$\int_\gamma F'(z)dz = F(\gamma(b)) - F(\gamma(a))$$

In particular, if γ *is a closed path (this means* $\gamma(\alpha)=\gamma(\beta)$*) then* $\int_\gamma F'(z)dz = 0$.

Big idea in proof of 7.1. Prove it for a smooth path γ by applying the fundamental theorem of calculus for real functions to the real and imaginary parts.

8.1 Estimation Lemma *Let* $f:U\to\mathbb{C}$ *be continuous (where* U *is some subset of* $\mathbb{C}$*), let* γ *be a path in* U*, and suppose* $|f(z)|<M$ *for all* $z\in\gamma^*$*. Let length(*γ*)=* L*. Then*

$$|\int_\gamma f(z)dz| \le ML.$$

Big ideas in proof of 8.1 First prove that for real-valued functions u and v of $t\in\mathbb{R}$,

$$(*) \qquad |\int_a^b (u(t)+iv(t))dt| \le \int_a^b |u(t)+iv(t)|dt$$

To do this this, write $\int_a^b (u(t)+iv(t))dt = Re^{i\theta}$ and show that $|\int_a^b(u(t)+iv(t))dt| = \int_a^b Re(e^{-i\theta}(u(t)+iv(t))dt$. Now apply $(*)$ to $|\int_\gamma f| = |\int_\alpha^\beta f(\gamma(t))\gamma'(t)dt|$.

8.2 Proposition *Let* γ *be a path in* $U\subset\mathbb{C}$ *and* $(F_n)_{n\ge0}$ *be a sequence of continuous functions* $U\to\mathbb{C}$ *converging uniformly on* γ^* *to a continuous function* F*. Then*

$$\int_\gamma F = \lim_{n\to\infty}\int_\gamma F_n$$

Big idea in proof of 8.2. Apply the Estimation Lemma toshow that

$$|\int_\gamma F - \int_\gamma F_n| = |\int_\gamma (F-F_n)| \le L(\gamma).sup_{z\in\gamma^*}|F(z)-F_n(z)|.$$

Now take the limit as n tends to infinity.

8.3 Corollary *If a series* $\sum_{m=0}^\infty f_m(z)$ *of continuous functions converges uniformly on* $\gamma*$ *to a continuous function, then*

$$\int_\gamma\left(\sum_{m=0}^\infty f_m(z)\right)dz = \sum_{m=0}^\infty\left(\int_\gamma f_m(z)dz\right)$$

9.1 Cauchy's Theorem for a triangle *If f is holomorphic on an open set U containing the triangle $\gamma*$, and its interior, then $\int_\gamma f(z)dz = 0$.*

Big ideas in proof of 9.1. Suppose the triangle has perimeter of length L, and that the integral is $I \neq 0$. Subdivide the triangle into four, each of perimeter $L/2$. One of these has integral of modulus at least $|I|/4$. Choose it and repeat. Inductively get a triangle of perimeter $L/2^n$ and on which the integral has modulus at least $|I|/4^n$. These triangles nest down to a point. Using differentiablity of f at this point, and the Estimation Lemma, we can get a contradiction and hence deduce that $I = 0$.

9.2 Theorem (Existence of Antiderivatives) *Let U be a convex open subset of $\mathbb{C}$ and let f be holomorphic on U. Then there exists a holomorphic function F on U such that $F' = f$.*

Big ideas in proof of 9.2. Choose a point $a \in U$, and for each $z \in U$ define

$$F(z) = \int_{[a,z]} f(w)dw$$

Now prove F is differentiable, with derivative f by observing that, by 9.1,

$$F(z+h) - F(z) = \int_{[z,z+h]} f(w)dw,$$

and applying the Estimation Lemma to show that $|\int_{[z,z+h]}(f(w) - f(z))dw|$ is small and hence that $\int_{[z,z+h]} f(w)dw$ is close to $\int_{[z,z+h]} f(z)dw = hf(z)$.

9.3 Corollary (Cauchy's Theorem for a convex region) *Let U be an open convex subset of $\mathbb{C}$ and f be holomorphic on U. Then $\int_\gamma f(z)dz = 0$ for every closed path γ in U (where $\gamma : [\alpha, \beta] \to U$ is said to be a closed path if $\gamma(\alpha) = \gamma(\beta)$).*

Big idea in proof of 9.3. Theorem 9.2 and the Fundamental Theorem of Calculus (7.1).

9.4 Cauchy's Theorem for a simple closed path *If f is holomorphic everywhere on and inside a simple closed path γ, then $\int_\gamma f(z)dz = 0$.*

Not proved in course.

9.5 Deformation Principle

(i) If γ_0 and γ_1 are homotopic closed paths in U and f is holomorphic on U then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$.

(ii) If γ_1 and γ_2 are paths in U between the same points a and b which are homotopic in U relative to their end points, and f is holomorphic on U, then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$.

Not proved in course.

9.6 Cauchy's Theorem for a simply-connected region *If* f *is holomorphic on a simply-connected region* U *then* $\int_\gamma f(z)dz = 0$ *for every closed path* γ *in* U*.*

Not proved in course.

9.7 Cauchy's Theorem for a multiply-connected region

If γ *is a simple closed path, and* $\gamma_1, \ldots, \gamma_n$ *are disjoint simple closed paths in the interior of* γ*, such that the interiors of* $\gamma_1, \ldots, \gamma_n$ *are disjoint, then if* f *is holomorphic on* γ^**, and on* $\gamma_1^*, \ldots, \gamma_n^*$*, and on the region between* γ^* *and* $\gamma_1^* \cup \ldots \cup \gamma_n^*$*, we have*

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \ldots + \int_{\gamma_n} f(z)dz$$

(provided γ *and* $\gamma_1, \ldots, \gamma_n$ *are all parametrized in the the same direction e.g. positively, that is, anti-clockwise.)*

Not proved in course.

SECTION III

10.1 Cauchy's Integral Formula

Let f *be holomorphic on and everywhere inside a simple closed path* γ*. Then for every point* a *inside* γ*,*

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz$$

Big idea in proof of 10.1 The deformation principle tells us that

$$\int_\gamma \frac{f(z)}{z-a}dz = \int_{C(a,r)} \frac{f(z)}{z-a}dz$$

But we know that

$$\int_{C(a,r)} \frac{f(a)}{z-a}dz = f(a)\int_{C(a,r)} \frac{1}{z-a}dz = f(a)\int_0^{2\pi} \frac{1}{re^{it}}ire^{it}dt = 2\pi i f(a)$$

so it suffices to prove that we can make

$$\int_{C(a,r)} \frac{f(z)-f(a)}{z-a}dz$$

arbitrarily small by taking r sufficiently small. We use the Estimation Lemma to prove this.

10.2 Liouville's Theorem

If f *is entire (that is, holomorphic everywhere on* $\mathbb{C}$*) and* f *is bounded on* $\mathbb{C}$*, then* f *is constant.*

Big idea in proof of 10.2 Choose any two points $a, b \in \mathbb{C}$, and take $R \geq 2max\{|a|, |b|\}$. Using Cauchy's Integral Formula, we have

$$f(a) - f(b) = \frac{1}{2\pi i}\int_{C(0,R)} f(z)\left(\frac{1}{z-a} - \frac{1}{z-b}\right)dz = \frac{1}{2\pi i}\int_{C(0,R)} f(z)\frac{a-b}{(z-a)(z-b)}dz$$

Now use the Estimation Lemma.

10.3 The Fundamental Theorem of Algebra

Let $p(z)$ be a non-constant polynomial (that is, a polynomial of degree ≥ 1) with coefficients in $\mathbb{C}$. Then there exists $w \in \mathbb{C}$ such that $p(w) = 0$.

Big idea in proof of 10.3 Suppose there is no solution in $\mathbb{C}$ to the equation $p(z) = 0$. Then the function $f(z)$ defined by $f(z) = 1/p(z)$ is entire. Choose R such that for all z with $|z| > R$ the terms $|a_0/z^n|, \ldots, |a_{n-1}/z|$ are all $< |a_n|/2n$. Now for all z with $|z| > r$

$$|p(z)| > \frac{|a_n|R^n}{2} \text{ and so } |f(z)| < \frac{2}{|a_n|R^n}.$$

Hence $f(z)$ is bounded on $\mathbb{C}$, and therfore it is constant by Liouville's Theorem.

10.4 Cauchy's formula for derivatives (extended form of Cauchy's Integral Formula)

Let f be holomorphic on and everywhere inside a simple closed path γ. Then at every point a inside γ, the nth derivative $f^{(n)}(a)$ exists for $n = 0, 1, 2, 3 \ldots$, and

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}dz$$

Proof of 10.4 is an immediate consequence of 11.2 (Taylor's Theorem) below.

11.1 Lemma *Let f be holomorphic on $D(a, R)$. Then there exist constants $c_n \in \mathbb{C}$ such that*

$$f(a+h) = \sum_{n=0}^{\infty} c_n h^n$$

and this series converges absolutely for all h with $|h| < R$. Moreover

$$c_n = \frac{1}{2\pi i}\int_{C(a,r)} \frac{f(z)}{(z-a)^{n+1}}dz$$

for any r with $0 < r < R$.

Big idea in proof of 11.1 Cauchy's Integral Formula gives

$$f(a+h) = \frac{1}{2\pi i}\int_{C(a,r)} \sum_{n=0}^{\infty} \frac{f(z)}{(z-a)^{n+1}}.h^n dz.$$

Expand $1/(z-(a+h))$ as a power series in h and apply the fact that $\int \sum = \sum \int$ (which follows from the uniform convergence of the terms in the sum of functions being integrated).

11.2 Taylor's Theorem

If f is holomorphic on the open set $U \subseteq \mathbb{C}$ then all the higher derivatives of f exist everywhere in U, and for all z in any disc $D(a,R) \subseteq U$, $f(z)$ is equal to the sum of the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

Moreover for any r with $0 < r < R$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{C(a,r)} \frac{f(z)}{(z-a)^{n+1}} dz$$

Big idea in proof of 11.2 By Lemma 11.1,

$$f(a+h) = \sum_{n=0}^{\infty} c_n h^n \quad \forall h \text{ with } |h| < R$$

so, writing $z = a+h$ we have

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \text{ for } |z-a| < R$$

By 5.6 and 5.7 any power series can be differentiated arbitrarily many times and for the power series above,

$$f^{(n)}(a) = n! c_n.$$

Finally by Lemma 11.1

$$c_n = \frac{1}{2\pi i} \int_{C(a,r)} \frac{f(z)}{(z-a)^{n+1}} dz$$

11.3 Corollary (Cauchy's Estimate)

If f is differentiable for $|z-a| < R$, if $0 < r < R$ and if $|f(z)| \leq M$ everywhere on the circle $|z-a| = r$, then

$$|f^{(n)}(a)| \leq \frac{M.n!}{r^n} \; \forall n \geq 0$$

Proof of 11.3

$$|f^{(n)}(a)| = |\frac{n!}{2\pi i} \int_{C(a,r)} \frac{f(z)}{(z-a)^{n+1}} dz| \leq \frac{n!}{2\pi} . \frac{M}{r^{n+1}} .2\pi r = \frac{M.n!}{r^n}$$

12.1 Laurent's Theorem

Let $A = \{z \in \mathbb{C} : R_1 < |z-a| < R_2\}$*, where* R_1 *and* R_2 *are constants such that* $0 \leq R_1 < R_2 \leq \infty$*, and let* $f \in H(A)$ *(that is,* f *is holomorphic on* A*). Then*

$$f(z) = \sum_{-\infty}^{\infty} c_n (z-a)^n \quad \forall z \in A$$

where

$$c_n = \frac{1}{2\pi i} \int_{C(a,r)} \frac{f(w)}{(w-a)^{n+1}} dw \quad \forall n \in \mathbb{Z}$$

for any r *such that* $R_1 < r < R_2$*.*

Big idea in proof of 12.1 By a change of coordinates assume that $a = 0$. Choose any $z \in A$ and let r_1 and r_2 be such that $R_1 < r_1 < |z| < r_2 < R_2$. By Cauchy's Integral Formula and the deformation principle

$$f(z) = \frac{1}{2\pi i} \int_{C(0,r_2)} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \int_{C(0,r_1)} \frac{f(w)}{w-z} dz.$$

Since $|z| < |w|$ for w on $C(0, r_2)$ and $|z| > |w|$ for w on $C(0, r_1)$, expanding $1/(w-z)$ in the two integrals gives

$$f(z) = \frac{1}{2\pi i} \int_{C(0,r_2)} \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}} f(w) dw - \frac{1}{2\pi i} \int_{C(0,r_1)} \sum_{m=0}^{\infty} \frac{-w^m}{z^{m+1}} f(w) dw.$$

Now invoke uniform convergence to interchange $\int$ with $\sum$ and complete the proof.

12.2 Lemma *If* f *is holomorphic on* $D'(a, R)$ *then for any positively oriented simple closed path* γ *around* a *in* $D'(a, R)$*,*

$$\int_\gamma f(z) dz = 2\pi i . res(f, a)$$

Proof of 12.2 By Laurent's Theorem, for any $r < R$

$$c_{-1} = \frac{1}{2\pi i} \int_{C(a,r)} f(z) dz$$

The result follows by the deformation principle (just deform $C(a, r)$ to γ).

12.3 The Residue Theorem *Let* γ *be a positively oriented simple closed path in* $\mathbb{C}$*. If* f *is holomorphic on* γ^* *and everywhere inside* γ^* *except at a finite number of isolated singularities* $z_1, \ldots, z_n$ *inside* γ^**, then*

$$\int_\gamma f(z) dz = 2\pi i \sum_{j=1}^{n} res(f, z_j)$$

Proof of 12.3 Immediate from 9.7 (Cauchy's Theorem for a multiply-connected region) and 12.2.

13.1 Lemma *(i) If* $f(z) = g(z)/(z-a)$ *where* g *is holomorphic at* a *with* $g(a) \neq 0$ *then* a *is a simple pole of* f *and* $Res(f,a) = g(a)$.
(ii) If $f(z) = h(z)/k(z)$ *where* h *and* k *are holomorphic at* a *with* $h(a) \neq 0$, $k(a) = 0$ *and* $k'(a) \neq 0$, *then* a *is a simple pole of* f *and* $Res(f,a) = h(a)/k'(a)$.

Big idea in proof of 13.1 Consider the Taylor series for g,h and k and hence compute the first few terms of the Laurent series for f in each case.

13.2 Lemma *if* $f(z) = g(z)/(z-a)^m$, *where* g *is holomorphic at* a *with* $g(a) \neq 0$, *then* a *is a pole of order* m *of* f *and*

$$Res(f,a) = \frac{g^{(m-1)}(a)}{(m-1)!}$$

Proof of 13.2 Immediate from dividing the Taylor series for g at $z = a$ by $(z-a)^m$.

14.1 Proposition *If a holomorphic function* $f(z)$ *on the punctured disc* $D'(a,R)$ *is bounded there, then* a *is a removable singularity of* f.

Idea in Proof of 14.1 The coefficient b_n of $(z-a)^{-n}$ in the Laurent series for f around a is given by:

$$b_n = \frac{1}{2\pi i}\int_{C(a,r)} f(z)(z-a)^{n-1}dz$$

By taking r arbitrarily small and applying the Estimation Lemma deduce that $b_n = 0$ (for all $n > 0$).

14.2 Proposition $f(z)$ *has a pole of order* m *at* a *if and only if* $1/f(z)$ *has a zero of order* m *at* a. *(More precisely, if and only if* $1/f(z)$ *has a removable singularity at* a*, which when removed gives a zero of order* m*)*

Proof of 14.2 Write $g(z) = (z-a)^m f(z)$. Consider the Laurent series for $f(z)$ at $z = a$ and the Taylor series for $g(z)$ at $z = a$, and observe that:
$f(z)$ has a pole of order m at $z = a$
$\Leftrightarrow$ $g(z)$ is holomorphic on $D(a,r)$ for some $r > 0$, and $g(a) \neq 0$
  (more precisely $g(z)$ has a removable singularity at $z = a$ which when removed gives $g(a) \neq 0$)
$\Leftrightarrow$ $1/g(z)$ is holomorphic on $D(a,r')$ for some $r' > 0$, and $1/g(a) \neq 0$
$\Leftrightarrow$ $1/f(z)$ (=$(z-a)^m/g(z)$) has a zero of order m at $z = a$.

14.3 Corollary *If* f *has a pole of order* m *at* a, *then* $\lim_{z\to a}|f(z)| = \infty$ *(that is, given any* $M > 0$ *there exists* $\delta > 0$ *such that* $|f(z)| > M$ *whenever* $|x-a| < \delta$*).*

14.4 Theorem (Weierstrass-Casorati) *If a is an isolated essential singularity of f, then in every neighbourhood of a the function f takes values arbitraily close to any assigned complex number w (that is, given any $r > 0, \epsilon > 0$ and $w \in \mathbb{C}$, there exists $\zeta \in D'(a, r)$ with $|f(\zeta) - w| < \epsilon|$).*

Proof of 14.4 Given any $w \in \mathbb{C}$, let $\phi(z) = f(z) - w$. Since its Laurent series differes from that of f only in the constant term, ϕ has an essential singularity at a. Thus it suffices to prove that if ϕ is any function with an essential singularity at a then for every $r > 0$ there exists $\zeta \in D'(a, r)$ with $|\zeta| < \epsilon$.

If a is a limit of zeros of ϕ this is obvious. If not, there exists r' such that $\phi(z) \neq 0$ for $z \in D'(z, r')$. Consider the function $1/\phi(z)$. *Either* there exists $\zeta \in D'(a, r')$ for which $1/|\phi(\zeta)| > 1/\epsilon$ (in which case we are finished) *or* $1/\phi(z)$ is bounded on $D'(a, r')$, in which case $1/\phi(z)$ has a removable singularity at a by 14.1. But then, by 14.2, $\phi(z)$ has (at worst) a pole at a, contradicting our hypothesis.

15.1 Theorem *A function $f(z)$ is meromorphic on the extended complex plane $\mathbb{C} \cup \{\infty\}$ if and only if $f(z)$ is a rational function (that is $f(z) = p(z)/q(z)$ for some polynomials $p(z)$ and $q(z)$).*

Big ideas in proof of 15.1 A rational function $p(z)/q(z)$ is meromorphic on $\mathbb{C} \cup \{\infty\}$ since:

- the only singularities of $p(z)/q(z)$ in $\mathbb{C}$ are the zeros of $q(z)$ and they are all poles;
- $z = \infty$ is a pole or removable singularity of $p(z)/q(z)$ (as $z = 0$ is a pole or removable singularity of $p(1/z)/q(1/z)$).

For the converse, given a function f which is meromorphic on $\mathbb{C} \cup \{\infty\}$:

- All the singularities of f in $\mathbb{C}$ are in some $D(0, R)$, since the singularity at ∞ is isolated. Suppose these are $z_1, \ldots, z_k$ and that they are poles of orders $n_1, \ldots, n_k$.
- Let $\psi(z) = (z - z_1)^{n_1} \ldots (z - z_k)^{n_k}$ and $g(z) = \psi(z)f(z)$. Then $g(z)$ is holomorphic on $\mathbb{C}$, and has a pole at ∞.
- The Taylor series for g at $z = 0$ converges on the whole of $\mathbb{C}$. But $g(1/z)$ can have only finitely many terms in $1/z$ since g has a pole at ∞. Hence $g(z)$ is a polynomial.

SECTION IV

16.1 Lemma *If f is holomorphic at a and $f'(a) \neq 0$ then f is conformal at a.*

Big idea in proof of 16.1 By the chain rule

$$\frac{(f \circ \gamma_2)'(t_2)}{(f \circ \gamma_1)'(t_1)} = \frac{f'(\gamma_2(t_2))\gamma_2'(t_2)}{f'(\gamma_1(t_1))\gamma_1'(t_1)} = \frac{f'(a)\gamma_2'(t_2)}{f'(a)\gamma_1'(t_1)} = \frac{\gamma_2'(t_2)}{\gamma_1'(t_1)}$$

17.1 Lemma *The automorphisms of $\hat{\mathbb{C}}$ are the maps of the form:*

$$\phi(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Big idea in proof of 17.1 If f is a rational map of degee d then $f^{-1}(w)$ consists of d distinct points, for almost every $w \in \hat{\mathbb{C}}$. So for f to be injective we need $d = 1$.

17.3 Proposition *A Mobius transformation is conformal at every point of* $\hat{\mathbb{C}}$*.*

Big idea in proof of 17.3 Differentiate and show the derivative is never zero.

17.4 Proposition *Every Möbius transformation* ϕ *is a composition of transformations of the three types:* $z \to az$ $a \neq 0$, $z \to z + b$, $z \to 1/z$

Big idea in proof of 17.4 Decompose into

$$z \to Z \to W \to w$$

where $Z = cz + d$, $W = 1/Z$, and $w = \alpha W + \beta$. Compute the values of α and β.

17.5 Corollary *Möbius transformations send circles to circles (where a 'circle through* ∞*' is a straight line in* $\mathbb{C}$*).*

17.6 Proposition *Let* z_1, z_2, z_3 *be any triple of distinct points in* $\hat{\mathbb{C}}$ *and* w_1, w_2, w_3 *also be a triple of distinct points in* $\hat{\mathbb{C}}$*. Then there exists a unique Möbius transformation* ϕ *such that* $\phi(z_j) = w_j$ *for* $j = 1, 2, 3$.

Big idea in proof of 17.6

$$\phi : z \to \left(\frac{z_1 - z_3}{z_2 - z_3}\right)\left(\frac{z_2 - z}{z_1 - z}\right)$$

maps z_1, z_2, z_3 to $\infty, 0, 1$ respectively. Similarly

$$\psi : w \to \left(\frac{w_1 - w_3}{w_2 - w_3}\right)\left(\frac{w_2 - w}{w_1 - w}\right)$$

maps w_1, w_2, w_3 to $\infty, 0, 1$. Thus $\psi^{-1} \circ \phi$ has the required property.

18.1 Proposition *Every automorphism of the plane* $\mathbb{C}$ *(that is to say holomorphic bijection* $\mathbb{C} \to \mathbb{C}$*) has the form*

$$\phi(z) = az + b$$

for some $a \in \mathbb{C}$ *with* $a \neq 0$*.*

Big idea in proof of 18.1: details not examinable. Every automorphism of the plane extends to an automorphism of the Riemann sphere.

19.1 Theorem *Let* f *be holomorphic in an open set* U*, with real and imaginary parts* u *and* v*. Then both* u *and* v *are harmonic in* U*.*

Big idea in proof of 19.1 Differentiate the Cauchy-Riemann equations with respect to x and y.

19.2 Theorem *Let $D \subset \mathbb{R}^2$ be an open disc and suppose that $u : D \to \mathbb{R}$ is harmonic. Then there exists a complex function f, holomorphic in D, such that $u = Re(f)$.*

Proof of 19.2 omitted in lectures because of lack of time, so non-examinable.

20.1 The Maximum Modulus Principle *If f is holomorphic on an open disc $D = D(a, r)$ and $b \in D$ is a maximum point for $|f|$, that is $|f(b)| \geq |f(z)|$ $\forall z \in D$, then f is constant on D.*

Big idea in proof of 20.1 Follows from Gauss' Mean Value Theorem which states that if f is holomorphic on $D(z_0, R)$, and $r < R$, then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

(Gauss' Mean Value Theorem follows directly from Cauchy's integral formula.)

20.2 Schwarz's Lemma *Let $f : D \to D$ be a holomorphic function of the unit disc to itself such that $f(0) = 0$. Then*

(i) $|f(z)| \leq |z|$ for all $z \in D$.

(ii) If for some $z_0 \neq 0$ we have $|f(z_0)| = |z_0|$ then there is some complex number μ of modulus 1 such that $f(z) = \mu z$.

Proof of 20.2 Since $f(0) = 0$ the Taylor series for $f(z)$ at $z = 0$ begins $f(z) = a_1 z + a_2 z^2 + \ldots$. Hence $f(z)/z = a_1 + a_2 z + a_3 z^2 + \ldots$ is holomorphic on the unit disc D. For any $r < 1$ choose z_0 with $|z_0| = r$. Now $|f(z_0)/z_0| < 1/r$ so by the Maximum Modulus Principle $|f(z)/z| \leq 1/r$ for all z with $|z| \leq |z_0| = r$. Letting r tend to 1 gives (i).

For (ii), if $\exists z_0 \in D$ with $|f(z_0)/z_0| = 1$ then by the Maximum Modulus Principle $f(z)/z$ is constant on D. Hence $f(z)/z = \mu$ for some constant μ of modulus 1.

20.3 Corollary *Let $f : D \to D$ be an automorphism of the unit disc. Then f is a Möbius transformation. Indeed if $a \in D$ is the unique point such that $f(a) = 0$, then*

$$f(z) = e^{i\theta} \frac{a - z}{1 - \bar{a}z}$$

for some real number θ.

Proof of 20.3 The Möbius transformation

$$g(z) = \frac{a - z}{1 - \bar{a}z}$$

sends D to D and it sends the point a to 0. So $f \circ g^{-1}$ is an automorphism h of D which has $h(0) = 0$. Apply 20.2(i) to h and to h^{-1} to deduce $|h(z)| = |z|$ for all $z \in D$. Part (ii) of 20.2 now gives $h(z) = e^{i\theta} z$ for some real θ. So $f = h \circ g$ has the required form.

20.4 Corollary The automorphisms of the complex upper half-plane H are the Möbius transformations of the form

$$z \to \frac{az+b}{cz+d}$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$

Proof of 20.4 Let ϕ be a Möbius transformation which is a bijection from H to the unit disc D. If f is an automorphism of H then $\phi \circ f \circ \phi^{-1}$ is an automorphism of D, and by 20.3 it is a Möbius transformation ψ. Hence $f(= \phi^{-1} \circ \psi \circ \phi)$ is a Möbius transformation. As f sends $\hat{\mathbb{R}}$ to $\hat{\mathbb{R}}$, preserving orientation, it has the stated form.

SECTION V

23.1 Proposition (Schwarz reflection principle) *Let U be a connected open set in $\mathbb{C}$ which is symmetric about the real axis, that is, $z \in U \Leftrightarrow \bar{z} \in U$. Let $U^+ = \{z \in U : Im(z) > 0\}$, $U^- = \{z \in U : Im(z) < 0\}$ and $U^0 = \{z \in U : Im(z) = 0\}$. Suppose $f : U^+ \cup U^0 \to \mathbb{C}$ is continuous, that $f(z)$ is real for all $z \in U^0$, and that f is holomorphic on U^+. Then f can be extended analytically to U^- by setting $f(z) = \overline{f(\bar{z})}$ for each $z \in U^-$.*

Statement and proof of 23.1 omitted from lectures because of lack of time, so not examinable

24.1 Lemma *Every double-periodic holomorphic function is constant.*

Proof of 24.1 The fundamental domain P is closed and bounded. So $f(P)$ is bounded. Hence f is bounded on $\mathbb{C}$, since for any $z \in \mathbb{C}$, $f(z) = f(z')$ for some $z' \in P$. Hence, by Liouville's Theorem, f is constant.

24.2 Proposition
(i) Every elliptic function of order 0 *is constant.*
(ii) There are no elliptic functions of order 1.

Proof of 24.2 Assume the fundamental parallelogram P chosen with no poles on its edges. Part (i) is now Lemma 24.1 (since a meromorphic f is holomorphic if and only if it has no poles). For part(ii), note that integrating f once around the boundary of P gives zero, since the integrals along opposite sides cancel. So by the Residue Theorem the sum of the residues of f inside P is zero. Hence f must have at least two simple poles, or at least one pole of order ≥ 2.

24.3 Proposition *For each $N \geq 3$*

$$F_N(z) = \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^N}$$

is elliptic of order N with respect to Ω

Idea in proof of 24.3 Once we know the sum is convergent (proof not expected), so F_N is a well defined (meromorphic) function, it follows from the expression for F_N that it is doubly periodic and of order N.

25.1 Picard's Little Theorem *If $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that there exist at least two points in $\mathbb{C}$ not in the image of f, then f is constant.*

Proof of 25.1 not examinable.