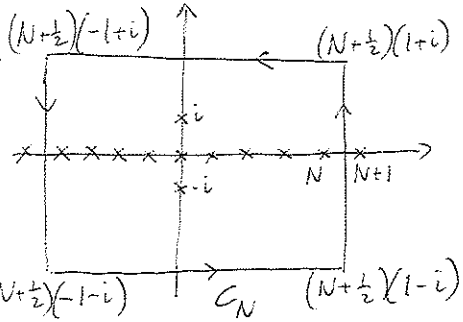
MTH6111 Complex Analysis 2010-11

Solutions to sheet 7

1. 


$f(z) = \dfrac{\cot \pi z}{z^2+1} = \dfrac{\cos \pi z}{(z^2+1)\sin \pi z}$ has singularities at $z = \pm i$ and at $z = n$ $(\forall n \in \mathbb{Z})$. (2)

To compute the residue of $f$ at $z=i$ observe that $f(z) = \dfrac{1}{(z-i)} \cdot g(z)$ where $g(z) = \dfrac{\cos \pi z}{(z+i)\sin \pi z}$. Since $g(i) \neq 0$, $z=i$ is a simple pole of $f$ and $\operatorname{Res}(f,i) = g(i) = \dfrac{1}{2i} \dfrac{\cos \pi i}{\sin \pi i} = \dfrac{1}{2i} \dfrac{e^{-\pi}+e^{\pi}}{2} \cdot \dfrac{2i}{e^{-\pi}-e^{\pi}} = \dfrac{1}{2} \dfrac{\cosh \pi}{-\sinh \pi} = \dfrac{-\coth \pi}{2}$

Similarly $\operatorname{Res}(f,-i) = \dfrac{-\coth \pi}{2}$ (2)

To compute residue of $f$ at $z=n$, observe that $f(z) = \dfrac{p(z)}{q(z)}$, where $p(z) = \dfrac{\cos \pi z}{z^2+1}$ and $q(z) = \sin \pi z$. Since $p(n) \neq 0$, $q(n) = 0$ but $q'(n) \neq 0$, $z=n$ is a simple pole of $f$ and $\operatorname{Res}(f,n) = \dfrac{p(n)}{q'(n)} = \dfrac{\cos \pi n}{n^2+1} \cdot \dfrac{1}{\pi \cos \pi n}$ (2)

By the Residue Theorem we deduce that

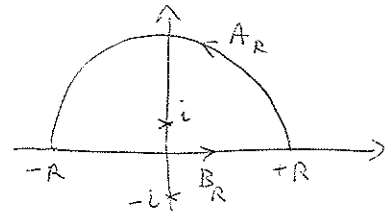
$$\int_{C_N} f(z)\,dz = 2\pi i \left( -\coth \pi + \sum_{n=-N}^{+N} \frac{1}{\pi(n^2+1)} \right) \qquad (2)$$

Assuming that $\lim_{N\to\infty} \int_{C_N} f(z)\,dz = 0$ we can deduce that

$$\boxed{\lim_{N\to\infty} \left( \sum_{n=-N}^{N} \frac{1}{n^2+1} \right) = \pi \coth \pi} \quad (*) \qquad (2)$$

[This is not quite the same as saying that $\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1} = \pi \coth \pi$ since $\sum_{n=-\infty}^{\infty}$ means $\lim_{M\to\infty} \sum_{n=-M}^{N}$. However it follows from (*) that $\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{1}{2}(\pi \coth \pi - 1)$ and from this that $\sum_{-\infty}^{\infty} \frac{1}{n^2+1} = \pi \coth \pi$.]

2. Let $f(z) = \dfrac{z^2}{(1+z^2)^2}$




The only singularity inside $A_R + B_R$ is at $z = i$. (1)

$f(z) = \dfrac{1}{(z-i)^2} g(z)$ where $g(z) = \dfrac{z^2}{(z+i)^2}$

So $z=i$ is a double pole of $f$, and $\operatorname{Res}(f,i) = g'(i)$.

But $g'(z) = \dfrac{(z+i)^2 \cdot 2z - z^2 \cdot 2(z+i)}{(z+i)^4}$ so $g'(i) = \dfrac{-8i+4i}{16} = \dfrac{-i}{4}$ (3)

On the semicircle $A_R$, $|f(z)| \leq \dfrac{R^2}{(R^2-1)^2}$ so $\left| \int_{A_R} f(z)\,dz \right| \leq \dfrac{\pi R^3}{(R^2-1)^2}$ which tends to zero as $R$ tends to $\infty$. (2)

The Residue Theorem tells us that

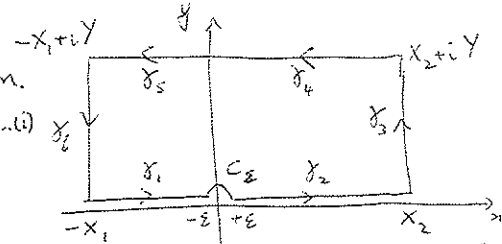
$$\int_{A_R+B_R} f(z)\,dz = 2\pi i \cdot \left(\frac{-i}{4}\right) = \frac{\pi}{2}$$

So we deduce that $\lim_{R\to\infty} \int_{-R}^{+R} \dfrac{x^2}{(1+x^2)^2}\,dx = \dfrac{\pi}{2}$ (2)

Since $\dfrac{x^2}{(1+x^2)^2}$ is an even function, it follows that $\boxed{\int_0^{\infty} \dfrac{x^2}{(1+x^2)^2}\,dx = \dfrac{\pi}{4}}$ (2)

3. Let $f(z) = \dfrac{e^{iz}}{z}$

Let $\gamma$ be the closed contour shown.




By Cauchy's Theorem $\int_{\gamma} f(z)\,dz = 0 \ldots$(i)

$-C_\varepsilon$ is the path $[0,\pi] \to \mathbb{C}$, $\theta \mapsto \varepsilon e^{i\theta}$

$f(z) = \dfrac{1}{z} + \phi(z)$ where $\phi(z) = i\left(1 + \dfrac{(iz)}{2!} + \dfrac{(iz)^2}{3} + \ldots\right)$ is holomorphic

so if we take $|z|$ small then $|\phi(z)|$ is bounded $(e.g. |\phi(z)| \leq 2)$

Hence, for $\varepsilon$ small, $\left| \int_{C_\varepsilon} \left(f(z) - \frac{1}{z}\right) dz \right| \to 0$ as $\varepsilon \to 0$.

But $\int_{C_\varepsilon} \frac{1}{z}\,dz = \int_0^{\pi} \frac{1}{\varepsilon e^{i\theta}} \cdot i\varepsilon e^{i\theta}\,d\theta = i\pi$

Thus $\lim_{\varepsilon\to 0} \int_{-C_\varepsilon} f(z)\,dz = -i\pi \ldots$ (ii)

p.t.o.

3 (continued)

For $\gamma_3$ we have $\left|\int_{\gamma_3} \frac{e^{i(X_2+iy)}}{X_2+iy}\,dz\right| \le \int_{\gamma_3}\left|\frac{e^{i(X_2+iy)}}{X_2+iy}\right|dz \le \int_{\gamma_3}\frac{e^{-y}}{X_2}\,dz$

$$= \int_0^Y \frac{e^{-y}}{X_2}\,dy < \frac{1}{X_2}$$

Similarly $\left|\int_{\gamma_6} f(z)\,dz\right| < \frac{1}{X_1}$

For $\gamma_4$ we have $\left|\int_{-\gamma_4} \frac{e^{i(x+iY)}}{x+iY}\,dz\right| \le \int_{-\gamma_4}\left|\frac{e^{ix-Y}}{x+iY}\right|dz \le \frac{e^{-Y}}{Y}\cdot X_2$

($\frac{e^{-Y}}{Y}$: max value; $X_2$: length of path)

Similarly $\left|\int_{\gamma_5} f(z)\,dz\right| < \frac{e^{-Y}}{Y}\cdot X_1$

Hence $\left|\int_{\gamma_3}+\int_{\gamma_4}+\int_{\gamma_5}+\int_{\gamma_6}\right| \le \frac{1}{X_1}+\frac{1}{X_2}+\frac{e^{-Y}}{Y}(X_1+X_2)$ ....(iii)

From (i) & (iii) we deduce that

$$\left|\int_{\gamma_1} f(z)\,dz + \int_{-C_\varepsilon} f(z)\,dz + \int_{\gamma_2} f(z)\,dz\right| \le \frac{1}{X_1}+\frac{1}{X_2}+\frac{e^{-Y}}{Y}(X_1+X_2)$$

Since the l.h.s. is indpt. of $Y$ it follows that (by letting $Y\to\infty$)

$$\left|\int_{\gamma_1}+\int_{-C_\varepsilon}+\int_{\gamma_2}\right| \le \frac{1}{X_1}+\frac{1}{X_2}$$

and now by letting $\varepsilon\to 0$ and using (ii) we can deduce that

$$\left|\int_{-X_1}^{0} f(x)\,dx + \int_0^{X_2} f(x)\,dx - i\pi\right| \le \frac{1}{X_1}+\frac{1}{X_2}$$

So, taking the imaginary part of the sum on the l.h.s., we have

$$\left|\int_{-X_1}^{0}\frac{\sin x}{x}\,dx + \int_0^{X_2}\frac{\sin x}{x}\,dx - \pi\right| \le \frac{1}{X_1}+\frac{1}{X_2}$$

It follows that $\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx$ exists and equals $\pi$

Finally, as $\frac{\sin x}{x}$ is even, it follows that $\int_0^{\infty}\frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

4. Let $\phi: z\to\frac{a_1z+b_1}{c_1z+d_1}$ $(a_1d_1-b_1c_1\neq 0)$

and $\psi: z\to\frac{a_2z+b_2}{c_2z+d_2}$ $(a_2d_2-b_2c_2\neq 0)$

Then a direct calculation shows that $\psi\phi: z\to\frac{a_3z+b_3}{c_3z+d_3}$

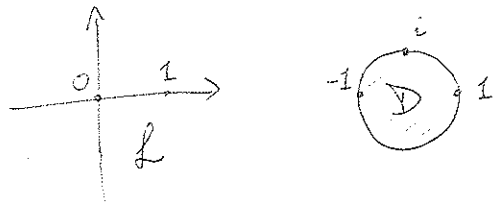
where $\begin{pmatrix}a_3 & b_3\\ c_3 & d_3\end{pmatrix} = \begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}$ (matrix product)

and $a_3d_3-b_3c_3\neq 0$ (since for matrices, $|AB|=|A||B|$)

Similarly, if we take $\begin{pmatrix}a_2 & b_2\\ c_2 & b_2\end{pmatrix} = \begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}^{-1}$ (matrix inverse)

it is easily checked that $\psi=\phi^{-1}$.

5. (i) Three point determine a Möbius transformation, so (for example) we may send




$0\to -1$, $1\to i$, $\infty\to 1$ } then $\mathbb{R}\to S^1$ and as orientation is preserved, the lower half-plane maps inside $S^1$

The transformation achieving this is $z\to\frac{az+b}{cz+d}$ where

$$\frac{b}{d}=-1 \qquad \frac{a+b}{c+d}=i \qquad \frac{a}{c}=1$$

(5)

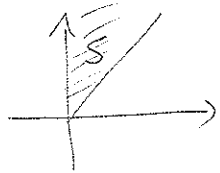
So $d=-b$, $c=a$ and $\frac{a+b}{a-b}=i$

Taking $a=1$ this yields $b+1=i(1-b)$ so $b=\frac{-1+i}{1+i}=i$

Hence $\boxed{z\to\frac{z+i}{z-i}}$ will do. (Check: This sends $-i\to 0$ which is inside $D$)

[Other answers possible.]

5 (ii)




$z\to z^4$ sends $S$ to $\ell$ (lower half-plane)

This map is bijective $S\leftrightarrow\ell$ and conformal (since the origin is not in $S$ and $z\to z^4$ has non-zero derivative except at 0)

Hence (using (i)) $\boxed{z\to\frac{z^4+i}{z^4-i}}$ sends $S$ conformally to $D$.

(5)