YOUNG TABLEAUX
AND THE ROBINSON-SCHENSTED-KNUTH CORRESPONDENCE

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by

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Contents

Declaration

All sentences or passages or passages quoted in this project dissertation from other people's work have been specifically acknowledged by clear cross referencing to author, work and pages. I understand that failure to do this amounts to plagiarism and will be considered grounds for failure in this module and the degree examination as a whole.

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Abstract

Young tableaux are fillings of the boxes of diagrams that correspond to partitions of positive integers, which are weakly increasing across rows and strictly increasing down columns. The aim of the project is to develop the basic combinatorics of Young tableaux, including the bumping and sliding algorithms that can be used to define the Plactic Monoid. Schensted [11] is the classic paper for this topic and it is here that an application for Young tableaux is given in finding increasing subsequences of a list of integers. Many of the results involving tableaux are proved by considering the word of a tableau and the elementary Knuth transformations which form equivalence classes. We will concentrate on the Robinson-Schensted-Knuth correspondence which allows us to associate an ordered pair of tableaux of the same shape (P,Q) to an arbitrary two-rowed array which is in fact a bijective correspondence between the set of pairs of tableaux of the same shape and the set of two-rowed arrays. The matrix-ball construction will allow us a more pictorial approach to the main correspondence and gives us a proof of the symmetry theorem which states that if a pair (P,Q) of tableaux correspond to an two-rowed array w then the pair (Q,P) corresponds to the array w tuned upside down. This is a surprising result as P and Q are treated very differently by the R-S-K correspondence. This approach is helpful in understanding certain properties which are difficult to show from the bumping description. We will conclude by showing some interesting applications of the correspondence and introducing the hook length formula for counting tableaux. The main reference for the project is Fulton's book [1] and the main structure is based on the first four chapters.

Chapter 1

Introduction

Alfred Young was born at Birchfield, Farnworth, near Wildnes, Lancashire, on 16 April 1873. He wrote his first paper in 1899 and continued to write and publish for over forty years; with the exception of his work on electromagnetism in 1918 every paper was devoted to the single theme of the algebra of groups. In 1900 he introduced the *Young tableau*, the method for which he is best remembered. He wrote a series of papers on quantitative substitutional analysis which arose out of the classical theory of invariants and contained his results in this area. In 1903 Frobenius used Young tableaux for the first time when he investigated representations of the symmetric group. In 1906 Young learnt of Frobenius's applications of his methods and in 1927 he published further work which extended what Frobenius had done. In his book *Theory of groups and quantum mechanics* Weyl also made use of Young tableaux and some of Young's ideas.

The Schensted and Schützenberger algorithms are combinatorial algorithms which deal with Young tableaux. The Schensted algorithm was originally found by Robinson [9], and rediscovered independently in a different form by Schensted [11]. The algorithm establishes a bijective correspondence between permutations and pairs of Young tableaux of the same shape. The interest that the algorithm enjoys today by combinatorialists is due to Schensted's paper of 1961 [11], where he looked at counting the longest increasing and decreasing subsequences of permutations. Only several years later was it realised that the algorithm in Robinson's paper and the algorithm in Schensted's paper were essentially the same even though they have very different definitions.

One major contribution to the subject was from Knuth [5] who replaced standard tableaux with tableaux which allowed repeats (non-standard) and made the generalization to two-rowed arrays. This defined a bijective correspondence between two-rowed arrays in lexicographic order and pairs of non-standard tableaux. Greene [3] extended Schensted's theorem concerning increasing and decreasing sequences. The paper allows us to directly find the shape of the Young tableaux corresponding to a permutation without the use of either of the two algorithms given by Schensted and Schützenberger.

A construction similar to the matrix ball construction in chapter 6 has been developed from Roby [8] by Fulton [1]. In his thesis Roby looked at the more broad area of *Differential Posets* of which partitions can be considered a part of. The symmetry theorem 5.5 was originally stated and proved by Knuth [5] although his approach relied on treating the limiting case of infinite permutations and Young tableaux. The matrix ball construction gives a more intuitive view on tableaux and allows a much easier proof of the symmetry theorem.

In the last chapter we shall look at some applications of the R-S-K correspondence which will allow us to count tableaux. The hook length formula will show us another way to count standard tableaux of shape λ with entries from $[n]$.

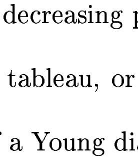
Chapter 2

Young tableaux, bumping and sliding

2.1 Young tableaux

A *Young diagram* is a collection of boxes, or cells, arranged in left-justified rows, with a (weakly) decreasing number of boxes in each row. Listing the number of boxes in each row gives a partition of the integer n that is the total number of boxes and conversely, every partition of n corresponds to a Young diagram. For example the partition of 12 into 4,3,3,2 corresponds to the Young diagram

Example 2.1 The Young diagram for (4,3,3,2)



A partition is usually denoted by λ with $\lambda \vdash n$ being used to say that λ is a partition of n. λ is given by a sequence of weakly decreasing positive integers, written λ=$(\lambda_1, \lambda_2, \ldots)$. In example 2.1 λ=(4,3,3,2). A Young tableau, or tableau is a filling, by positive integers with entries not necessarily distinct, of a Young diagram that is

- Weakly increasing across each row

- Strictly increasing down each column

We say that the tableau is a tableau on the diagram λ, or that λ is the shape of the tableau. A standard tableau is a tableau in which the entries are the numbers 1 to n each occurring once.

Example 2.2 For the partitions (4,3,3,2) of 12 and (4,3,2) of 9 we have

Tableau

1	2	2	3
3	4	6	
4	5	7	
6	6		

Standard tableau

1	4	6	7
2	5	8	
3	9		

2.2 Schensted's bumping algorithm

The Schensted "bumping" algorithm is the first of the two algorithms that we will look at and was first proposed by Schensted [11]. He defined a way that a positive integer can act on a tableau to create a new tableau with one extra entry than before. Suppose we have a tableau T and a positive integer x, then we define the action T $\leftarrow x$ by these three simple steps;

(1) Look at the first row of T and find the smallest number that is larger than x, replace this number with x. If the smallest number larger than x occurs more than once in the row then choose the one furthest to the left. If no such number is larger than x then simply place x at the end of the first row.

(2) If an integer say y was replaced by x in the first row then bump y into the second row using the same method as above. If there is no row to add y to, then y has been bumped out of the bottom, in which case it forms a new row with one entry.

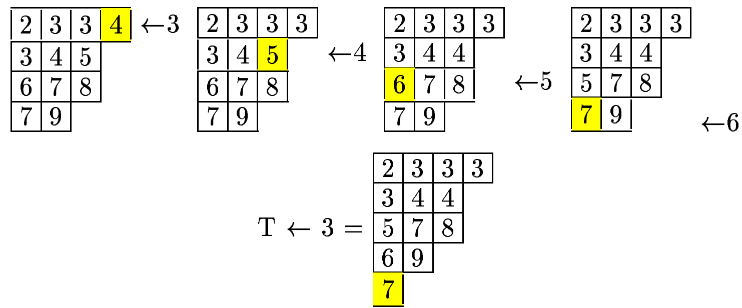
(3) Repeat the process on each row of the tableau until either some number gets added to the end of a row or until it is bumped out of the bottom.

Example 2.3 If

T =

2	3	3	4
3	4	5	
6	7	8	
7	9		

and x=3, then



We want to show that this action is well defined and to do this all we have to do is ensure that we get a Young tableau out at the end of the process. First we will need to show that the shape of T ← x is a Young diagram and then show that the numbers inside are in the right order for a tableau.

Theorem 2.4 *If T is a tableau then T ← x is a also a tableau.*

Proof It is really quite simple to prove that the result is a Young diagram as we only have to consider the situation in which two rows in T are the same length. If two rows of the tableau are not the same length then the upper most row is longer than the lower row. When the box is added it will go at the end of either row which will still leave us with a weakly decreasing number of boxes in each row, which is exactly what we want for a Young diagram. Consider the two rows of identical length. Suppose a number is displaced from the first row, then there are two things that can happen to it, it can displace the number under it or some number to its left. It can't go to the right due to the weakly increasing rows and strictly increasing columns.

a	b	c	d	e
f	g	h	i	j

Above we have a part of a tableau, so we have a≤b≤c≤d≤e and that f≤g≤h≤i≤j. Also due to the strictly increasing columns we have that a<f, b<g, c<h, d<i, and e<j. Suppose x was bumped into the space c, then c will be bumped down to the next row. It would be impossible for c to go in the space i as h is greater than c and less than or equal to i. So c must bump h or some number to the left of h.

Now we must show that T $\leftarrow$ x has weakly increasing rows and strictly increasing columns. When a number x is inserted into a row the number to the left of it is less than x and the number to the right is greater than or equal to x, thus we have a weakly increasing row as nothing has happened to the rest of the row. Consider the two rows and the insertion of x.

$$\begin{array}{|c|c|c|c|c|}\hline a & b & c & d & e \\ \hline f & g & h & i & j \\ \hline \end{array} \leftarrow x \qquad \begin{array}{|c|c|c|c|c|}\hline a & b & x & d & e \\ \hline f & g & h & i & j \\ \hline \end{array} \leftarrow \mathrm{c}$$

There are two possibilities for the position h. The first one is that it will be displaced by c, in which case x is strictly less than c so we are ok. If h is not displaced by c the c will go to the left of it in some place. Here x is strictly less than h as c is strictly less than h and $x\leq$ c. So when a number, say c, is displaced from one row to the next it ends up either in the same column in the row below, or in column to the left in the row below. So it is either under the number which displaced it, so strictly bigger (as c>h) or otherwise under a number to the left, of which c is even larger than. So we have that two consecutive numbers in a column form an increasing sequence if either one of them has just been inserted into its present position. Of course if no bumping has occurred on the two numbers then they are simply as they were in the original tableau T, so are therefore in strictly increasing order. We thus have the shape of a Young diagram, weakly increasing rows and strictly increasing columns. Bumping is a well defined operation that takes a tableau T and an integer x and gives us another tableau T $\leftarrow$ x. ∎

A useful property of this bumping operation is that it is invertible. If we have a tableau $T \leftarrow x$ and we know which box was added to the diagram then we can find the original tableau T. We quite simply run the bumping operation backwards. Suppose y is the box that was added to the diagram, then we can look for its position in the row above like this. Look for the entry in the row above y that is farthest to the right and strictly less than y. Now put y in this box and bump the entry say z up to the next row above and continue until an entry is bumped out of the top row. This entry is x and the resulting tableau is T.

2.3 Monoid Structure

We have looked at the bumping operation and shown how it works with relation to a tableau T and integer x. We have proved that it is well defined and that the operation may be reversed by running it backwards. Now we are going to look at how this operation may be used to form an associative monoid on the set of tableaux. First though we will look at what happens when we have two successive bumpings on a tableau T. This will tell us something about where new boxes get added when we have these two successive bumpings. Looking at all these bumping routes may seem a little pointless but when we look at the main part of the proof for the Robinson-Schensted-Knuth correspondence we see that they play an important role.

When an integer x is bumped into a tableau T, then in T, a number of the integers get bumped down to the next row. We will call R the set of elements that get bumped from a row and B the box that the last element lands. Quite logically we will call R the *bumping route* of the row insertion, and call B, the box added to the diagram the *new box*. In example 2.3 the bumping route consists of the shaded boxes and the new box contains the number 7.

2	3	3	4
3	4	4	
5	7	8	
6	9		
7			

It is clear to see that the bumping route has exactly one box in each of the rows going down until we reach the new box. What we are going to do is compare the routes that two bumpings take and show that we get some interesting results for two integers x and y that are bumped into the tableau. First though we will need to set up some terminology.

Definition 2.5 *We say that a route R is strictly left of a route R' if all the boxes of R are to the left of R' for the rows in which R' has a box. We will also describe weakly left similarly.*

Lemma 2.6 *Two successive bumpings of integers x and y on a tableau T will give us a tableau $((T \leftarrow x) \leftarrow y)$ and we will have two bumping routes R and R' and two new boxes B and B' with;*

- *If $x \leq y$, then R is strictly left of R', and B is strictly left of and weakly below B'.*
- *If $x > y$, then R' is weakly left of R, and B' is weakly left of and strictly below B.*

Proof To show this we need to apply a similar technique to earlier. If $x \leq y$ then let us suppose that x bumps a number say u from the first row. This will end up in the row beneath either underneath x or to the left of it. Remember here that as u has been bumped by x then it was the smallest number in the first row that was larger than x, so $u > x$. Suppose v is bumped out of the first row by y, so $v > y$ and we have $v > y \geq x$. The element v bumped by y must lie strictly to the right of the box where x bumped u to, since the elements in that box or to the left are no larger than x, which v certainly is. One conclusion that we can, make is that $u \leq v$ and this argument is continued for each row. We see here that the route for R cannot stop above that of R', and if R' stops first, the route for R never moves to the right, so the box B must be strictly left of and weakly below B'.

Alternatively, if $x > y$, and u and v are bumped respectively, then v must end up to the left of u or bumping v. In both of these cases we end up with $u > v$, and so the argument is again continued for each row. When $x > y$ the route R' must continue at least one row below that of R. ∎

We can define another type of diagram which is obtained by considering two Young tableaux.

Definition 2.7 *A skew diagram is the diagram that we get by removing a smaller Young diagram from a larger one that contains it. If the two diagrams correspond to partitions* $\lambda=(\lambda_1, \lambda_2, \ldots)$ *and* $\mu=(\mu_1, \mu_2, \ldots)$*, we will write* $\mu \subset \lambda$*. The resulting shape is denoted by* λ/μ*.*

A *skew tableau* is just like before with Young tableaux. Fill the boxes of the skew diagram with positive integers which are weakly increasing in rows and strictly increasing in columns.

Theorem 2.8 *Let T be a tableau of shape* λ*, and let T be operated on by some* $x_1,\ldots,x_i$*.*

$$U = ((\ldots(T \leftarrow x_1) \leftarrow x_2) \leftarrow \ldots \leftarrow x_i)$$

with U having shape μ*.*

• *If* $x_1 \leq x_2 \leq \ldots \leq x_i$ *then no two boxes in* μ/λ *are in the same column. If* $x_1 > x_2 > \ldots > x_i$*, then no two boxes of* μ/λ *are in the same row. Also, let's say that we begin with U and there is a Young diagram, shape* λ*, contained in* μ*, with i boxes in* μ/λ*.*

• *If we get that no two boxes in* μ/λ *are in the same column , then there is a unique tableau T of shape* λ*, and unique* $x_1 \leq x_2 \leq \ldots \leq x_i$ *such that* $U = ((\ldots(T \leftarrow x_1) \leftarrow x_2) \leftarrow \ldots \leftarrow x_i$*. Similarly if no two boxes in* μ/λ *are in the same row, then there is a unique tableau T of shape* λ*, and a unique* $x_1 > x_2 > \ldots > x_i$ *such that* $U = ((\ldots(T \leftarrow x_1) \leftarrow x_2) \leftarrow \ldots \leftarrow x_i$*.*

Proof The first point made in theorem 2.8 comes from looking at two successive bumpings in a tableau looked at in Lemma 2.6. For proof of the second point we need to look at reverse bumping. If μ/λ has no two boxes in the same column then start the reverse bumping on U using the boxes in μ/λ. Begin from the right most box and work to the left.

Example 2.9 Consider the tableaux given below.

U =

1	2	3	4
5	6	7	
8	9	10	
11			

S =

1	2	3
5	6	7
8		

R =

			4
	9	10	
11			

R is the skew tableau formed from S and U.

It is quite easy to see how the reverse bumping gives us a tableau T and a set of integers which are bumped out. Lemma 2.6 on row bumping guarantees that the resulting sequence satisfies $x_1 \leq \ldots \leq x_i$. If the skew tableau μ/λ has no two boxes in the same row, then do i reverse bumpings, starting with the box which is furthest to the left. The Lemma again tells us that what comes out of the top satisfies $x_1 > \ldots > x_i$ ∎

We have already seen that we can successively bump two elements into a tableau, and we have reverse bumped a whole string of elements but what happens if we take this further by bumping in even more elements? We can form a *product tableau* T•U given any two tableaux T and U. All that we do is successively bump in each element of the tableau starting from the bottom and working left to right up to the top. So;

$$T\bullet U = ((\ldots(T \leftarrow x_1) \leftarrow x_2) \leftarrow \ldots \leftarrow x_i)$$

So if we have T and U and we want the product of the two then we just treat U as a string of integers that need to be bumped in to T.

Example 2.10 For example;

T =

1	2	3	5
2	4	6	
3	5	7	
8	8		
9			

and U=

3	4	6
4		

then T•U= (((T←4)←3)←4)←6

T←4 =

1	2	3	4
2	4	5	
3	5	6	
7	8		
8			
9			

(T ←4)←3 =

1	2	3	3
2	4	4	
3	5	5	
6	8		
7			
8			
9			

((T←4)←3)←4 =

1	2	3	3	4
2	4	4		
3	5	5		
6	8			
7				
8				
9				

$$(((T \leftarrow 4) \leftarrow 3) \leftarrow 4) \leftarrow 6 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 4 & 6 \\ \hline 2 & 4 & 4 \\ \cline{1-3} 3 & 5 & 5 \\ \cline{1-3} 6 & 8 \\ \cline{1-2} 7 \\ \cline{1-1} 8 \\ \cline{1-1} 9 \\ \cline{1-1} \end{array}$$

The product T•U can be shown to be associative and so the product operation makes the set of tableaux into an associative monoid, with the empty tableau being the unit. The proof of this associativity will be given when we look at words in the Plactic Monoid in the next chapter.

2.4 Shützenberger's sliding algorithm

The jeu de taquin (or "teasing game") of Shützenberger is another way of constructing a product on the set of tableaux. We can define an inside or outside corner as follows.

Definition 2.11 *If we have a skew tableau on μ/λ then an inside corner is a box in μ such that the boxes below and to the right are not in μ. An outside corner is a box in λ such that neither box below or to the right is in λ.*

Example 2.12 Consider the skew tableau given below;

$$S = \begin{array}{|c|c|c|c|c|c|} \hline & & & & 5 & 6 \\ \hline & & & 4 \\ \cline{1-4} & & & 5 \\ \cline{1-4} & & 7 & 8 \\ \cline{1-4} 9 & 10 & 11 \\ \cline{1-3} 12 & 13 \\ \cline{1-2} \end{array}$$

In example 2.12 the inside corners are the fourth box in the first row, the third box in the third row and the second box in the fourth row. The outside corners are the last boxes in the first, fourth, fifth and sixth rows. To perform the operation we take a skew tableau S and an inside corner which can be thought of as a hole or an empty box.

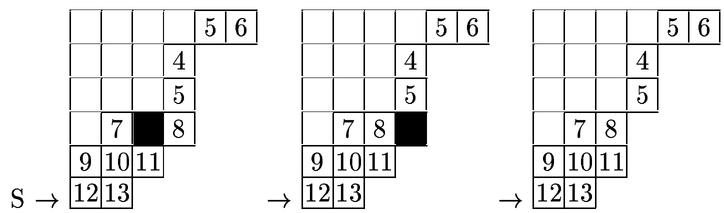
- Slide the smaller of its two neighbours, to the right or below, into the empty box.

• If only one of these two neighbours is in the skew diagram then we choose that one.

• In the situation that the two neighbours have the same value then we choose the one below. This is very similar to bumping where we considered elements to the left as smaller than those elements to the right if two elements were the same.

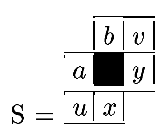
• We repeat the process with the new inside corner (hole). If we get to a situation in which there are no boxes to the right or below, then we remove the box from the diagram.

If we consider the skew diagram given in example 2.12 then carry out the sliding for the second box in the fourth row.

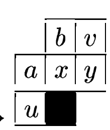


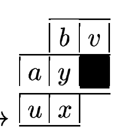
Theorem 2.13 *The Schützenberger sliding algorithm is well defined.*

Proof Since the box which is added is an inside corner, and that which is removed is an outside corner, the shape that results is certainly a skew diagram. To show that the result of an operation is a skew tableau we need to show that the rows remain weakly increasing and that the columns remain strictly increasing. We only need worry about what happens in a select few boxes given below, where some of the labelled boxes might not exist in a particular skew tableau.



There are two possibilities here either $x \leq y$ or $x > y$

If $x \leq y$ then S→ 

Whereas if $x > y$ then S→ 

Let us consider the first scenario to begin with. The only thing that we need to ensure here is that $a \leq x \leq y$. As we have a tableau then $a < u \leq x$ and we have that $x \leq y$, so $a < u \leq x \leq y$ and therefore $a \leq x \leq y$ as required.

In the second case we have to prove that $b < y < x$. We have that $y < x$ and that $b \leq v < y$, so $b \leq v < y < x$ and therefore $b < y < x$. So the sliding rule is well defined and takes a skew tableau to a skew tableau and maintains the tableau conditions at each step. ∎

The sliding operation is very similar to the bumping operation in that they are both reversible. Just like before we run the operation backwards, if we have a skew tableau and we know the box that was removed then we can arrive at the starting tableau with the chosen inner corner. The empty box or hole will move up or to the left, changing places with the larger of the two entries. Opposite to before, if the two entries are the same then choose the one above. When the empty box become an inside corner then we stop.

If we carry on choosing inside corners and sliding them out through the operation then eventually we will end up with a tableau with no inside corners. This is called the *rectification* of the skew tableau and the whole process is called the jeu de taquin. The nice thing about the process is that no matter what order we choose the inner corners in then we still get the same rectified tableau at the end. The proof of this is something that we will be looking at later on, as we will need some more apparatus first. If we have a skew tableau, S, then we can denote the rectification of S by *Rect(S)*. The really interesting thing is that the Schensted bumping and the Schützenberger sliding operations are very closely related.

Given two tableaux T and U we can form a skew tableau U $*$ T by taking the tableau U and placing it beneath the T and creating some empty squares like below, to form a skew tableau U $*$ T.

Example 2.14 Consider the tableaux T and U;

Let $T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \cline{1-3} 3 & 5 & 7 \\ \cline{1-3} 8 & 8 \\ \cline{1-2} 9 \\ \cline{1-1}\end{array}$ and $U = \begin{array}{|c|c|c|}\hline 3 & 4 & 6 \\ \hline 6 \\ \cline{1-1}\end{array}$

then $T * U = \begin{array}{|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & 3 & 4 & 6 \\ \hline \ & \ & \ & \ & 4 \\ \cline{1-5} 1 & 2 & 3 & 5 \\ \cline{1-4} 2 & 4 & 6 \\ \cline{1-3} 3 & 5 & 7 \\ \cline{1-3} 8 & 8 \\ \cline{1-2} 9 \\ \cline{1-1}\end{array}$ and $U * T = \begin{array}{|c|c|c|c|c|c|c|}\hline \ & \ & \ & 1 & 2 & 3 & 5 \\ \hline \ & \ & \ & 2 & 4 & 6 \\ \cline{1-6} \ & \ & \ & 3 & 5 & 7 \\ \cline{1-6} \ & \ & \ & 8 & 8 \\ \cline{1-5} \ & \ & \ & 9 \\ \cline{1-4} 3 & 4 & 6 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}$

$\text{Rect } (T * U) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 3 & 4 & 6 \\ \hline 2 & 4 & 4 \\ \cline{1-3} 3 & 5 & 5 \\ \cline{1-3} 6 & 8 \\ \cline{1-2} 7 \\ \cline{1-1} 8 \\ \cline{1-1} 9 \\ \cline{1-1}\end{array}$

If we form the rectification of T $*$ U, then we get a tableau. The surprising result is that Rect(T $*$ U) = T $\bullet$ U which is the operation that we looked at with Schensted's bumping algorithm. For example 2.14 we can see this as T and U are the same matrices used in example 2.10 earlier in the chapter. The proof that Rect(T $*$ U) = T $\bullet$ U will be looked at later on in the next few chapters when we consider the word of a tableau.

Chapter 3

The Plactic Monoid

In this chapter we are going to study the *word* of a tableau. This is a way of encoding a tableau with a series of integers which is less visual than the tableau itself but is very important to the proofs of many of the properties of the tableaux. The Schensted operations were originally invented to study the sequences of integers, but here we will look at it the other way round. We will see what the bumping and sliding does to the associated words.

3.1 Words

We write a word as a sequence of integers and write $w{\cdot}w'$ or ww' for the word which is the juxtaposition of the two words w and w'. Given a tableau or skew tableau T we define the word of T, denoted by W(T) by reading the entries of T from left to right and bottom to top.

- Start with the bottom row
- Write down it entries from left to right
- Go to the next row up and repeat
- Continue until you get the last entry in the top row.

In fact we have already looked at the word of a tableau when we considered the product T•U. The whole process is clearly invertible and it is very easy to obtain the original tableau. All we do is break the word whenever one entry is strictly greater than the next. These pieces are the rows of T, read from bottom to top.

Example 3.1 The word 6 8 4 6 6 2 3 3 5 1 1 1 2 4 7 breaks into 6 8|4 6 6|2 3 3 5|1 1 1 2 4 7 which is the tableau

1	1	1	2	4	7
2	3	3	5		
4	6	6			
6	8				

It is worth noting that not every word comes from a tableau. 7 8|4 5 6 7 8|2 4 5 5|1 2 3 4 is a word but its pieces do not have weakly increasing length. Also we need strictly increasing columns in Young tableaux which is not guaranteed in words. Skew tableau though are slightly more interesting in that many different skew tableaux may determine the same word. Unlike Young tableaux every word arises from some skew tableau. We break the word into increasing pieces and put the pieces in rows, each row placed above and entirely to the right of the row below.

Example 3.2 For the word 7 8 4 5 6 7 8 2 4 5 5 1 2 3 4 we would have the skew tableau given by;

											1	2	3	4
							2	4	5	5				
		4	5	6	7	8								
7	8													

Note that we do not always put in the empty boxes in the top left as these can always be filled in if we require.

3.2 Bumping on words

What does the Schensted bumping algorithm do to the word of a tableau? Suppose an element x is row-inserted (bumped) into a row. The easiest way to show what happens is to break up the words into its pieces first, 6 8 4 6 6 2 3 3 5 1 1 1 2 4 7 can be written as (68)(466)(2335)(111247)

How might we row insert 3?

$$(68)(466)(2335)(111247)\cdot 3 \longmapsto (68)(466)(2335)\cdot 4(111237) \longmapsto (68)(466)\cdot 5(2334)(111237)$$

$$\longmapsto (68)\cdot 6(456)(2334)(111237) \longmapsto (8)(66)(456)(2334)(111237)$$

More generally suppose v and w are words and x' is an integer that is to be bumped out by x. We have that x' is strictly larger than x and so the word $u{\cdot}x'{\cdot}w$ becomes $u{\cdot}x{\cdot}w$ and x' is bumped to the next row. The resulting tableau has word $x'{\cdot}u{\cdot}x{\cdot}w$. So the basic algorithm is therefore;

$$(u{\cdot}x'{\cdot}w){\cdot}x \longmapsto x'{\cdot}u{\cdot}x{\cdot}w \text{ if } u{\leq}x{<}x'{\leq}w$$

Here u and w are weakly increasing, with $u{\leq}w$ meaning that every letter in u is smaller than or equal to every letter in w. We may look at this bumping procedure at an even closer level when considering tableaux. When we row-insert an element x in a tableau T, we start by trying to put x at the end of the first row, testing x against the last entry of the row to see if that entry is larger. If it is not, we put x at the end. If the last entry z of the row is larger, and the entry y before it also is larger than x, we move x one step to the left and repeat the process. Breaking this down we see;

$$\begin{aligned} u{\cdot}x'{\cdot}w{\cdot}x = ux'w_1\ldots w_{q-1}w_q x &\longmapsto ux'w_1\ldots w_{q-1}xw_q \text{ for } x < w_{q-1} \leq w_q \\ &\longmapsto ux'w_1\ldots w_{q-2}xw_{q-1}w_q \text{ for } x < w_{q-2} \leq w_{q-1} \\ &\vdots \\ &\longmapsto ux'w_1xw_2\ldots w_{q-1}w_q \text{ for } x < w_1 \leq w_2 \\ &\longmapsto ux'xw_1\ldots w_{q-1}w_q \text{ for } x < x' \leq w_1 \end{aligned}$$

If we look at each level of the transformation, then the result is that, each stage depends on only three consecutive letters. If we look at the three letters then the first remains where it is and the last two interchange. The basic transformation is;

$$\text{K: } yzx \longmapsto yxz \text{ if } x < y \leq z$$

We call this the elementary Knuth transformation K. This transformation was looked at by Knuth [5]. There is also another elementary Knuth transformation which we shall denote by K'. Suppose that x bumps x' and the x' moves successively to the left.

$$\begin{aligned} u{\cdot}x'{\cdot}x{\cdot}w = u_1 \dots u_{p-1}u_p x x' w &\longmapsto u_1 \dots u_{p-1} x' u_p x w \text{ for } u_p \leq x < x' \\ &\longmapsto u_1 \dots x' u_{p-1} u_p x w \text{ for } u_{p-1} \leq u_p < x' \\ &\quad \vdots \\ &\longmapsto u_1 x' u_2 u_3 \dots u_p x w \text{ for } u_2 \leq u_3 < x' \\ &\longmapsto x' u_1 u_2 \dots u_p x w \text{ for } u_1 \leq u_2 < x' \end{aligned}$$

And so we denote the second elementary Knuth transformation, K', by

$$\text{K'}: xzy \longmapsto zxy \text{ if } x \leq y < z$$

If we look at row bumping in a tableau then we can illustrate what these Knuth transformations really mean.

$$\begin{array}{|c|c|}\hline y & z \\ \hline \end{array} \bullet \begin{array}{|c|}\hline x \\ \hline \end{array} = \begin{array}{|c|c|}\hline x & z \\ \hline y \\ \cline{1-1} \end{array} \quad \text{and so } yzx \longmapsto yxz \text{ if } x < y \leq z$$

$$\begin{array}{|c|c|}\hline x & z \\ \hline \end{array} \bullet \begin{array}{|c|}\hline y \\ \hline \end{array} = \begin{array}{|c|c|}\hline x & y \\ \hline z \\ \cline{1-1} \end{array} \quad \text{and so } xzy \longmapsto zxy \text{ if } x \leq y < z$$

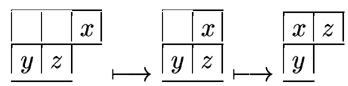
Both of these elementary Knuth transformations, K and K', can be thought of as having inverses. This is easily done by running the map backwards. So we can interchange the two neighbours on one side of a letter y so long as they meet the either of the conditions in K or K'. Suppose that we have two words w and w'. We call the two words Knuth equivalent if one can be changed into the other by a sequence of elementary Knuth transformations. We write $w \equiv w'$ to denote that the two words w and w' are Knuth equivalent. What we have shown here is that for any tableau T and positive integer x;

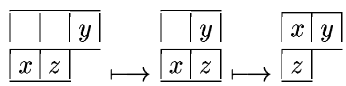
Theorem 3.3 *$W(T) \leftarrow x \equiv W(T){\cdot}x$.*

Proposition 3.4 *In chapter 2 we defined that the way to construct T•U was by successively row inserting the letters of U into T. So we have $W(T \bullet U) \equiv W(T){\cdot}W(U)$*

3.3 Sliding on words

We have looked at the Schensted bumping and how this affects Knuth equivalence but what about the Schützenberger sliding. Luckily this procedure also preserves Knuth equivalence. This can be seen by looking at some simple cases;


and so $yzx \longmapsto yxz$ if $x < y \leq z$ (K)


and so $xzy \longmapsto zxy$ if $x \leq y < z$ (K')

What we are going to show is that the Knuth equivalence class of a word is unchanged by each step of the Schützenberger sliding procedure. As before the word of a tableau is defined as reading the entries that occur from left to right and bottom to top. First we will consider a horizontal slide and the consequences that this has on the word of a tableau. In fact the word here remains the same as we read from left to right.

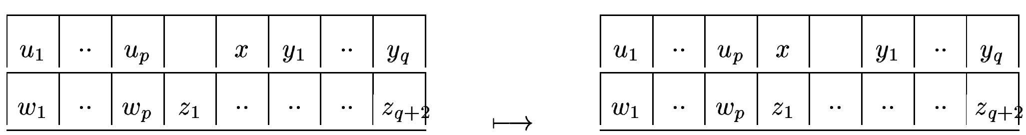
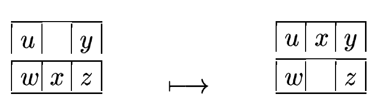


Figure 3.1: The effect of a horizontal slide on a tableau

So we have shown that for a horizontal slide the Knuth equivalence class of a word is left unchanged. Consider the vertical slide given below;



Due to the tableau's properties, and the fact that we are moving vertically, we have that $u < w \leq x \leq y < z$. During the slide we get the word of the tableau changing from $wxzuy$ to $wzuxy$. What we want to show is that these two words are Knuth equivalent and therefore that the sliding procedure preserves the Knuth equivalence of the words of a skew tableau. Using K and K' we show $wxzuy \equiv wzuxy$. Note here that with K we swap the two letters to the right and with K' we swap the two letters to the left.

$wxzuy \equiv wxuzy$ as $u < y < z$ (K)

$\equiv wuxzy$ as $u < w \leq x$ (K')

$\equiv wuzxy$ as $x \leq y < z$ (K)

$\equiv wzuxy$ as $u < x < z$ (K')

For a formal proof we need to look at a general tableau. Consider the tableau below which is similar to the one we looked at earlier but now we will perform a vertical slide.

$$\begin{array}{|c|c|c|c|c|c|c|} \hline u_1 & .. & u_p & & y_1 & .. & y_q \\ \hline w_1 & .. & w_p & x & z_1 & .. & z_q \\ \hline \end{array} \quad \longmapsto \quad \begin{array}{|c|c|c|c|c|c|c|} \hline u_1 & .. & u_p & x & y_1 & .. & y_q \\ \hline w_1 & .. & w_p & & z_1 & .. & z_q \\ \hline \end{array}$$

Figure 3.2: The effect of a vertical slide on a tableau

As we would expect, this tableau has all the usual properties that we would associate with a tableau and so we have weakly increasing rows and strictly increasing columns. Also as we are sliding the x upwards then we can also say that $x \leq y_1$. So what we want to show is that the two words $wxzuy$ and $wzuxy$ are Knuth equivalent. If u and w are both the identity word (i.e. there is nothing there) then we have to show;

$xz_1 \dots z_q y_1 \dots y_q \equiv z_1 \dots z_q x y_1 \dots y_q$

What we have here is y_1 being inserted into a row with entries $x, z_1, \dots, z_q$. We have to ask ourselves which of these is the smallest number which is larger than y_1. Thankfully we know this already as we can use the properties of a tableau. We have that $x \leq y_1 < z_1 \leq z_2 \dots \leq z_q$ and so z_1 is bumped. We saw earlier that $\mathrm{W}(\mathrm{T} \leftarrow x) \equiv \mathrm{W}(\mathrm{T}) \cdot x$ and so we can say that $xz_1 \dots z_q y_1 \equiv z_1 x y_1 z_2 \dots z_q$ giving us that;

$(xz_1 \dots z_q y_1)(y_2 \dots y_q) \equiv (z_1 x y_1 z_2 \dots z_q)(y_2 \dots y_q)$

If we continue to bump y_2 into the row with entries $x, y_1, z_2, \ldots, z_q$ then for the same reason as before we end up bumping z_2 into the next row. Now we can say that $xy_1z_2 \ldots z_qy_2 \equiv z_2xy_1y_2z_3 \ldots z_q$ giving us that;

$(z_1xy_1z_2 \ldots z_qy_2)(y_3 \ldots y_q) \equiv (z_1z_2xy_1y_2z_3 \ldots z_q)(y_3 \ldots y_q)$

If we continue to bump in each of the y_i's then what we eventually get is that;

$xz_1 \ldots z_qy_1 \ldots y_q \equiv z_1 \ldots z_qxy_1 \ldots y_q$ and so $xzy \equiv zxy$

In the case when u and w are identity words we have therefore shown that $wxzuy \equiv wzuxy$. So really what we have been looking at is a base case where $p = 0$. If we now extend this to a situation where $p \geq 1$ and assume that $wxzuy \equiv wzuxy$ holds for smaller p then we can proceed to a proof by induction. Let;

$$w' = w_2 \ldots w_p, \qquad u' = u_2 \ldots u_p$$

$$\text{And so} \qquad wxzuy = w_1w'xzu_1u'y$$

Row inserting u_1 into the row with the word $w_1w'xz$ we get w_1 being bumped as this is the smallest letter that is larger than u_1. We therefore get $w_1w'xzu_1 \equiv w_1u_1w'xz$. And thus;

$wxzuy = w_1w'xzu_1u'y \equiv w_1u_1w'xzu'y.$

We have already assumed that for $p-1$ $w'xzu'y \equiv w'zu'xy$ so we have;

$wxzuy = w_1u_1w'xzu'y \equiv w_1u_1w'zu'xy$ (a)

The clever part here is to approach the problem from the other side. We want to show that $wxzuy \equiv wzuxy$ and what we have is that $wxzuy \equiv w_1u_1w'zu'xy$. We can write that $wzuxy = w_1w'zu_1u'xy$. If u_1 is row inserted into the word $w_1w'z$ then we get that w_1 is bumped, thus $w_1w'zu_1 \equiv w_1u_1w'z$. So we have;

$wzuxy = w_1w'zu_1u'xy \equiv w_1u_1w'zu'xy$ (b)

Bringing parts (a) and (b) together we get $wzuxy \equiv wxzuy$ as required. The case for any vertical slide follows quite easily from the situation that we just looked at by consulting figure 3.3.

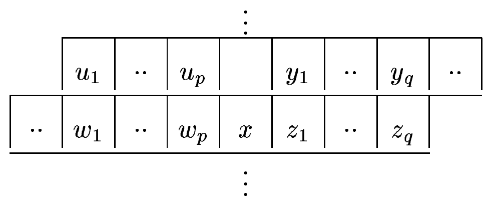


Figure 3.3: Part of a skew tableau

Figure 3.3 shows the important part of a generic skew tableau, which may be extended upwards, downwards, to the left or to the right. The only parts which are affected by a slide have been shown to be Knuth equivalent earlier.

3.4 Consequences of Knuth equivalence

Proposition 3.5 *If one skew tableau can be obtained from another by a sequence of Schützenberger slides then their words are Knuth equivalent.*

Theorem 3.6 *Every word is Knuth equivalent to the word of a unique tableau.*

If we consider Theorem 3.3 then we see that every word is Knuth equivalent to at least one tableau, we just construct the obvious tableau by bumping in the letters of the word in successive order. If $w = w_1 w_2 \dots w_p$ then construct the tableau:

$$((\dots((w_1 \leftarrow w_2) \leftarrow w_3) \leftarrow \dots) \leftarrow w_{p-1}) \leftarrow w_p$$

There is a special name for this procedure and the resulting tableau. The procedure is called the *canonical* procedure and we can denote the tableau by P(w). The proof of theorem 3.6 will be in the next chapter. To show that this P(w) is unique will take us until the end of the next chapter but if we assume that the theorem is true for the moment then we can make some interesting observations.

Theorem 3.7 *The rectification of a skew tableau S is the unique tableau whose word is Knuth equivalent to the word of S. If S and S' are skew tableaux, then Rect(S) = Rect(S') if and only if W(S) $\equiv$ W(S').*

The proof here follows directly from theorem 3.3 and proposition 3.6.

In chapter 2 we showed two different ways to form a product of two tableaux T and U. Related closely to these two is a third way, the word of a tableau. As before we define the product of two words by juxtaposition.

Theorem 3.8 *The three constructions of the product of two tableaux agree*

To prove this all we need to show is that W(T•U) = W(Rect(T∗U)) ≡ W(T)·W(U). For the bumping construction, T•U, we showed this in proposition 3.4. When the operation is sliding, with the rectification, we need to show that W(Rect(T∗U))≡ W(T)·W(U) and this follows directly from proposition 3.5. ∎

Once we have proved the uniqueness part of theorem 3.6 then we can say that we have proved the three claims made in chapter 2. We can now say that the product makes the set of tableaux into an associative monoid with the empty tableau being the unit in this monoid. We can also say that if we start with a given tableau then all choices of inside corners lead to the same rectified tableau. Also we have shown above that the two definitions of a product on a tableau agree, i.e. T•U = Rect(T∗U)

3.5 The Plactic Monoid

For the last part of this chapter we are going to take a look at what we mean by the Plactic monoid. A monoid is a set M together with a binary operation $\cdot$ on it that satisfies the following two conditions:

- the operation is associative: given any three elements a, b and c in M,

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

- there is an element ϕ in M (the identity element) such that for any element a in M,

$$a \cdot \phi = \phi \cdot a = a$$

Let $M = M_m$ be the set of Knuth equivalence classes of words on the alphabet $[m] = \{1, \ldots, m\}$. The product on this set is determined by the juxtaposition of words, as if $w \equiv w'$ and $v \equiv v'$ then we have $w \cdot v \equiv w' \cdot v \equiv w' \cdot v'$. This makes M into an associative monoid with the identity represented by the empty word. More formally, the words form a free monoid F with the operation and identity the same as in M. The map from F to M that takes a word to its equivalence class is a homomorphism of monoids; $M = F/R$, where R is the equivalence relation generated by the Knuth relations K and K'. M was called the Plactic monoid by Lascoux and Schützenberger [6]. What we have done amounts to saying that the monoid of tableaux is isomorphic to the Plactic monoid M.

Chapter 4

Increasing Sequences and the Proof of Uniqueness

4.1 Weakly Increasing Sequences

The main aim of this chapter will be to prove the uniqueness part of theorem 3.6. We will do this by considering Schensted [11] and Greene [3]. The original purpose of Schensted's work was to look at finite sequences of integers. He concentrated on the problem of the determination of the number of sequences of length n, consisting of the integers 1,2,...,m, which have longest weakly increasing sequences of length α. If $w = x_1 x_2 \ldots x_q$ then a weakly increasing sequence is a subsequence that we can extract from the word such that;

$$x_{i_1} \leq x_{i_2} \leq x_{i_3} \leq \ldots \leq x_{i_l} \qquad \text{with } i_1 < i_2 < \ldots < i_l$$

Example 4.1 If we consider the word

$$w = 9883572461235$$

Then a weakly increasing sequence of w of length four is 2 2 3 5. In fact this is the longest weakly increasing sequence of the word w. If we consider the tableau P(w);

$$\mathrm{P}(w) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \cline{1-3} 3 & 5 & 7 \\ \cline{1-3} 8 & 8 \\ \cline{1-2} 9 \\ \cline{1-1} \end{array}$$

then we see that the length of the first row is also four. This is no coincidence.

Proposition 4.2 *The length of the longest increasing sequence of a word w is the length of the first row of $P(w)$.*

We will consider the proof of this later on in the chapter.

Definition 4.3 *Consider a sequence σ and a weakly increasing sequence σ_1 that is extracted from σ. Suppose we now ignore all the numbers that were used in the sequence σ_1 and we manage to extract another weakly increasing sequence σ_2 from σ with the remaining numbers. If there are no more numbers to extract another sequence then we can write σ as the disjoint union of two weakly increasing sequences σ_1 and σ_2.*

$$\sigma = \sigma_1 \uplus \sigma_2$$

Definition 4.4 *If w is a word then a subsequence σ of w is k weakly increasing if, as a set, it can be written as;*

$$\sigma = \sigma_1 \uplus \sigma_2 \uplus \ldots \uplus \sigma_k$$

Where the σ_i are weakly increasing subsequences of w and $\uplus$ is the disjoint union.

Let L(w,k) be the length of w's longest k-weakly increasing sequence. L(w,1) is the length of the longest weakly increasing sequence that can be extracted from w and denote this by α, thus L(w,1) $=\alpha$. For w = 9 8 8 3 5 7 2 4 6 1 2 3 5 then α = 4. Suppose that we have extracted a weakly increasing sequence from w. Now we can consider extracting a further disjoint weakly increasing sequence from w. L(w,2) is the largest number that can be obtained when we sum the lengths of 2 disjoint weakly increasing sequences extracted from w. Note though that the first sequence does not have to be maximal in length.

For example for our w=9 8 8 3 5 7 2 4 6 1 2 3 5 we see that the longest 1-,2-,3-,4- and 5- weakly increasing subsequences are given by;

$$1\ 2\ 3\ 5$$

$$2\ 4\ 6\ 1\ 2\ 3\ 5 = 1\ 2\ 3\ 5 \uplus 2\ 4\ 6$$

$$3\ 5\ 7\ 2\ 4\ 6\ 1\ 2\ 3\ 5 = 1\ 2\ 3\ 5 \uplus 2\ 4\ 6 \uplus 3\ 5\ 7$$

$$8\ 8\ 3\ 5\ 7\ 2\ 4\ 6\ 1\ 2\ 3\ 5 = 1\ 2\ 3\ 5 \uplus 2\ 4\ 6 \uplus 3\ 5\ 7 \uplus 8\ 8$$

$$9\ 8\ 8\ 3\ 5\ 7\ 2\ 4\ 6\ 1\ 2\ 3\ 5 = 1\ 2\ 3\ 5 \uplus 2\ 4\ 6 \uplus 3\ 5\ 7 \uplus 8\ 8 \uplus 9$$

Thus $L(w,1) = 4$, $L(w,2) = 7$, $L(w,3) = 10$, $L(w,4) = 12$ $L(w,5) = 13$.

Recall that $P(w)$ has the partition $\lambda = (4,3,3,2,1)$, so that;

$$\lambda_1 = 4 \qquad \lambda_1 + \lambda_2 = 4+3 = 7$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 4+3+3 = 10 \qquad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 4+3+3+2 = 12$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 4+3+3+2+1 = 13$$

Thus we can see a pattern emerging.

Theorem 4.5 ***Greene's theorem***

Given a word w of a tableau T with partition $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_k)$. Then, for any k,

$$L(w,k) = \lambda_1 + \lambda_2 + \ldots + \lambda_k$$

Proof If we have a word w of a tableau T then we can easily find $L(w,k)$ for each k. If we consider any increasing sequence taken from w then the numbers which are taken are from the tableau from left to right, never going down the tableau. The letters must be taken from different columns and so $L(w,1)$ is the total number of columns which is the number of boxes in the first row. In a similar fashion $L(w,k)$ is the number of boxes in the first k rows. Any disjoint union of k sets of boxes in a Young diagram, each of which contains at most one box in a column, can have no more boxes than there are in the first k rows. For example if we have k such sets then we can find k sets with the same number of boxes. These are all taken from the first row, by replacing lower boxes by higher boxes in these k rows. ∎

Theorem 4.6 *If w and w' are two Knuth equivalent words, then for all k*

$$L(w,k) = L(w',k)$$

Proof We consider the two elementary Knuth transformations K and K'. First K. Suppose that $w = x_1 \dots yxz \dots x_n$ and $w' = x_1 \dots yzx \dots x_n$ and that w' is obtained from w by the elementary Knuth transformation K, so that $x < y \leq z$. Then L(w,k) $\geq$ L(w',k). Since x and z are out of order in w', they can never be in the same component of a k-weakly increasing subsequence. Similarly for K'. Here we would have that $v = x_1 \dots xzy \dots x_n$ and $v' = x_1 \dots zxy \dots x_n$ and that v' is obtained from v by the elementary Knuth transformation K', so that $x \leq y < z$. Here we also have that L(v,k) $\geq$ L(v',k). Since x and z are out of order in v', they can never be in the same component of a k-weakly increasing subsequence.

To show that L(w,k) $\leq$ L(w',k) we consider K again. We only need to show that any k-weakly increasing subsequence of w has a corresponding k-weakly increasing subsequence in w' of the same length. Let;

$$\bar{w} = w_1 \uplus w_2 \uplus \dots \uplus w_k$$

be a k-weakly subsequence of w. If x and z are not in the same w_i, for any i, then the w_i are also increasing subsequences of w' and we are finished. Now suppose that $x \in w_1$ $z \in w_1$ (Without loss of generality). If $y \notin \bar{w}$, then let;

$$w_1' = w_1 \text{ with } x \text{ replaced by } y.$$

Since we have that $x < y \leq z$, then w_1' is still weakly increasing and

$$\bar{w}' = w_1' \uplus w_2 \uplus \dots \uplus w_k$$

is a sequence of the right length. Finally, suppose that $y \in \bar{w}$. Note we cannot have $y \in w_1$ as this would mean that x, y and z are in w_1. This would be impossible as $x < y \leq z$. In w x, y and z appear in the order yxz which means that they cannot all be in the same subsequence. So suppose that $y \in w_2$. Let;

$$\bar{w}_1 = \text{the subsequence of } w_1 \text{ up to and including } x,$$

$$\bar{\bar{w}}_1 = \text{the subsequence consisting of the rest of } w_1,$$

$$\bar{w}_2 = \text{the subsequence of } w_2 \text{ up to and including } y,$$

$$\bar{\bar{w}}_2 = \text{the subsequence consisting of the rest of } w_2$$

It is useful to note here that $w_i = \bar{w}_i \cdot \bar{\bar{w}}_i$ for $i = 1, 2$

We now construct

$$w_1' = \bar{w}_1 \cdot \bar{\bar{w}}_2 \qquad \text{and} \qquad w_2' = \bar{w}_2 \cdot \bar{\bar{w}}_1$$

Which are weakly increasing as $x < y < \min\bar{\bar{w}}_2$ and $y < z$ respectively, where $y < \min\bar{\bar{w}}_2$ means that y is less than the smallest number in $\bar{\bar{w}}_2$. Also we have;

$$\bar{w}' = w_1' \uplus w_2' \uplus w_3 \uplus \ldots \uplus w_k$$

is a subsequence of w' as x and z are no longer in the same component subsequence. Since its length is correct then we have proved the required inequality that $\mathrm{L}(w,k) \leq \mathrm{L}(w',k)$.

To do the same for v and v' where we consider the elementary Knuth transformation K', we consider $v = x_1 \ldots xzy \ldots x_n$ and $v' = x_1 \ldots zxy \ldots x_n$ where $x \leq y < z$. The idea goes along the same line as above, taking care that x and z are now on the left hand side of y instead of the right.

So we have shown that $\mathrm{L}(w,k) = \mathrm{L}(w',k)$ and that $\mathrm{L}(v,k) = \mathrm{L}(v',k)$. So if two words w and w' which are Knuth equivalent then

$$\mathrm{L}(w,k) = \mathrm{L}(w',k) \quad \blacksquare$$

Incidentally the fact that $\mathrm{L}(w,k) = \lambda_1 + \lambda_2 + \ldots + \lambda_k$ does not mean that we can find a k-weakly increasing subsequence of maximum length

$$\bar{w} = w_1 \uplus w_2 \uplus \ldots \uplus w_k$$

such that the length of w_i is λ_i for all i.

Example 4.7 We consider

$$w = 4\ 9\ 13\ 17\ 11\ 2\ 6\ 12\ 15$$

Then

$$\mathrm{P}(w) = \begin{array}{|c|c|c|c|c|} \hline 2 & 6 & 11 & 12 & 15 \\ \hline 4 & 9 & 17 \\ \cline{1-3} 15 \\ \cline{1-1} \end{array}$$

Here we have that $\mathrm{L}(w,2) = 5+3 = 8$ but there is only one 2-weakly increasing subsequence of w having length 8, namely

$$\bar{w} = 4\ 9\ 13\ 17 \uplus 2\ 6\ 12\ 15$$

It is easy to check that we cannot represent $\bar{w}$ as the disjoint union of two increasing subsequences of lengths 5 and 3.

From theorem 4.5 and theorem 4.6 we can tell the shape of any tableau whose word w is Knuth equivalent to a given word w'. If we are given a word w then we would like to be able to tell the tableau that it has come from or what the tableau $\mathrm{P}(w)$ has as its entries without having to use Schensted's procedure.

Definition 4.8 *Consider the largest letter in w (or the furthest to the right if there are two or more) and call this l.*

Remove l from w and call this new word $w_\circ$. This word $w_\circ$ will give a diagram with one less box than w. By comparing $\mathrm{L}(w,k)$ and $\mathrm{L}(w_\circ,k)$ we can tell the row of the tableau that l was in. We clearly have that l is at the end of this row otherwise we would not have a Young tableau. If we continue in this way then we can recover the whole table.

For example consider the word $w = 6\ 3\ 4\ 6$. The largest letter in w is 6 and we choose the one which is furthest to the right. Thus we have that $w_\circ = 6\ 3\ 4$. $\mathrm{L}(w,1) = 3$ and $\mathrm{L}(w_\circ,1) = 2$, and so our 6 is at the end of the first row of $\mathrm{P}(w)$. Denote $w_{\circ\circ}$ to be the same word as w except that the largest and second largest entry have been removed. For $w = 6$

3 4 6 we have $w_{\circ\circ} = 3\ 4$. It is clear then that as L($w_{\circ\circ}$,1) = 2 and L($w_\circ$,1) = 2 the we must compare the next row of P(w). L($w_\circ$,2) = 3 and L($w_{\circ\circ}$,2) = 2 giving us that our 6 is at the end of the second row of P(w). If we define $w_{\circ\circ\circ}$ to be the same word as w but with the largest, second largest, third largest entry taken off w, for our example we have $w_{\circ\circ\circ} = 3$. Comparing L($w_{\circ\circ}$,1) = 2 and L($w_{\circ\circ\circ}$,1) = 1 we have that the entry 4 must be the second to last entry at the end of the first row of P(w). The length of the first row of P(w) is three and so our 3 must go at the beginning of the first row. Thus we have that;

$$\text{For } w = 6\ 3\ 4\ 6 \qquad \mathrm{P}(w) = \begin{array}{|c|c|c|}\hline 3 & 4 & 6 \\ \hline 6 \\ \cline{1-1}\end{array}$$

4.2 Proof of Uniqueness (Theorem 3.6)

What we are going to show is that removing the largest letters from Knuth equivalent words leaves Knuth equivalent words and then use this to prove the uniqueness part of theorem 3.6.

Theorem 4.9 *Let w and w' be Knuth equivalent words with $w_\circ$ and $w'_\circ$ being the result of removing the p largest and the q smallest letters from each respectively. We have that $w_\circ$ and $w'_\circ$ are Knuth equivalent words.*

Proof We are going to use induction on the length of a word to show that if we move the smallest or largest letters from w and w' then we still get Knuth equivalent words. Consider as before w and w' where the latter is obtained from the former by an elementary Knuth transformation K. Let;

$$w = x_1 \ldots yxz \ldots x_n \qquad \text{and} \qquad w' = x_1 \ldots yzx \ldots x_n$$

If the element removed from w is not one of x, y or z then the Knuth equivalence of $w_\circ$ and $w'_\circ$ is clear. Suppose that the letter that is removed is x, y or z. In fact it must be z as we have that $x < y \leq z$. If z is removed then we have $w_\circ = w'_\circ$. Thus $w_\circ \equiv w'_\circ$ when we consider removing the largest letter from w and w' with w being related to w' by K. To prove the case for K' and the smallest letter we consider a similar approach which is left as an exercise.

We will now complete the proof of theorem 3.6. We will show that if a word w is Knuth equivalent to the word W(T) of a tableau T, then T is uniquely determined by w. We will use induction on the length of a word. The base case is the situation in which we have only one letter in a word. Clearly T is uniquely determined by w. We know that the shape λ of a tableau is uniquely determined by w. So

$$\lambda_k = \mathrm{L}(w,k) - \mathrm{L}(w,k-1)$$

We have from earlier that l is the largest letter (or furthest to the right if there are two or more) occurring in w and that $w_\circ$ is the word left when l is removed. Define $T_\circ$ to be the tableau obtained by removing l from T. By theorem 4.9 $w_\circ$ is Knuth equivalent to $\mathrm{W}(T_\circ)$. Through induction on the length of the word, $T_\circ$ is the unique tableau whose word is Knuth equivalent to $w_\circ$. As we know the shape of T and the shape of $T_\circ$, the only possibility for T is that it is obtained from $T_\circ$ by putting l in the remaining box. ∎

Chapter 5

The Robinson-Schensted-Knuth Correspondence

5.1 The Robinson-Schensted Correspondence

The Robinson-Schensted-Knuth correspondence is the result that we have been leading up to all the way through. It gives a one-to-one correspondence between matrices with non-negative integer entries and pairs of tableaux of the same shape. A version of the correspondence, for permutations was given by Robinson [9], and later shown independently by Schensted [11], who also extended it to arbitrary words. Knuth [5] made the generalization to two-rowed arrays, or matrices.

Let $\mathrm{P}(w)_i$ be the result of the canonical procedure from using letters $x_1 \ldots x_i$ from a word $w = x_1 x_2 \ldots x_i \ldots x_j$. The tableau $\mathrm{P}(w)$ corresponding to a word $w = x_1 x_2 \ldots x_j$ is formed from the canonical procedure

$(\ldots((x_1 \leftarrow x_2) \leftarrow x_3) \ldots \leftarrow x_j)$

The tableau $\mathrm{Q}(w)_i$ corresponding to the same word w is the array which is obtained by putting i in the square which is added to the shape of $\mathrm{P}(w)_i$ when x_i is inserted into $\mathrm{P}(w)_{i-1}$. $\mathrm{Q}(w)_i$ is certainly a tableau as the box which is added to $\mathrm{P}(w)_{i-1}$ is an outside corner, and so the new entry in $\mathrm{Q}(w)_i$ is larger than the entries above or to its left.

Example 5.1 Suppose that $w = 4\ 9\ 13\ 17\ 11\ 2\ 6\ 12\ 15$, then the tableaux constructed by the bumping procedure are

$$P(w)_1 = \begin{array}{|c|}\hline 4\\\hline\end{array} \qquad P(w)_2 = \begin{array}{|c|c|}\hline 4 & 9\\\hline\end{array} \qquad P(w)_3 = \begin{array}{|c|c|c|}\hline 4 & 9 & 13\\\hline\end{array} \qquad P(w)_4 = \begin{array}{|c|c|c|c|}\hline 4 & 9 & 13 & 17\\\hline\end{array}$$

$$P(w)_5 = \begin{array}{|c|c|c|c|}\hline 4 & 9 & 11 & 17\\\hline 13 \\\cline{1-1}\end{array} \qquad P(w)_6 = \begin{array}{|c|c|c|c|}\hline 2 & 9 & 11 & 17\\\hline 4 \\\cline{1-1} 13 \\\cline{1-1}\end{array} \qquad P(w)_7 = \begin{array}{|c|c|c|c|}\hline 2 & 6 & 11 & 17\\\hline 4 & 9 \\\cline{1-2} 13 \\\cline{1-1}\end{array}$$

$$P(w)_8 = \begin{array}{|c|c|c|c|}\hline 2 & 6 & 11 & 12\\\hline 4 & 9 & 17 \\\cline{1-3} 13 \\\cline{1-1}\end{array} \qquad P(w)_9 = P(w) = \begin{array}{|c|c|c|c|c|}\hline 2 & 6 & 11 & 12 & 15\\\hline 4 & 9 & 17 \\\cline{1-3} 13 \\\cline{1-1}\end{array}$$

The corresponding $Q(w)_i$ tableaux are,

$$Q(w)_1 = \begin{array}{|c|}\hline 1\\\hline\end{array} \qquad Q(w)_2 = \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array} \qquad Q(w)_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3\\\hline\end{array} \qquad Q(w)_4 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4\\\hline\end{array}$$

$$Q(w)_5 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4\\\hline 5 \\\cline{1-1}\end{array} \qquad Q(w)_6 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4\\\hline 5 \\\cline{1-1} 6 \\\cline{1-1}\end{array} \qquad Q(w)_7 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4\\\hline 5 & 7 \\\cline{1-2} 6 \\\cline{1-1}\end{array}$$

$$Q(w)_8 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4\\\hline 5 & 7 & 8 \\\cline{1-3} 6 \\\cline{1-1}\end{array} \qquad Q(w)_9 = Q(w) = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 9\\\hline 5 & 7 & 8 \\\cline{1-3} 6 \\\cline{1-1}\end{array}$$

We saw in Chapter 2 that the Schensted bumping algorithm was invertible if we knew the position in which the last box was added. So by reversing the steps in the Schensted algorithm we can recover the original word w from the pair (P,Q). It is very easy to go from the pair of tableaux (P_i,Q_i) to the pair (P_{i-1},Q_{i-1}). Note which box the largest element of Q_i is in, and then apply the reverse bumping to the corresponding box in P_i. The reverse bumping will bump an integer out of the tableau, this is the letter x_i of the word w. Now we remove the i^{th} letter from Q and we have a pair of tableau (P_{i-1},Q_{i-1}) and the i^{th} letter of w. We can carry on with this procedure to get the word w. What is useful is that we can use any pair of tableaux with the same shape so long as one of them has entries with no repeats, i.e. a standard tableau. We can find the word w from which P was formed, in the canonical procedure, by using the method that was described above. What we have is a bijective correspondence between words w (letters from $1, 2, \ldots, m$ of length i) and ordered pairs (P,Q) of tableaux (entries from $1, 2, \ldots, n$ of the same shape λ and i boxes) with Q being standard. This is called the *Robinson-Schensted Correspondence.*

5.2 Robinson Correspondence

If we consider the special case of when $i = n$ and when the letters of w are from the set $1, 2, \ldots, n$ each occurring once then we get the Robinson correspondence. We have a bijective correspondence between standard tableaux (P,Q) and the Symmetric Group S_n when w is considered as the permutation which takes i to the i^{th} letter of w. Consider the Symmetric group S_3 then the corresponding tableaux to the elements of S_3 are as follows.

Example 5.2 The elements of S_3 are,

$$\pi_1 = 1\ 2\ 3 \qquad \pi_2 = 2\ 1\ 3 \qquad \pi_3 = 2\ 3\ 1$$

$$\pi_4 = 3\ 1\ 2 \qquad \pi_5 = 3\ 2\ 1 \qquad \pi_6 = 1\ 3\ 2$$

$$\mathrm{P}(\pi_1) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \qquad \mathrm{Q}(\pi_1) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$$

$$\mathrm{P}(\pi_2) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array} \qquad \mathrm{Q}(\pi_2) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array}$$

$$\mathrm{P}(\pi_3) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array} \qquad \mathrm{Q}(\pi_3) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array}$$

$$\mathrm{P}(\pi_4) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array} \qquad \mathrm{Q}(\pi_4) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array}$$

$$\mathrm{P}(\pi_5) = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \qquad \mathrm{Q}(\pi_5) = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

$$\mathrm{P}(\pi_6) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array} \qquad \mathrm{Q}(\pi_6) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array}$$

An interesting point to make here is that when we have an element π_i which is an involution (i.e. a permutation such that $\pi = \pi^{-1}$) in S_n then the tableaux P and Q that are formed are identical to each other. The reason for this will become apparent when we have considered the symmetry theorem and the matrix of an array.

5.3 Robinson-Schensted-Knuth Correspondence

Knuth [5] generalized the Robinson-Schensted correspondence to an arbitrary ordered pair (P,Q) of the same shape, where P has entries from $1, \ldots, n$ and Q has entries from $1, \ldots, m$.

The same reverse bumping procedure can be used as before to give us a two-rowed array $\begin{pmatrix} u_1 & u_2 & \ldots & u_j \\ v_1 & v_2 & \ldots & v_j \end{pmatrix}$ where v_i is the entry that is bumped from the top row of P_i and u_i is the corresponding entry in Q_i.

In the case of the Robinson correspondence we had Q had n distinct entries from 1,...,n. In this case the set of integers u_j, from the array above, would just be j so we have the array $\begin{pmatrix} 1 & 2 & \ldots & n \\ v_1 & v_2 & \ldots & v_n \end{pmatrix}$ which carries the same information as the word $w = v_1 v_2 \ldots v_j$. We can treat this array as a word so long as its top row consists of the numbers 1,2,...,n in order and the v_j are also distinct integers from 1,2,...,n. We can write our permutations that we looked at earlier in this way. For example $\pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

Before we can look at Knuth's correspondence a little closer we need to define what we mean by *lexicographic* order. If we are given two tableau P and Q and we do reverse bumping to get an array, then what properties does this array have? An easy property to spot first of all is that the u_i's are in weakly increasing order, so we have:

$$(1) \qquad u_1 \leq u_2 \leq \ldots \leq u_j$$

Another property of the array can be obtained if we look at lemma 2.6 about the row-bumping route of an integer. This describes the bumping routes of two successive row-insertions. If we have that $u_{i-1} = u_i$, the we would have that the box B' that is removed from P_i lies strictly to the right of the box that is removed from P_{i-1} in the next step of the reverse bumping. So now we are in the situation described in case one of lemma 2.6 This tells us that the entry v_i is at least as large as v_{i-1}, so we also have:

$$(2) \qquad v_{i-1} \leq v_i \qquad \text{if} \qquad u_{i-1} = u_i$$

Definition 5.3 *We say that a two-rowed array* $\gamma = \begin{pmatrix} u_1 & u_2 & \ldots & u_j \\ v_1 & v_2 & \ldots & v_j \end{pmatrix}$ *is in lexicographic order if our conditions (1) and (2) hold.*

$$(1) \qquad u_1 \leq u_2 \leq \ldots \leq u_j$$

$$(2) \qquad v_{i-1} \leq v_i \qquad \textit{if} \qquad u_{i-1} = u_i$$

Lexicographic ordering is the way in which words are ordered in a dictionary. When we compare two words in a dictionary we order them first by their first letters and if these are the same we move on to the second letters and so on. Given an array in lexicographic order we can construct a pair of tableaux with the same shape (P,Q). For $i \leq j$ let P_i be the result of the canonical procedure from using letters $v_1 \dots v_i$ from an array $\begin{pmatrix} u_1 & u_2 & \dots & u_j \\ v_1 & v_2 & \dots & v_j \end{pmatrix}$. The tableau P corresponding to the array $\begin{pmatrix} u_1 & u_2 & \dots & u_j \\ v_1 & v_2 & \dots & v_j \end{pmatrix}$ is formed from the canonical procedure

$(\dots((v_1 \leftarrow v_2) \leftarrow v_3) \dots \leftarrow v_j)$

The tableau Q_i corresponding to the same array is the tableau which is obtained by putting u_i in the square which is added to the shape of P_i when v_i is inserted into P_{i-1}. It is easy to see that P_i is a tableau as it is formed from the canonical procedure. To show that Q_i is a tableau we use induction. Q_1 is a tableau with entry u_1. To see that Q_i is a tableau, we show that if u_i is placed under an entry u_l in Q_{i-1} then u_i is strictly larger than u_l. We do this by considering the case when $u_i = u_l$ as it is impossible for $u_i < u_l$ as we have the array in lexicographic order. If $u_l = u_i$ then $u_l = u_{l+1} = \dots = u_{i-1} = u_i$, from (1). So from (2) we have $v_l \leq v_{l+1} \leq \dots \leq v_i$. From lemma 2.6 we have that the box added for u_i will not be in the same column as u_l and so we have a contradiction.

Theorem 5.4 ***R-S-K*** *The operations defined in this chapter set up a bijective correspondence between two-rowed lexicographic arrays γ and ordered pairs of tableaux (P,Q) with the same shape. Under this correspondence:*

(1) γ has j entries in each row $\Longleftrightarrow$ P and Q have j boxes each. The entries of P are the elements of the bottom row of γ and the entries of Q are the elements of the top row of γ

(2) γ is a word $\Longleftrightarrow$ Q is a standard tableau.

(3) γ is a permutation $\Longleftrightarrow$ P and Q are standard tableau.

In fact if we are given any two-rowed array then we find the unique lexicographic array by putting its vertical pairs in lexicographic order. Two arrays are considered the same if they have the same pairs (u_i, v_i) each occurring the same number of time in each array. We could also say that their corresponding lexicographic arrays are the same. So to conclude we can associate an ordered pair (P,Q) to an arbitrary two-row array.

When we construct P and Q we treat the two rows of the array very differently. For the entries in the bottom row we bump them into a tableau P and for the first row we just place them in the tableau Q. The interesting fact is that these two operations are more closely related then we would expect. This is illustrated in the following theorem.

5.4 Symmetry Theorem

Theorem 5.5 ***Symmetry Theorem*** *If an array* $\begin{pmatrix} u_1 & u_2 & \dots & u_j \\ v_1 & v_2 & \dots & v_j \end{pmatrix}$ *corresponds to the pair of tableaux (P,Q), then the array* $\begin{pmatrix} v_1 & v_2 & \dots & v_j \\ u_1 & u_2 & \dots & u_j \end{pmatrix}$ *corresponds to the pair (Q,P).*

Example 5.6 As an illustration of this theorem consider the array; $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 13 & 17 & 11 & 2 & 6 & 12 & 15 \end{pmatrix}$ we showed earlier that this array corresponds to the pair of tableaux

$$Q(w) = \begin{array}{lllll} 1 & 2 & 3 & 4 & 9 \\ 5 & 7 & 8 \\ 6 \end{array} \quad \text{and} \quad P(w) = \begin{array}{lllll} 2 & 6 & 11 & 12 & 15 \\ 4 & 9 & 17 \\ 13 \end{array}$$

If we consider the array $\begin{pmatrix} 4 & 9 & 13 & 17 & 11 & 2 & 6 & 12 & 15 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}$ then by putting this in lexicographic order we get the array $x = \begin{pmatrix} 2 & 4 & 6 & 9 & 11 & 12 & 13 & 15 & 17 \\ 6 & 1 & 7 & 2 & 5 & 8 & 3 & 9 & 4 \end{pmatrix}$ then the tableaux constructed by the bumping procedure are

$$P(w)_1 = \begin{array}{l} 6 \end{array} \quad P(w)_2 = \begin{array}{l} 1 \\ 6 \end{array} \quad P(w)_3 = \begin{array}{ll} 1 & 7 \\ 6 \end{array} \quad P(w)_4 = \begin{array}{ll} 1 & 2 \\ 6 & 7 \end{array}$$

$$P(w)_5 = \begin{array}{lll} 1 & 2 & 5 \\ 6 & 7 \end{array} \quad P(w)_6 = \begin{array}{llll} 1 & 2 & 5 & 8 \\ 6 & 7 \end{array} \quad P(w)_7 = \begin{array}{llll} 1 & 2 & 3 & 8 \\ 5 & 7 \\ 6 \end{array}$$

$$P(w)_8 = \begin{array}{lllll} 1 & 2 & 3 & 8 & 9 \\ 5 & 7 \\ 6 \end{array} \quad P(w)_9 = P(w) = \begin{array}{lllll} 1 & 2 & 3 & 4 & 9 \\ 5 & 7 & 8 \\ 6 \end{array}$$

The corresponding $Q(w)_i$ tableaux are,

$$Q(w)_1 = \begin{array}{l} 2 \end{array} \quad Q(w)_2 = \begin{array}{l} 2 \\ 4 \end{array} \quad Q(w)_3 = \begin{array}{ll} 2 & 6 \\ 4 & \end{array} \quad Q(w)_4 = \begin{array}{ll} 2 & 6 \\ 4 & 9 \end{array}$$

$$Q(w)_5 = \begin{array}{lll} 2 & 6 & 11 \\ 4 & 9 & \end{array} \quad Q(w)_6 = \begin{array}{llll} 2 & 6 & 11 & 12 \\ 4 & 9 & & \end{array} \quad Q(w)_7 = \begin{array}{llll} 2 & 6 & 11 & 12 \\ 4 & 9 & & \\ 13 & & & \end{array}$$

$$Q(w)_8 = \begin{array}{lllll} 2 & 6 & 11 & 12 & 15 \\ 4 & 9 & & & \\ 13 & & & & \end{array} \quad Q(w)_9 = Q(w) = \begin{array}{lllll} 2 & 6 & 11 & 12 & 15 \\ 4 & 9 & 17 & & \\ 13 & & & & \end{array}$$

Here we can see that we have the same pair of tableaux that we got earlier in example 5.6. We will look at the proof of the symmetry theorem in the next chapter through the *Matrix-ball construction.* Before the introduction of this construction we will look at matrices and show how the information in a two-rowed array may be converted into a simple m x n matrix. An equivalence class of two-rowed arrays can be identified with a collection of pairs of elements (i,j), i *in* $[m]$, j *in* $[n]$, each occurring with some nonnegative multiplicity. We can represent this data in an m x n matrix A whose (i,j) entry is the number of times $\begin{pmatrix} i \\ j \end{pmatrix}$ occurs in the array.

Example 5.7 If we consider the array $\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 \\ 2 & 2 & 3 & 1 & 1 & 2 & 3 & 1 & 1 & 1 & 3 & 3 & 1 & 3 & 3 \end{pmatrix}$

then it can be identified with the matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}$$

with the tableaux P and Q being

$$P = \begin{array}{lllllllll} 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 \\ 2 & 2 & 2 & 3 & 3 & & & & \\ 3 & & & & & & & & \end{array} \qquad Q = \begin{array}{lllllllll} 1 & 1 & 1 & 2 & 3 & 3 & 3 & 4 & 4 \\ 2 & 2 & 2 & 3 & 4 & & & & \\ 3 & & & & & & & & \end{array}$$

We will be looking at example 5.7 very closely in the next chapter. Now we can restate the R-S-K correspondence in a different form. We can say that the R-S-K correspondence is then a correspondence between matrices with nonnegative integer entries and ordered pairs of tableaux (P,Q) of the same shape. We can see that if we have an m x n matrix, then P has entries in $[n]$ and Q has entries in $[m]$. For the matrix in example 5.7 we see that P would have entries in [3] and Q would have entries in [4] as we have a 3 x 4 matrix. Looking even closer at the matrix we can tell even more about P and Q. The ith row sum is the number of times i appears in Q. By looking at A we see that its first row sum is 3 which is exactly the number of times 1 occurs in Q. Similarly the j column sum is the number of times j occurs in P. By looking at A we see that the third column sum is 6 which is exactly the number of times 3 occurs in the tableau P. There are other properties that can be found also. Consider the case when we have exactly one 1 in each row with the other entries being zero, this is the matrix of a word.

Example 5.8 For the word $w = 4\ 9\ 13\ 17\ 11\ 2\ 6\ 12\ 15$ the corresponding array is $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 13 & 17 & 11 & 2 & 6 & 12 & 15 \end{pmatrix}$ and this has the matrix

$$\begin{bmatrix} 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \end{bmatrix}$$

A natural operation on a matrix could be to take its transpose. This transpose would correspond to the array turned upside down. We can therefore restate the Symmetry theorem again. If a matrix A corresponds to the tableau pair (P,Q) then the transpose A^t corresponds to the pair (Q,P). If we consider symmetric matrices then we notice that $A = A^t$ and therefore we can say that symmetric matrices correspond to pairs of the form (P,P).

Chapter 6

Matrix-ball Construction

To go from an array to a pair of tableaux means that we have apply the bumping procedure several times to a tableau, which requires, especially for large tableaux, successive rewriting of tableaux. In this chapter we shall look at a geometric construction which will allow us to go from an array to a pair of tableaux much quicker as we will not have to keep rewriting tableaux. The motive for looking at this construction though will be to prove the Symmetry theorem given at the end of chapter 5. At the end of the chapter we will look at a construction which will allow us to go from a pair of tableaux (P,Q) to the original array w without having to do any reverse bumping. We shall give a recipe for assigning to an m x n matrix A a pair (P,Q) = (P(A),Q(A)) of tableaux which we call the *matrix-ball* construction.

6.1 Matrix of Balls

Let A be an m x n integer matrix, we will associate a diagram to A as follows, for example, consider the matrix A discussed at the end of the last chapter in example 5.7.

$$A=\begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}$$

Construct a table with 2 vertical lines and 3 horizontal lines as below. For the general m x n matrix we would have $n-1$ vertical lines and $m-1$ horizontal lines. To fill the table

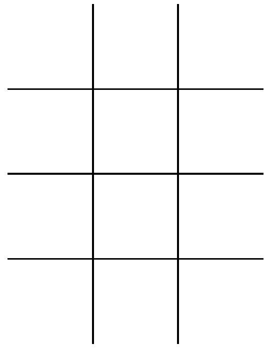


Figure 6.1: The empty table for A

consider the (i,j) entry of A. Place this many balls in the (i,j) position of the table. If there is more than one ball in a position then place them diagonally from top left to bottom right. In matrix A there is a 3 in position (3,1). In the table there will be three balls in the (3,1) position. Consider the table given in figure 6.2 for the matrix A . It is therefore quite easy

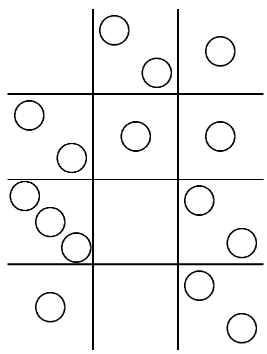


Figure 6.2: Matrix of balls for A before numbering

to go from a matrix A to the table with balls in the corresponding positions. What we are going to do now is number each of the balls with a positive integer. This is done as follows.

Definition 6.1 *We say that a ball is in a position upper left to another if it is weakly above and weakly to the left of it.*

We number the balls starting from upper left to lower right. We give a ball the smallest number that is larger than all the numbers which number balls to the upper left. If there is more than one ball in a position in the table then we number them with consecutive integers with the ball to the top left getting the smallest numbering. A ball is numbered 1 if there are no balls to the upper left of it. A ball is numbered k if $k-1$ is the number of the upper left ball in the same position, or if the ball is the first one in a given position, and $k-1$ is the largest number occurring in a ball to the upper left. This numbering may sound complicated but it isn't really and it is best shown by an illustration. Start with the first row and first column of the table. Number a ball 1 if it is the first one in its row or column. Now label the next one in the column or row 2 and the next 3 etc. For our matrix A given in example 5.7 we have numbered the first row and column of the corresponding table in figure 6.3. Now we consider the second row and second column. The ball in the second row

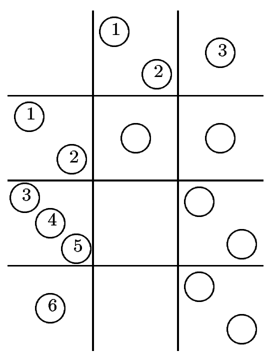


Figure 6.3:

and second column must be numbered so that it is one more than any ball to the upper left. Note that in this example there is a ball numbered 2 to the left and above. We therefore number our ball 3, as 3 is the smallest integer which is larger than all the numbers which number balls to the upper left. The ball in the third column and second row is numbered 4, as there is a 3 above it and a 3 to the left. Now we look at the third row and column. The first ball in this column must be numbered 6 as there is a ball in a position to the left which is numbered 5. The next ball may be numbered 7, and the next two 8 and 9. The resulting table is given in figure 6.4 and we call it $A^{(1)}$.

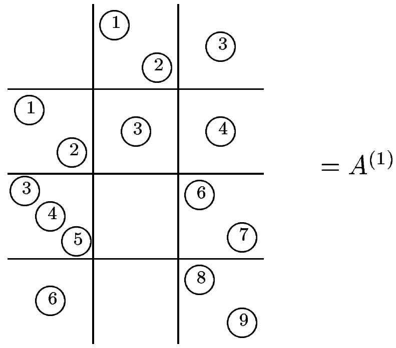


Figure 6.4:

6.2 Forming Tableaux from the Matrix of Balls

In this section we are going to see how the matrix ball construction allows us to form two tableaux P and Q of the same shape from the matrix A and we shall denote these tableaux by P(A) and Q(A). This will be done a row at a time starting with the first rows of the tableaux P(A) and Q(A). To get the first row of P(A) we simply list the left most columns where each number appears in the matrix-ball construction. To get the first row of Q(A) we just list the top-most row where each number occurs. So the ith entry of the first row of P(A) is the number of the left-most column where a ball numbered i occurs. Similarly the ith entry of the first row of Q(A) is the number of the upper-most row where a ball numbered i occurs. For example 5.7 we see that the left most column in which a 1 occurs in $A^{(1)}$ is the first column and the upper most row in which a 1 in $A^{(1)}$ occurs is the first row. Therefore the first entry in P(A) is a 1 and the first entry in Q(A) is a 1. We continue this for each of the i elements in the first row. For our matrix A in example 5.7 this gives us the first row of P(A) as (1,1,1,1,1,1,3,3,3) and the first row of Q(A) as (1,1,1,2,3,3,3,4,4).

To find the second row of the tableaux P(A) and Q(A) we need to form a new matrix ball construction which is done as follows. We look carefully at sets of balls in our original matrix with the same number. Suppose we have a number p, and there are $l > 0$ balls with this number on them in our original matrix. We then put $l - 1$ balls in our new matrix as follows. The l balls in $A^{(1)}$ all lie in a string from upper right to lower left with each of then occurring in a different column and row. Connect all the l balls by perpendicular lines as follows. Start from the ball furthest to the bottom of the matrix. Draw a horizontal line from this ball to the right until you get to the column in which the next ball numbered p is situated. Now at a right angle draw a vertical line up to the second ball numbered p. Now repeat the same procedure for the second and third balls numbered p and up to the end so that you reach the lth ball. For a general matrix this is shown in figure 6.5. Where the

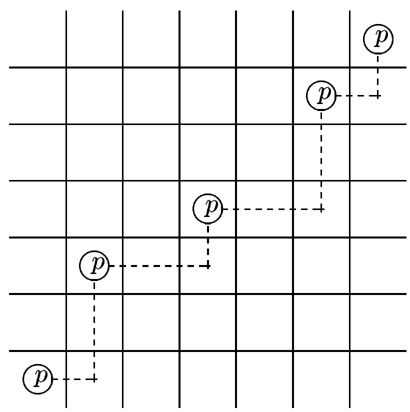


Figure 6.5:

dotted lines meet at right angles we place our $l-1$ balls in these positions reordering within a position to get them in a diagonal string from the top left of a position to the bottom right. We discard what was in $A^{(1)}$ before and we end up with a new matrix with $l-1$ balls. For the string of balls in figure 6.5 we get the new diagram as given in figure 6.6.

For each number p we repeat the process and we put all of the new balls in the matrix. For our example 5.7 we get the matrix of balls given in figure 6.7. We number the new matrix of balls in the same way that we did before. We therefore get a new matrix called $A^{(2)}$.

Definition 6.2 *From the matrix ball construction we define the derived matrix A^{α} to be a matrix having for its (i,j) entry the number of balls in the (i,j) position of the new matrix of balls, $A^{(2)}$*

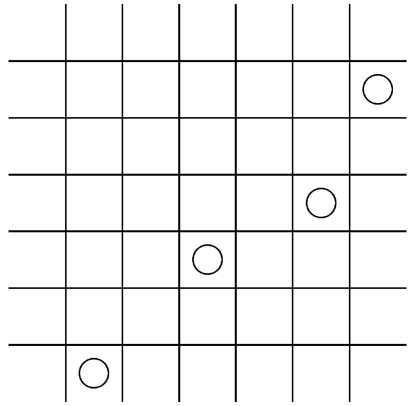


Figure 6.6:

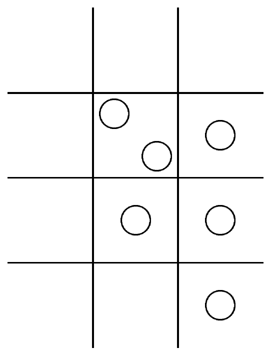
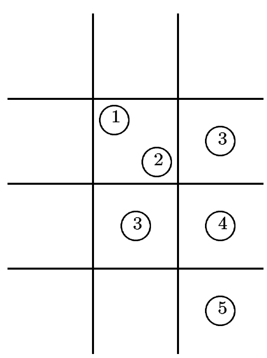


Figure 6.7:

For our example 5.7 this would give;

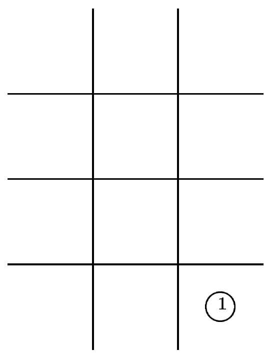
$$A^{\alpha} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



$= A^{(2)}$

We can now read off the second rows of our tableaux $\mathrm{P}(A)$ and $\mathrm{Q}(A)$. This is done in the same way as before. That is the *ith* entry of the second row of P(A) is the number of the left-most column where a ball numbered i occurs. The *ith* entry of the second row of Q(A) is the number of the upper-most row where a ball numbered i occurs. This gives us the second row of P(A) to be (2,2,2,3,3) and the second row of Q(A) to be (2,2,2,3,4).

To get the third row of P(A) we continue with the matrix ball construction to get $A^{(3)}$ which is the matrix of balls corresponding to $(A^{\alpha})^{\alpha}$. From $A^{(3)}$ we can easily read off the third row of P(A) and Q(A) which in our example we find to be (3) and (3). We get the same tableaux P and Q that we looked at towards the end of chapter 5 in example 5.7 when we considered the word $w = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 \\ 2 & 2 & 3 & 1 & 1 & 2 & 3 & 1 & 1 & 1 & 3 & 3 & 1 & 3 & 3 \end{pmatrix}$ corresponding to A.



$= A^{(3)}$

$$\mathrm{P}(A)=\mathrm{P}(w)= \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 3 & 3 \\ \cline{1-5} 3 \\ \cline{1-1} \end{array} \qquad \mathrm{Q}(A)=\mathrm{Q}(w)= \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 & 3 & 3 & 4 & 4 \\ \hline 2 & 2 & 2 & 3 & 4 \\ \cline{1-5} 3 \\ \cline{1-1} \end{array}$$

For our example we have shown that for our A corresponding to w we have P(A)=P(w) and Q(A)=Q(w). What we want to prove in general is that the matrix-ball construction agrees with the construction given in the R-S-K correspondence in chapter 5.

Theorem 6.3 *If the matrix A corresponds to a two-rowed array w, then (P(A),Q(A)) = (P(w),Q(w))*

Proof We begin by looking at what happens when there is only one entry in the matrix A. Suppose this is in position (x,y); this will have come form the array $w = \begin{pmatrix} x \\ y \end{pmatrix}$. This array corresponds to the tableaux

$$\mathrm{Q}(w)=\boxed{\mathrm{x}} \qquad \mathrm{P}(w)=\boxed{\mathrm{y}}$$

This tells us that there is one pair in the array and one ball in $A^{(1)}$. This ball will be numbered 1 and it will be in xth column and yth row. Therefore;

$$\mathrm{Q(A)} = \boxed{\mathrm{x}} \qquad \mathrm{P(A)}=\boxed{\mathrm{y}}$$

Suppose we have a matrix A with a number of entries in different positions.

Definition 6.4 *Define the* ***last*** *position of A to be the position (x,y) as follows. Pick the lowest row of A which is non-zero. Choose the entry of this row which is furthest to the right and non empty. This is the last position of A. For example consider the matrix*

$$B = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In B the last position is the position (6,3) which has a 1 in it.

Consider the general matrix A. Suppose we subtract 1 from the last entry in A and leave all the other entries the same and call our new matrix A_β. Note that A_β is exactly the same as A except for the fact that in its last position the value is 1 less than that of A. Suppose that $w = \begin{pmatrix} u_1 & u_2 & \dots & u_j \\ v_1 & v_2 & \dots & v_j \end{pmatrix}$ is the lexicographic array corresponding to A. It is clear then that the last position $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u_j \\ v_j \end{pmatrix}$ but what is the corresponding word w_β of the matrix A_β? Consider subtracting 1 from the last entry of A, this means that the length of the array w_β is one less than that of w and $w_\beta = \begin{pmatrix} u_1 & u_2 & \dots & u_{j-1} \\ v_1 & v_2 & \dots & v_{j-1} \end{pmatrix}$. What we will do is a proof by induction on the total of entries in the matrix A going from $j-1$ entries to j entries. Assume that to obtain $\mathrm{P}(A_\beta)$ we just bump in the elements $v_1 \leftarrow v_2 \leftarrow \dots \leftarrow v_{j-1}$. Assume also that $\mathrm{Q}(A_\beta)$ is obtained by placing $u_1,\dots,u_{j-1}$ in the new boxes. To prove theorem 6.3 then we only need to prove the following proposition, since the theorem will follow by induction on the sum of the entries in A.

Proposition 6.5 *$P(A) = P(A_\beta) \leftarrow y$ and $Q(A)$ is obtained from $Q(A_\beta)$ by placing x in the box that is in $P(A)$ but not in $P(A_\beta)$*

Recall that A_β is obtained from A by taking one away from the last entry. It is therefore clear to see that the diagram $A^{(1)}$ has one more ball than $A_\beta^{(1)}$ which is in the xth row and yth column of the diagram. Suppose that this extra ball in $A^{(1)}$ is numbered p, that is p is the largest number of a ball in the (x,y) position of $A^{(1)}$. Let us first suppose that there are no other balls in $A^{(1)}$ numbered p. It is quite clear then that $A^\alpha = A_\beta^\alpha$. This neatly gives us that all the rows of P(A) and P(A_β) are the same after the first row. Similarly the same goes for Q(A) and Q(A_β). It is clear that there are no balls in $A^{(1)}$ numbered greater than p as any such balls would be located to the lower right. We need to see how to get from P(A_β) to P(A) The first row of P(A) lists the left most columns of balls numbered one to p, with the last entry being y. The first row of P(A_β) lists the left most columns of balls numbered 1 to $p-1$. These numbers can be no larger than y as they are the same numbers which are in the first $p-1$ positions of P(A). Thus if we bump y into P(A_β) then it will simply be added on to the end. This is where it is in P(A) and so P(A) = P(A_β) $\leftarrow y$. Similarly, by considering upper-most rows of balls numbered 1 to p, Q(A) is obtained from Q(A_β) by adding an x to the end of the first row. When there is only one ball numbered p in $A^{(1)}$ we have proved proposition 6.5.

Suppose there are other balls in $A^{(1)}$ numbered with a p. All of these balls must be strictly above and strictly to the right of our first ball numbered p. Consider the diagram in figure 6.8, it illustrates in general what sort of situation we are in.

Our first ball numbered p is in the xth row and the yth column. Let the next ball to the right of the yth column be the second ball numbered p and denote its row by x' and its column by y'. The fact that the first row of P(A) is the first row of P(A_β)$\leftarrow y$ is a corollary to the following claim.

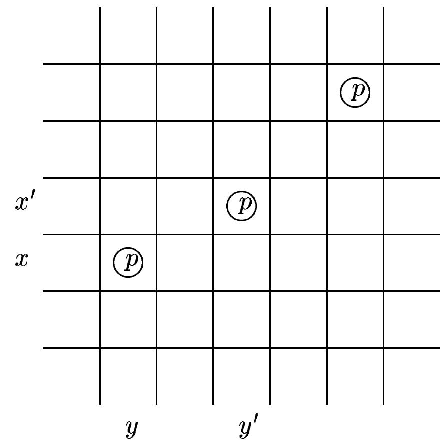


Figure 6.8:

Claim 6.6 *When y is row-inserted into the first row of $P(A_\beta)$, the element y' is bumped from the pth box.*

Proof To get the entries of the first row of $\mathrm{P}(A_\beta)$ we list the numbers of the left-most columns of $A_\beta^{(1)}$ which contain the balls numbered 1,2,3,.... Therefore we can do the reverse and find the left most columns of $A_\beta^{(1)}$ which contain the balls numbered 1,2,3,... by listing the first row of $\mathrm{P}(A_\beta)$. The first $p-1$ entries of the first row of $\mathrm{P}(A)$ and $\mathrm{P}(A_\beta)$ are the same and it is only the *pth* entry which is different. These rows are weakly increasing from left to right and so in the first row of $\mathrm{P}(A)$ the first $p-1$ entries are less than or equal to the *pth* entry which is y. The first $p-1$ entries of the first row of $\mathrm{P}(A_\beta)$ are therefore less than or equal to y. The *pth* entry which is y', is larger than y and so it is therefore this one which is bumped. ∎

To get the rest of $\mathrm{P}(A)$ we could consider it row by row, but it is in fact easier to look at the whole of $\mathrm{P}(A)$ below the first row which is $\mathrm{P}(A^\alpha)$. We can do the same for $\mathrm{P}(A_\beta)$ and so we look at $\mathrm{P}(A_\beta^\alpha)$. To prove proposition 6.5 we therefore need to show that $\mathrm{P}(A^\alpha)$ $= \mathrm{P}((A_\beta)^\alpha) \leftarrow y'$. Also we need to show that the new box in $\mathrm{P}((A_\beta)^\alpha) \leftarrow y'$ is the box in $\mathrm{Q}(A^\alpha)$ that is not in $\mathrm{Q}((A_\beta)^\alpha)$. This part follows from the inductive assumption for A_β so long as we can prove the following claim.

Claim 6.7 *The last position of A^α is (x,y'), and $(A^\alpha)_\beta = (A_\beta)^\alpha$*

If we look at the diagram given in figure 6.9 we can see that the (x,y') entry of A^α does exist, it will be where the dashed lines meet, and that A^α does not have any entries below the x row. If there is another nonzero entry of A^α in the x row then it must come from two balls of $A^{(1)}$ labelled with an l which is strictly less than p. The first ball must occur in the x row, but weakly to the left of column y, in the area shown by the diagonal lines from upper left to lower right. The second ball must occur in a position strictly to the upper right of the first ball. The ball numbered p in the (x',y') position must have a ball numbered l lying strictly to the upper left of it, which is shown in the area with the diagonal lines going from bottom left to top right. Thus the second ball numbered l cannot lie in a column labelled larger than y'. The entry of A^α that arises from these two balls lies weakly to the left of the position (x,y'). Thus we have shown that the last position of A^α is (x,y').

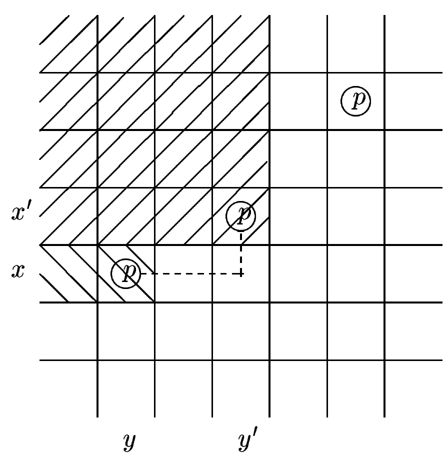


Figure 6.9:

To see that $(A^\alpha)_\beta = (A_\beta)^\alpha$ we only have to consider balls that are numbered p. The diagram below in figure 6.10 shows what A^α looks like.

To get $(A^\alpha)_\beta$ we simply remove the ball in the (x,y') position. To get A_β we just remove the ball in the (x,y) position from A. To get $(A_\beta)^\alpha$ we just form the new matrix of balls and we therefore get what is in figure 6.11.

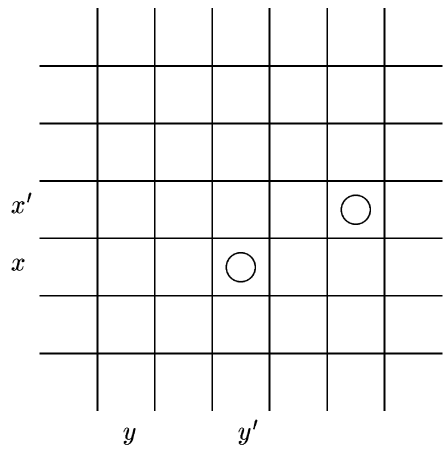


Figure 6.10:

It is clear then that $(A^{\alpha})_{\beta} = (A_{\beta})^{\alpha}$. We have now proved theorem 6.3 which basically tells us that the matrix-ball construction is well defined and that it can give us the pair of tableaux P and Q from the array w. The main motivation to looking at the matrix-ball construction was to prove the symmetry theorem given at the end of the last chapter and we are now in a position to do this.

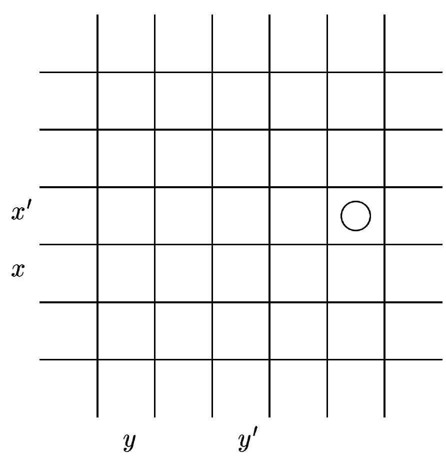


Figure 6.11:

6.3 Proof of the Symmetry Theorem (Theorem 5.5)

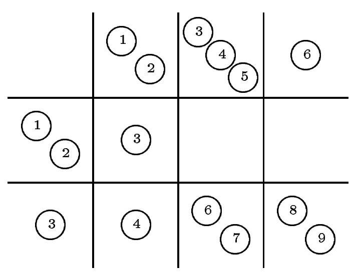
Theorem 6.8 ***Symmetry Theorem***

If a matrix A corresponds to the tableau pair (P,Q) then the transpose A^τ corresponds to the pair (Q,P).

Proof Let (P',Q') be the tableaux for A^τ, What we are going to show is that $P' = Q$ and that $Q' = P$. The Matrix-ball construction is symmetric in the rows and columns of the matrix A. If we have a diagram for a matrix A then the diagram for the matrix A^τ is obtained by swapping the rows for columns and vice versa. Consider our matrix A, it is easy to get A^τ.

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix} \qquad A^\tau = \begin{bmatrix} 0 & 2 & 3 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$

The associated matrix-ball diagram to A^τ with balls in the corresponding positions is;



$= A^{\tau(1)}$

Figure 6.12:

The point here is that the balls are numbered exactly as before, except that the diagram is now transposed. A ball in $A^{\tau(1)}$ will have the same number of balls to the upper left of it as the corresponding ball in $A^{(1)}$ and so the two balls will have the same numbering. As we have swapped rows for columns the upper-most row in which a ball numbered p occurs in $A^{(1)}$ is now the left-most column in which a ball numbered p occurs in $A^{\tau(1)}$. Similarly the left-most column in which a ball numbered p occurs in $A^{(1)}$ is now the upper most row in which a ball numbered p occurs in $A^{\tau(1)}$. So the first rows of the tableaux correspond, we have the first row of P' equals the first row of Q, also we have that the first row of Q' equals the first row of P. It is clear that $(A^\tau)^\alpha = (A^\alpha)^\tau$ by considering two balls in a (i_1, j_1) and (i_2, j_2) position in A both numbered p as in figure 6.13. In A^α there will be a ball in the (i_1, j_2) position and in $(A^\alpha)^\tau$ the ball will be in the (j_2, i_1) position. If we consider A^τ then there will be two balls in the (j_1, i_1) and (j_2, i_2) positions. In $(A^\tau)^\alpha$ there will be a ball in the (j_2, i_1) position which is the same as $(A^\alpha)^\tau$. ∎

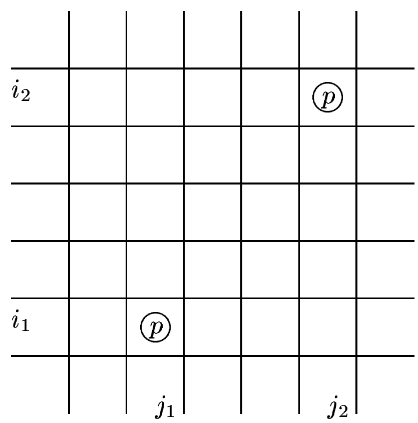


Figure 6.13:

6.4 Forming a Matrix from Young Tableaux

We will now see how we can go directly from a pair of Young tableaux (P,Q) of the same shape to its matrix A without using Schensted's bumping algorithm. What is interesting is that now through using this method we can work out the R-S-K correspondence without having to do any bumping of integers in to tableaux.

Earlier we assigned to a matrix A a pair of rows of the same length and a matrix B $= A^\alpha$. This was done for each of the rows of P and Q until we got an empty matrix A^α. Suppose we assigned to A the rows $(v_1, \ldots, v_r)$ and $(u_1, \ldots, u_r)$. The matrix $B = A^\alpha$ must have the property that in $B^{(1)}$, all balls numbered p lie weakly below and weakly to the right of the position in the $u_p th$ row and the $v_p th$ column. The R-S-K correspondence gives us a bijective correspondence between pairs of tableaux and matrices with integer entries. The matrix-ball construction allows us to go from a matrix A to a pair of tableaux (P,Q). We saw in chapter 5 that this could be done though by bumping in integers from the corresponding array w to form the pair (P,Q). We have seen that the bumping procedure is reversible, taking us from a pair of tableaux (P,Q) to the array w. We are going to show in this section how to do something similar with the matrix ball construction. We need to show that given a pair of rows $(v_1, \ldots, v_r)$ and $(u_1, \ldots, u_r)$ and a matrix B with the property described, then we can recover the matrix A.

Before we assigned a number to each ball in the diagram by considering how many balls were upper-left of it, now we will do things a little differently. We now number balls in the diagram that is associated to B in the following manner. A ball in diagram with no ball lying weakly below it or weakly to the left of it is numbered with the largest integer p such that the position in the $u_p th$ row and the $v_p th$ column is strictly above and strictly to the right of it. As soon as all the balls strictly below or strictly to the right of a given ball have been numbered, it is numbered with the largest p such that all the balls weakly below or weakly to the right of it have numbers larger than p, and such that the position in the $u_p th$ row and $v_p th$ column is strictly above and strictly to the right of it.

When we numbered our balls in the original matrix-ball construction we found that the process was much easier to follow if we considered an example and so we shall do the same here.

Example 6.9 Suppose that the rows of the tableaux are (1,1,1,1,1,1,,3,3,3,) and (1,1,1,2,3,3,3,4,4) and that the matrix B is

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

For our example we construct a table with two vertical lines and three horizontal lines as in figure 6.14 below. For the general $m \times n$ matrix we would have $n-1$ vertical lines and $m-1$ horizontal lines, in our example we have $m=4$ and $n=3$.

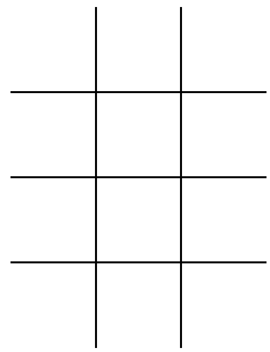


Figure 6.14: The diagram associated to B

To fill the table, consider the (i,j) entry of B which we shall denote by B_{ij}. Place this many balls in the (i,j) position of the table. In addition to this we put the number p in the (u_p,v_p) position, as this helps us with the numbering of the balls and we call these *Tableaux entry numbers.*

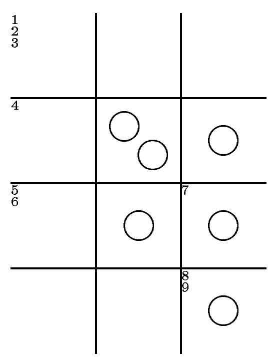


Figure 6.15: The matrix of balls before numbering

To number the balls in the diagram we start with the ball which has no other balls strictly below it or strictly to the right of it. For figure 6.15 the ball in the fourth row and third column must be numbered with a 6 as this is the largest *tableaux entry number* which is strictly left of and strictly above it. The ball in the third row and third column is numbered less than 6 as there is a ball numbered with a 6 strictly below it, also the largest *tableaux entry number* which is strictly above and to the left of it which is less than 6 is 4. Therefore the ball is numbered 4. The ball in the third row and second column is numbered less than four as there is a ball numbered with a 4 strictly to the right of it, also the largest *tableaux entry number* strictly above and to the left of it which is less than 4 is 3. Therefore the ball is numbered 3. The next few balls are easily numbered in a similar manner. Therefore we have the diagram given in figure 6.16

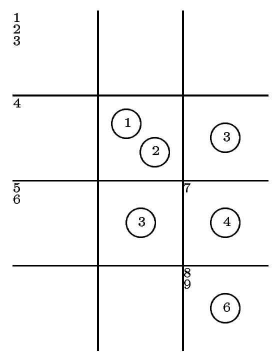


Figure 6.16: The matrix of balls after numbering

We continue with the general case and now that we have numbered our matrix of balls we still need to go further to recover our matrix A. In our matrix of balls we have several configurations of the form given in figure 6.17, which are the strings of balls of the same number p. We now replace these $l-1$ balls with l balls as follows. We start with the ball in the lowest row and we draw a vertical dotted line until we get to the row where the next ball numbered p is. We now draw a horizontal line until we meet the next ball numbered p this is repeated until we get to the ball numbered p in the highest row. From here we draw a vertical dotted line until we get to the row in which the *tableaux entry number* p is. We now draw a dotted horizontal line to the *tableaux entry numberp* and then a dotted vertical line to the lowest row in which there is a ball numbered p. Now we draw a dotted horizontal line until we meet the ball numbered with a p in this row. The diagram will all be like the one given in figure 6.18.

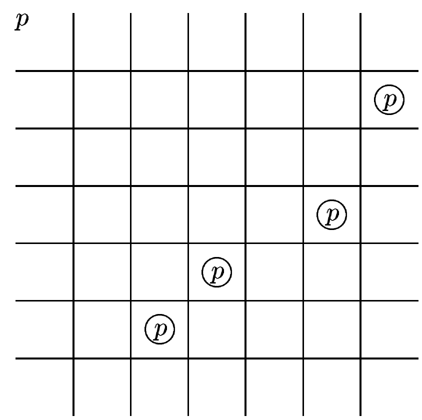


Figure 6.17:

Where the dotted lines meet at right angles we place the l balls, except at the *tableaux entry number* p, and this helps us form the matrix A. If $l = 1$ then we just place one ball in the position of the *tableaux entry number* p. We number these balls with a p and discard what was there before and we get a diagram as in figure 6.19.

We do this for each configuration with the balls numbered p and then of course sort the balls in each position into numerical order top left to bottom right. The result is a matrix of balls with the numbering that we saw in section 6.1. For our example 6.9 we get the matrix ball construction in figure 6.20. From here we can read off our matrix A.

$$A=\begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}$$

This is the same matrix that we discussed in example 5.7, which means that for our example the construction allows us to go from the tableaux pair to the matrix A which in turn gives us our original word w.

In the general case, if we are given a pair of tableaux (P,Q) and we want to find the corresponding matrix A then we first look at the last rows of P and Q. If these rows are $(v_1, \ldots, v_r)$ and $(u_1, \ldots, u_r)$ then we place a ball numbered p in the $u_p th$ row and the $v_p th$ column of a diagram. This is our matrix B and we do the construction shown in this section

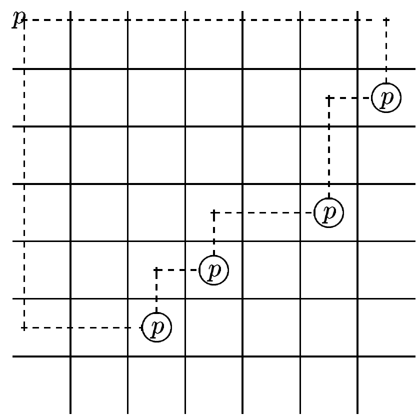


Figure 6.18:

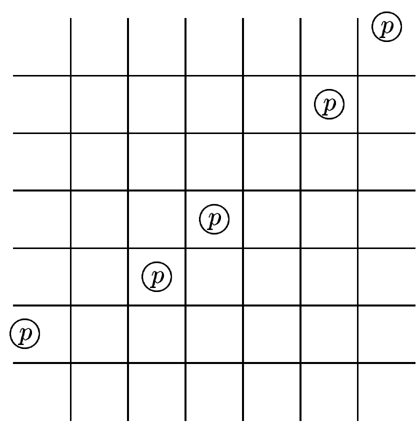


Figure 6.19:

with the second to last rows of P and Q to get a matrix A_1 say. We then take $B = A_1$ and use the third last rows of P and Q to get a new matrix A_2. We carry on doing so until we can't go any further. Once we have got to this stage we have the matrix A which will in turn give us the word w corresponding to the pair (P,Q).

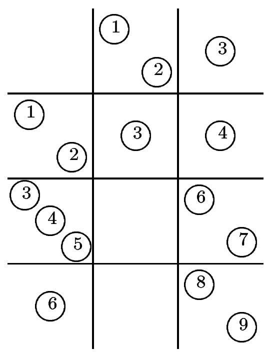


Figure 6.20:

Chapter 7

Applications of the R-S-K Correspondence

In this chapter we will be looking at some of the applications of the Robinson-Schensted-Knuth correspondence between pairs of tableaux and two rowed arrays which was stated in theorem 5.4. We will see how the Robinson correspondence allows us to count standard tableaux. We use the fact that $| S_n | = n!$ and the bijective Robinson correspondence between elements of S_n and pairs of standard young tableaux. The hook formula is very simple to use and we shall look at how we can count standard tableaux of a shape λ. We will then touch on a little representation theory by writing elements of S_n as matrices and forming a product on pairs of standard tableaux with entries from $[n]$ so that the group formed is isomorphic to the group S_n.

7.1 Counting Tableaux

Definition 7.1 *We define f^λ to be the number of standard tableaux of a given shape λ, where λ is a partition of n .*

We saw in example 5.2 which was given in chapter 5 that when $\pi \in S_3$ was an involution then the corresponding tableaux pair (P,Q) were actually the same. So if $\pi = \pi^{-1}$ then the tableaux pair would be (P,P). This can be deduced from the following more general result:

Corollary 7.2 *if $\pi \in S_n$ corresponds to the tableaux pair (P,Q) the element $\pi^{-1} \in S_n$ corresponds to the tableaux pair (Q,P).*

Proof Taking the inverse of a permutation π is the same as turning the corresponding array w upside down. The result therefore follows from the symmetry theorem 5.5. ∎

Now we can see why in example 5.2 the elements which were involutions corresponded to tableaux pairs of form (P,P). This fact will allow us to count tableaux.

Corollary 7.3 *Let Inv_n denote the set of involutions of S_n. We have a bijection between Inv_n and the set of standard tableaux of shape λ , where λ runs over all partitions of n. Counting either side gives:*

$$\sum_{\lambda \vdash n} f^{\lambda} = \mid Inv_n \mid$$

Counting the number of involutions in the group S_n is easier than at first seems as the following theorem shows.

Theorem 7.4 *Let $n \in \boldsymbol{N}$*

$$\sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{(n-2k)! \cdot 2^k k!} = \mid Inv_n \mid$$

Proof If $\pi \in S_n$ is an involution then it must have the form $\pi = (a_1 b_1) \cdot (a_2 b_2) \cdot \ldots \cdot (a_k b_k)$ where the a_i and b_i are distinct in $[n]$. There are n choices for a_1 and then $(n-1)$ choices for b_1 and $(n-2)$ choices for a_2 and then so on until we get to a_k which there are $(n-2(k-1))$ choices for. There are $(n-2k+1)$ choices for b_k and so there are $\frac{n!}{(n-2k)!}$ choices in total. What we have to remember though is that these transpositions (a_i, b_i) may be re-ordered, for example the two involutions $\pi_1 = (34)(12)$ and $\pi_2 = (34)(12)$ are the same element in S_n. Also we have that the transposition (a_i, b_i) is the same as the transposition (b_i, a_i) in S_n. For an involution $\pi \in S_n$ as above there are $k!$ different ways of writing it down in the above form due to reordering of the pairs (a_i, b_i). There are 2^k different ways to write each of these due to the reordering within the transpositions. So the number of involutions made from k transpositions is $\frac{n!}{(n-2k)! \cdot 2^k k!}$. We can do this for k running from zero (the identity element) up to $[\frac{n}{2}]$. We only go up to $[\frac{n}{2}]$ as if n is an odd number then we can only form π from $\frac{n-1}{2}$ transpositions. ∎

Lemma 7.5 *Since the number of permutations of* n *elements is* $n!$*;*

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = \mid S_n \mid = n!$$

Proof This follows from the Robinson correspondence for standard tableaux given in chapter 5. ∎

The Robinson correspondence restricts us to standard tableaux but thankfully we can look at non standard tableaux through the Robinson-Schensted correspondence. The Robinson-Schensted correspondence sets up a bijective correspondence between words w (letters from $1, 2, \ldots, m$ of length i) and ordered pairs (P,Q) of tableaux (entries from $1, 2, \ldots, n$ of the same shape λ and i boxes) with Q being standard.

Definition 7.6 *We define* $d_\lambda(m)$ *to be the number of tableaux on the shape* λ *whose entries are taken from* $[m]$.

Theorem 7.7 *From Fulton [1] Let* $n, m \in \boldsymbol{N}$

$$\sum_{\lambda \vdash n} d_\lambda(m) \cdot f^\lambda = m^n$$

Proof The number of words with n letters taken from the set $[m]$ is m^n, thus the proof follows from the Robinson-Schensted correspondence. ∎

The Robonson-Schensted-Knuth corrsespondence is concerned with pairs of tablaeux (P, Q) of the same shape λ but without the restriction that either tableaux had to be standard. If we look back to the end of chapter 5 just before example 5.8 we discussed what this meant for the matrix A which corrrespnds to the tableaux pair (P, Q). The Robinson-Schensted-Knuth correspndence tells us that the number of $m \times n$ matrices with non negative integer entries that sum to i is the sum of $d_\lambda(m) \cdot d_\lambda(n)$ over partition of i.

Theorem 7.8 *From Fulton [1]*

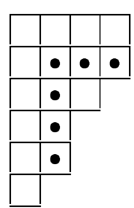
$$\sum_{\lambda \vdash i} d_\lambda(m) \cdot d_\lambda(n) = \binom{i+mn-1}{i}$$

Proof(Fulton [1]) The number of $k-$tuples $(a_1, \ldots, a_k)$ of non negative integers that sum to i is $\binom{i+k-1}{k-1} = \binom{i+k-1}{i}$. This is shown by considering each subset $\{b_{<} b_2 < \ldots < b_{k-1}\}$ of [k + i -1]. We set $b_0 = 0$ and $b_k = k + i$, and construct $(a_1, \ldots, a_k)$ by letting $a_j = b_j - b_{j-1} - 1$. ∎

7.2 Hook Length Formula

There is a very simple product formula for f^λ, the number of standard tableaux of shape λ and it involves objects called hooks. For a Young diagram of shape λ, each box determines a hook, which consists of that box and all boxes in its row to the right of the box or in its column below the box. For a box in the i, j position we define its hook to be this collection of boxes and we denote it by $H_{i,j}$. The corresponding hooklength is the size of $H_{i,j}$ so that $\mid H_{i,j} \mid = h_{i,j}$

Example 7.9 Consider the tableaux below



$H_{2,2}$ is the set of the boxes with dots in and the number of boxes with dots in is

$$\mid H_{2,2} \mid = h_{2,2} = 6.$$

Theorem 7.10 ***(Hook Formula [2])***

If $\lambda \vdash n$, *then*

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda}(h_{i,j})}$$

Suppose we want to calcualte the number of standard Young tableaux of shape $\lambda = (4,4,3,2,2,1)$ which is a partition of 16 as given in example 7.9. The hook lengths are given below in the table which we call $H(4,4,3,2,2,1)$

$$H(4,4,3,2,2,1) = \begin{array}{cccc} 9 & 7 & 4 & 2 \\ 8 & 6 & 3 & 1 \\ 6 & 4 & 1 & \\ 4 & 2 & & \\ 3 & 1 & & \\ 1 & & & \end{array}$$

Therefore

$$f^{(4,4,3,2,2,1)} = \frac{16!}{9 \cdot 8 \cdot 6 \cdot 4 \cdot 3 \cdot 1 \cdot 7 \cdot 6 \cdot 4 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 1 \cdot 2 \cdot 1} = 500500$$

Thus there are 500500 different standard young tableaux of shape $\lambda = (4,4,3,2,2,1)$

7.3 An Interesting Product

In this section we are going to show how we can form a product on pairs of tableaux. Given two pairs of standard tableaux (P_1, Q_1) and (P_2, Q_2) with entries from $[n]$ then we can form their product $(P_1, Q_1) \star (P_2, Q_2)$ which will be a new pair of tableaux (P_3, Q_3). This is done by considering the $n \times n$ square matrix A_1 which corresponds to the tableaux pair (P_1, Q_1). To form the product $(P_1, Q_1) \star (P_2, Q_2)$ we simply take the product of the two square matrices which correspond, A_1 and A_2, to give a matrix A_3 which corresponds to the tableaux pair (P_3, Q_3).

Example 7.11 Consider the group S_3 that was looked at in example 5.2. Here we found the corresponding tableaux pair (P,Q) for each element in S_3 and they are repeated below. The elements of S_3 are,

$$\pi_1 = 1\ 2\ 3 \qquad \pi_2 = 2\ 1\ 3 \qquad \pi_3 = 2\ 3\ 1$$

$$\pi_4 = 3\ 1\ 2 \qquad \pi_5 = 3\ 2\ 1 \qquad \pi_6 = 1\ 3\ 2$$

$$P(\pi_1) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \qquad Q(\pi_1) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$$

$$P(\pi_2) = \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \qquad Q(\pi_2) = \begin{array}{cc} 1 & 3 \\ 2 & \end{array}$$

$$P(\pi_3) = \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \qquad Q(\pi_3) = \begin{array}{cc} 1 & 2 \\ 3 & \end{array}$$

$$P(\pi_4) = \begin{array}{cc} 1 & 2 \\ 3 & \end{array} \qquad Q(\pi_4) = \begin{array}{cc} 1 & 3 \\ 2 & \end{array}$$

$$P(\pi_5) = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \qquad Q(\pi_5) = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$P(\pi_6) = \begin{array}{cc} 1 & 2 \\ 3 & \end{array} \qquad Q(\pi_6) = \begin{array}{cc} 1 & 2 \\ 3 & \end{array}$$

The corresponding matrices A_i to the element π_i and the tableux pair (P_i, Q_i) are found easily from π_i by writing each π_i as a word. If we consider the element $\pi_4 = 3\ 1\ 2$ the we can write this as the word $w_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ To get the matrix A_4 we simple form a 3×3 matrix whose $\begin{pmatrix} i \\ j \end{pmatrix}$ entry is the number of times $\begin{pmatrix} i \\ j \end{pmatrix}$ occurs in the array w_4. Thus we have that for each of the elements π_i we have the matrices given below.

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} A_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The nice thing about these matrices is that they have an entry in every row and every column which means that they are all nonsingular. Furthermore as each entry is 1 then the determinant will also be 1 which means that if we invert the matrices we will get another matrix back, ie $A_i^{-1} = A_j^{-1}$ for some $j \in [n]$. For our group S_3 the inverse of each of the elements are given below.

$$A_1^{-1} = A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_2^{-1} = A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_4^{-1} = A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad A_3^{-1} = A_4 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A_5^{-1} = A_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad A_6^{-1} = A_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

As matrix multiplication is associative we have a group of matrices A_i with entries from $[n]$ which is isomorphic to the group S_n by the bijective Robinson correspondence. For the moment though we will denote our group by A_{S_3} For our example 7.11 we have the Cayley table given below in figure 7.1

A_{S_3}	A_1	A_2	A_3	A_4	A_5	A_6
A_1	A_1	A_2	A_3	A_4	A_5	A_6
A_2	A_2	A_1	A_5	A_6	A_3	A_4
A_3	A_3	A_6	A_4	A_1	A_2	A_5
A_4	A_4	A_5	A_1	A_3	A_6	A_2
A_5	A_5	A_4	A_6	A_2	A_1	A_3
A_6	A_6	A_3	A_2	A_5	A_4	A_1

Figure 7.1: The Cayley table for A_{S_3}

Now we are in a position to find what the product of two standard tableaux pairs with (P, Q) with entries from $[n]$ are. We simply look at the corresponding matrices in figure 7.1. For example consider the tableaux pair $\left(\young(13,2)\ \ \young(13,2)\right)$ and the tableaux pair $\left(\young(12,3)\ \ \young(13,2)\right)$. We can now form the product $\left(\young(13,2)\ \ \young(13,2)\right) \star \left(\young(12,3)\ \ \young(13,2)\right)$, which can be found by looking at the product $A_2 \cdot A_4 = A_6$. Thus we know that $\left(\young(13,2)\ \ \young(13,2)\right) \star \left(\young(12,3)\ \ \young(13,2)\right) = \left(\young(12,3)\ \ \young(12,3)\right)$. We can continue this for all of the possible pairs of standard Young tableaux with entries from $[n]$ and for our case where $n = 3$ we have the Cayley table in figure 7.2

$\star$	$\young(123)\ \young(123)$	$\young(13,2)\ \young(13,2)$	$\young(13,2)\ \young(12,3)$	$\young(12,3)\ \young(13,2)$	$\young(1,2,3)\ \young(1,2,3)$	$\young(12,3)\ \young(12,3)$
$\young(123)\ \young(123)$	$\young(123)\ \young(123)$	$\young(13,2)\ \young(13,2)$	$\young(13,2)\ \young(12,3)$	$\young(12,3)\ \young(13,2)$	$\young(1,2,3)\ \young(1,2,3)$	$\young(12,3)\ \young(12,3)$
$\young(13,2)\ \young(13,2)$	$\young(13,2)\ \young(13,2)$	$\young(123)\ \young(123)$	$\young(1,2,3)\ \young(1,2,3)$	$\young(12,3)\ \young(12,3)$	$\young(13,2)\ \young(12,3)$	$\young(12,3)\ \young(13,2)$
$\young(13,2)\ \young(12,3)$	$\young(13,2)\ \young(12,3)$	$\young(12,3)\ \young(12,3)$	$\young(12,3)\ \young(13,2)$	$\young(123)\ \young(123)$	$\young(13,2)\ \young(13,2)$	$\young(1,2,3)\ \young(1,2,3)$
$\young(12,3)\ \young(13,2)$	$\young(12,3)\ \young(13,2)$	$\young(1,2,3)\ \young(1,2,3)$	$\young(123)\ \young(123)$	$\young(13,2)\ \young(12,3)$	$\young(12,3)\ \young(12,3)$	$\young(13,2)\ \young(13,2)$
$\young(1,2,3)\ \young(1,2,3)$	$\young(1,2,3)\ \young(1,2,3)$	$\young(12,3)\ \young(13,2)$	$\young(12,3)\ \young(12,3)$	$\young(13,2)\ \young(13,2)$	$\young(123)\ \young(123)$	$\young(13,2)\ \young(12,3)$
$\young(12,3)\ \young(12,3)$	$\young(12,3)\ \young(12,3)$	$\young(13,2)\ \young(12,3)$	$\young(13,2)\ \young(13,2)$	$\young(1,2,3)\ \young(1,2,3)$	$\young(12,3)\ \young(13,2)$	$\young(123)\ \young(123)$

Figure 7.2: The Cayley table for pairs of standard tableaux with entries from [3]

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