Planet Migration via Planetesimal Scattering

Total energy before scattering (ignoring mutual interactions)

$$E_T = -\frac{GM_* m_{p1}}{2a_1} - \frac{GM_* m_{p2}}{2a_2}$$

Now mass m_{p1} scatters a_2 inward and reduces a_2 by a factor 0.5, conserving energy:

$$E_T = -\frac{GM_* m_{p1}}{2a_1} - \frac{GM_* m_{p2}}{2a_2} = -\frac{GM_* m_{p1}}{2a_1'} - \frac{GM_* m_{p2}}{a_2}$$

$$\therefore m_{p2}\left(\frac{1}{2a_2} - \frac{1}{a_2}\right) = m_{p1}\left(\frac{1}{2a_1'} - \frac{1}{2a_1}\right)$$

Assume that $a_1 \simeq a_2$ for initial semimajor axes:

$$m_{p2}\left(\frac{1}{2a_1} - \frac{1}{a_1}\right) = m_{p1}\left(\frac{1}{2a_1'} - \frac{1}{2a_1}\right)$$

$$m_{p2}\left(\frac{a_1 - 2a_1}{2a_1^2}\right) = m_{p1}\left(\frac{2a_1 - 2a_1'}{4a_1'a_1}\right)$$

$$\therefore -\frac{M_{p2}}{2a_1} = M_{p1}\frac{(a_1 - a_1')}{2a_1'a_1}$$

$$\therefore -M_{p2} = M_{p1}\frac{(a_1 - a_1')}{a_1'}$$

$$M_{p2} = \frac{(a_1' - a_1)}{a_1'} M_{p1}$$

If mass M_{p2} is scattered inward then mass M_{p1} is scattered outward by an amount Δa_1 where

$$a_1' = a_1 + \Delta a_1$$

$$M_{p2} = \frac{\Delta a_1}{a_1 + \Delta a_1} M_{p1}$$

Now we want to calculate $\frac{\Delta a_1}{a_1}$ as a function of the scattered mass M_{p2}:

$$M_{p2} = \frac{\Delta a_1 / a_1}{1 + \Delta a_1 / a_1} M_{p1}$$

$$M_{p2}(1 + \Delta a_1/a_1) = \frac{\Delta a_1}{a_1} M_{p1}$$

$$\therefore \frac{\Delta a_1}{a_1}(M_{p2} - M_{p1}) = -M_{p2}$$

$$\therefore \frac{\Delta a_1}{a_1} = \left(\frac{M_{p2}}{M_{p1} - M_{p2}}\right)$$

If M_{p2} is a planetesimal then $M_{p1} - M_{p2} \simeq M_{p1}$ since $M_{p2} \ll M_{p1}$

$$\therefore \boxed{\frac{\Delta a_1}{a_1} \simeq \frac{m_{p2}}{m_{p1}}}$$

For substantial migration we require $\frac{\Delta a_1}{a_1} \simeq 1$, and therefore we require multiple scatterings with planetesimals such that $\sum_i m_{p2} \simeq m_{p1}$.

Migration will stop when outermost planet reaches the outer edge of the planetesimal disc.

For Solar System with Neptune at $\sim$ 30 AU we assume disc edge at $\sim$ 35 AU.

If Neptune began at 15 AU then the disc mass in planetesimals is given approximately by

$$M_D^{outer} = 2\pi \Sigma_0 \int_{15}^{35} R . R^{-3/2} \, dR = \left[4\pi \Sigma_0 R^{1/2} \right]_{15AU}^{35AU}$$

Take total disc mass $\quad M_D = 2\pi \Sigma_0 \int_0^{40} R^{1/2} dR$

$$= 0.015 \, M_\odot \text{ (MMSN).}$$

$$\therefore \; 4\pi \Sigma_0 \sqrt{40} = 0.015, \quad \Sigma_0 = \frac{0.015 \, M_\odot}{4\pi \sqrt{40}}.$$

$$\Rightarrow M_D^{outer} = 4\pi \Sigma_0 \left[\sqrt{35} - \sqrt{15} \right] \simeq 4\pi \Sigma_0 (6-4) \simeq 8\pi \Sigma_0$$

$\therefore \; M_D^{outer} \simeq 5\times10^{-3} M_\odot$

In Solids $M_D^{outer} \simeq 5\times10^{-5} M_\odot$

$1 M_\oplus = 3\times10^{-6} M_\odot$

$\therefore$ MMSN has $\sim$ 15-20 $M_\oplus$ of Solids.

Efficient Conversion of grains into larger bodies explains a Massive Kuiper belt !!