MTH 6111 Complex Analysis 2010-11

Solutions 9

See final sheet of these solutions for "how to deal with ∞"

1. z is a fixed point of $\phi \iff \dfrac{az+b}{cz+d} = z$

$\iff az+b = cz^2+dz$

$\iff cz^2+(d-a)z-b = 0$

This has solutions $z = \dfrac{(a-d)\pm\sqrt{(d-a)^2+4bc}}{2c}$ (1)

(i) There is just one solution in $\mathbb{C}\cup\{\infty\} \iff (d-a)^2+4bc = 0$

$\iff d^2+a^2-2ad+4bc = 0$

$\iff d^2+a^2+2ad-4 = 0$ (using $ad-bc=1$)

$\iff |a+d| = 2$ (2)

This solution is necessarily real (or ∞) since it is $z = \frac{a-d}{2c}$.

(ii) There are two solutions in $\mathbb{R}\cup\{\infty\} \iff (d-a)^2+4bc > 0 \iff d^2+a^2+2ad > 4$

$\iff |a+d| > 2$ (2)

(iii) There are two, non-real, complex conjugate solutions $\iff (d-a)^2+4bc < 0$

$\iff d^2+a^2+2ad < 4 \iff |a+d| < 2$

But if the solutions are complex conjugate then exactly one of them is in the upper $\frac{1}{2}$-plane H. (3)

(iv) For unique fixed point ∞ we need $c=0$ and $d=a$ (1)
so ϕ has the form $\boxed{\phi(z) = z+b}$

For unique fixed point 0 we need $a=d$ and $b=0$
so ψ has the form $\boxed{\psi(z) = \dfrac{z}{cz+1}}$ (1)

Now $\psi\circ\phi(z) = \dfrac{z+b}{c(z+b)+1} = \dfrac{z+b}{cz+(bc+1)}$

Thus $\psi\circ\phi$ is hyperbolic (by (ii)) $\iff |bc+2| > 2$ (1)
and elliptic (by (iii)) $\iff |bc+2| < 2$

[Note that we cannot have $\psi\circ\phi$ parabolic unless either b or c is zero, in which case either ϕ or ψ is the identity.]

2. Let (z_n) tend to ∞, and let $w_n = f(z_n)$ for each n.

(*) If (w_n) does not tend to ∞ then $\exists R$ s.t. $\forall n\ \exists m > n$ with $|w_m| \le R$.

[This is because the definition of $\lim_{n\to\infty} w_n = \infty$ is the following:
$\forall R\ \exists n$ s.t. $\forall m > n\ |w_m| > R$]

But (*) $\Rightarrow (w_n)$ has an infinite subsequence, say (w_{n_k}) in $\overline{D}(0,R)$.
As $\overline{D}(0,R)$ is closed and bounded, some infinite subsequence of (w_{n_k}) converges to some point, say $W \in \overline{D}(0,R)$ (by Bolzano-Weierstrass)

But now, by the continuity of f^{-1}, the corresponding subsequence of (z_n) converges to $f^{-1}(W) \in \mathbb{C}$.

This contradicts the fact that (z_n) tends to ∞.

Thus if (z_n) tends to ∞, then $(f(z_n))$ tends to ∞.

3. $u(x,y) = xe^x\cos y - ye^x\sin y \Rightarrow \dfrac{\partial u}{\partial x} = xe^x\cos y + e^x\cos y - ye^x\sin y$

$\Rightarrow \dfrac{\partial^2 u}{\partial x^2} = xe^x\cos y + 2e^x\cos y - ye^x\sin y$

and $\dfrac{\partial u}{\partial y} = -xe^x\sin y - ye^x\cos y - e^x\sin y \Rightarrow \dfrac{\partial^2 u}{\partial y^2} = -xe^x\cos y - 2e^x\cos y + ye^x\sin y$

So $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ ie u is harmonic.

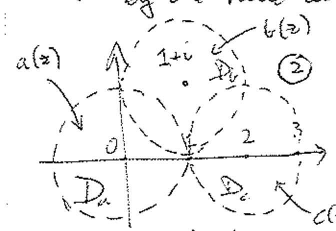
For v to be a harmonic conjugate of u we need $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = xe^x\cos y + e^x\cos y - ye^x\sin y$

So $v = xe^x\sin(y) + ye^x\cos y + k(x)$ but we also need $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$, so $k(x) = $ const.

So, taking $k(x)=0$, $v = \boxed{xe^x\sin y + ye^x\cos y}$ is a harmonic conjugate of u.

Finally, $f(z) = u(x,y) + iv(x,y) = xe^x(\cos y + i\sin y) + iye^x(\cos y + i\sin y) = \boxed{ze^z}$
is a holomorphic function with $Re(f) = u$.

4. By the ratio test: $a(z) = \sum_{n=0}^{\infty} z^n$ converges abs $\iff |z|<1$ so the disc of convergence is $D(0,1)$. (1)



(2)

$b(z) = \sum_{n=0}^{\infty} i^{n+1}(z-i-1)^n$ converges abs $\iff |z-i-1|<1$ so the disc of conv. is $D(1+i,1)$. (1)

$c(z) = \sum_{n=0}^{\infty}(-1)^{n+1}(z-2)^n$ converges abs $\iff |z-2|<1$ so the disc of conv. is $D(2,1)$. (1)

Call the three discs D_a, D_b, D_c.

In $D_a\cap D_b$ the series $a(z)$ (being a geometric series) sums to $\frac{1}{1-z}$
and the series $b(z) = i\sum_{n=0}^{\infty}(i(z-i-1))^n$ sums to:

$\dfrac{i}{1-i(z-i-1)} = \dfrac{i}{-iz+i} = \dfrac{1}{1-z}$ (2)

In $D_b\cap D_c$ the series $b(z)$ again sums to $\frac{1}{1-z}$
and the series $c(z) = (-1)\sum_{n=0}^{\infty}(-(z-2))^n$ sums to:

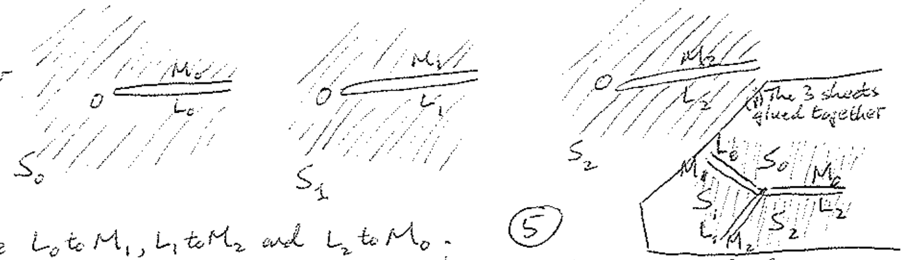
$\dfrac{-1}{1+(z-2)} = \dfrac{1}{1-z}$ (2)

Thus $a(z) = b(z)$ where their discs overlap, and $b(z) = c(z)$ where their discs overlap.

The open discs $D(0,1)$ and $D(2,1)$ do not overlap as they have radii 1 and their centres are distance 2 apart. (1)
(Note that $1/_{1-z}$ has a singularity at $z=1$.)

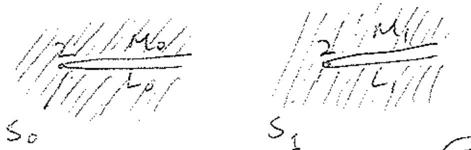
5(i) For a Riemann surface for $z^{1/3}$ we take 3 copies S_0, S_1, S_2 of the complex plane slit from $z=0$ to ∞ along the real axis:

(Here M_j & L_j denote the upper & lower edges of the cut)



Glue L_0 to M_1, L_1 to M_2 and L_2 to M_0. 5

(ii) For a Riemann surface for $(z-2)^{\frac{1}{2}}$ we take 2 copies S_0, S_1 of the complex plane slit from $z=2$ to ∞ along the real axis:



Glue L_0 to M_1, and M_0 to L_1 5

6 (i) If z lies on the unit circle, then $z\bar{z}=1$ (i.e. $x^2+y^2=1$) so $\bar{z}=\frac{1}{z}$ $\therefore f(\frac{1}{\bar{z}})=f(z)$. But we are assuming that if z lies on the unit circle then so does $f(z)$, so we also have $\overline{f(z)}=\frac{1}{f(z)}$

Thus $\dfrac{1}{\overline{f(\frac{1}{\bar{z}})}} = \dfrac{1}{\overline{f(z)}} = f(z)$.

It now follows by the Schwarz reflection principle that if we define F by:

$$F(z) = \begin{cases} f(z) & \text{if } z \in \overline{D} \\ \dfrac{1}{\overline{f(\frac{1}{\bar{z}})}} & \text{if } z \in \hat{\mathbb{C}} - D \text{ and } f(\frac{1}{\bar{z}}) \neq 0 \end{cases}$$

then F is holomorphic at all these points.

(ii) f has no zeros on the unit circle (since $|f(z)|=1$ there) so f cannot be the constant function zero on $\overline{D}$, and hence the zeros of f on $\overline{D}$ are isolated. Suppose these zeros are $z_1, \dots, z_n$ (there are finitely many as $\overline{D}$ is closed and bounded).

Now $\lim_{z \to \frac{1}{\bar{z}_j}} F(z) = \infty$ (since $\lim_{z \to z_j} f(z) = 0$)

So the points $\frac{1}{\bar{z}_j}$ are poles and hence $F: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is meromorphic.

7. Unfortunately there's not enough room on this sheet for a solution....!

For more information on this problem (the "Riemann Hypothesis") see:

H.M. Edwards "Riemann's Zeta Function", Dover 2001 (paperback version of 1974 Academic Press hardback)

or: Marcus Du Sautoy "The Music of the Primes", Harper Collins 200

How to deal with ∞ for a Möbius transformation $z \xrightarrow{\phi} \dfrac{az+b}{cz+d}$

① ∞ in the domain of ϕ

$\phi(\infty)$ is defined to be $\phi j(0)$, where $j(z)=\frac{1}{z}$.

If $\phi(z)=\dfrac{az+b}{cz+d}$ then $\phi j(z) = \dfrac{\frac{a}{z}+b}{\frac{c}{z}+d} = \dfrac{a+bz}{c+dz}$ so $\phi j(0)=\frac{a}{c}$

i.e. $\boxed{\phi(\infty)=\frac{a}{c}}$

② ∞ in the range of ϕ

$\phi(z)=\infty$ is defined to mean $j\phi(z)=0$, where $j(z)=\frac{1}{z}$.

If $\phi(z)=\dfrac{az+b}{cz+d}$ then $\phi(z)=\infty \Leftrightarrow \dfrac{cz+d}{az+b}=0 \Leftrightarrow \boxed{z=-\frac{d}{c}}$

③ $\phi(\infty)=\infty$

This means $j\phi j(0)=0$.

If $\phi(z)=\dfrac{az+b}{cz+d}$ then $j\phi j(z)=\dfrac{c+dz}{a+bz}$ so $\phi(\infty)=\infty \Leftrightarrow j\phi j(0)=0 \Leftrightarrow \boxed{c=0}$

Similar rules apply to a rational function $z \xrightarrow{f} \dfrac{p(z)}{q(z)}$

i.e. ① $f(\infty)$ is defined to be $fj(0)$

② $f(z)=\infty \Leftrightarrow q(z)=0$

③ $f(\infty)=\infty \Leftrightarrow jfj(0)=0 \Leftrightarrow \dfrac{q(1/z)}{p(1/z)}=0$ for $z=0$