MTH6111 Complex Analysis **2010-11**

Exercises 9

This is the last exercise sheet for the course. Attempt all questions. Hand in your solutions to Question 1 and EITHER Question 4 OR Question 5 for coursework credit and feedback.

***1.** Let
$$\phi : z \to \frac{az+b}{cz+d}$$
be a non-identity automorphism of the (open) upper half-plane H, with a, b, c and $d \in \mathbb{R}$, and supposed *normalised* so that $ad - bc = 1$. Show that
(i) ϕ has exactly one fixed point on the boundary $\mathbb{R} \cup \{\infty\}$ of H if and only if $|a + d| = 2$ (in this case ϕ is said to be *parabolic*);
(ii) ϕ has two fixed points on $\mathbb{R} \cup \{\infty\}$ if and only if $|a + d| > 2$ (in this case ϕ is said to be *hyperbolic*);
(iii) ϕ has a fixed point in H if and only if $|a + d| < 2$ (in this case ϕ is said to be *elliptic*).
(iv) Write down the general form of a parabolic automorphism ϕ of H having fixed point ∞, and the general form of a parabolic automorphism ψ of H having fixed point 0. Show that for such ϕ and ψ the composition $\psi \circ \phi$ can be either elliptic or hyperbolic.

2. Prove that every continuous bijection $f : \mathbb{C} \to \mathbb{C}$ with continuous inverse f^{-1} can be extended to a continuous bijection $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$.
HINT: This is the step we missed out in lectures when we proved that every automorphism of $\mathbb{C}$ *has the form* $z \to az + b$. *It suffices to show that if* $(z_n)_{n\in\mathbb{N}}$ *is any sequence of points tending to* ∞ *then* $(f(z_n))_{n\in\mathbb{N}}$ *tends to* ∞. *So suppose* $(z_n)_{n\in\mathbb{N}}$ *tends to* ∞ *but* $(f(z_n))_{n\in\mathbb{N}}$ *does not. Deduce that some infinite subsequence of* $(f(z_n))_{n\in\mathbb{N}}$ *remains within a closed disc* $\overline{D(0, R)}$ *and obtain a contradiction by applying* f^{-1} *to this subsequence.*

3. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be the function $u(x, y) = xe^x \cos y - ye^x \sin y$. Show that u is harmonic, find a harmonic conjugate v for u, and find a holomorphic function f such that $u = Re(f)$.

Please turn over

***4.** Let

$$a(z) = \sum_{n=0}^{\infty} z^n; \quad b(z) = \sum_{n=0}^{\infty} i^{n+1}(z-i-1)^n; \quad c(z) = \sum_{n=0}^{\infty} (-1)^{n+1}(z-2)^n$$

Find the disc of convergence of each of these series and sketch them. By summing the series, prove that $a(z) = b(z)$ where their discs overlap, and $b(z) = c(z)$ where their discs overlap. Do the discs of $a(z)$ and $c(z)$ overlap?

***5.** Describe Riemann surfaces for the following functions:
(i) $f(z) = z^{1/3}$;
(ii) $f(z) = (z-2)^{1/2}$.

6. Let f be a function which is holomorphic from the unit disc D to itself and extends continuously on the boundary to a function which sends the unit circle to itself.
(i) Show that at all points $z \in \hat{\mathbb{C}} \setminus D$ such that $\overline{f(1/\bar{z})} \neq 0$ the function f has an analytic continuation defined by $f(z) = 1/\overline{f(1/\bar{z})}$ for $z \in \mathbb{C} \setminus D$. *(By the Schwarz reflection principle it will suffice to show that the two definitions of f agree on the unit circle.)*
(ii) Deduce that f extends to a meromorphic function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$. *(You may assume the zeros of a non-constant holomorphic function are isolated.)*

7. **Prize question** *For the value of the prize see:*
http://www.claymath.org/millennium/Riemann_Hypothesis/

Let $\zeta(s)$ be *Riemann's zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } s \in \mathbb{C}$$

that is to say, the branch of the function defined by this formula which agrees with the natural definition of the sum when s is real and > 1. This branch is holomorphic except for a simple pole at $s = 1$, and it has zeros at $s = -2, -4, -6, \ldots$.

Prove that all other zeros of the function ζ lie on the line $Re(s) = 1/2$.

Deadline for handing in solutions to question 1 and question 4 or question 5: 12 noon on Friday 25th March 2011. *(To me or to the yellow box on the second floor of the Maths building.)* SB 17/03/11