Queen Mary
University of London

M. Sc. Examination by course unit 2010

ASTM 735 Extrasolar Planets and Astrophysical Discs

Duration: 3 hours

Date and time: 27 May 2010, 18:15–21:15

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorized use of material stored in pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): R.P.Nelson

Section A: You should attempt ALL questions. Marks awarded are shown next to each question.

Question 1 The virial theorem for an isolated, self-gravitating, non-magnetised cloud of volume V is

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \;=\; 4\mathcal{K} \;+\; 2E_g \;+\; 6\int_V P\,\mathrm{d}V \;\;,$$

where I is the moment of inertia of the fluid about its centre of mass, t is time, $\mathcal{K}$ is the total kinetic energy of the fluid, E_g is the internal gravitational potential energy and P is the gas pressure.

(a) An isolated, spherical, uniform density interstellar cloud has a constant temperature, T, and chemical composition throughout its interior, and is in a state of solid body rotation with angular velocity Ω. Show that the virial theorem for this cloud can be expressed as

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \;=\; \frac{4}{5}MR^2\Omega^2 \;-\; \frac{6GM^2}{5R} \;+\; \frac{6\mathcal{R}MT}{\mu} \;\;,$$

where μ is the mean molecular weight, G is the gravitational constant and $\mathcal{R}$ is the gas constant. [4]

(b) State the condition on $\mathrm{d}^2 I/\mathrm{d}t^2$ which must be satisfied for gravitational collapse of the cloud to ensue. Explain briefly why this condition applies. [2]

(c) Show that the condition for the cloud to collapse may be written as

$$M \;>\; \frac{2R^3\Omega^2}{3G} \;+\; \frac{5\mathcal{R}TR}{\mu G} \;\;.$$

[2]

(d) The cloud has a temperature $T = 10$K, a rotation rate $\Omega = 10^{-13}$ rad s^{-1} and a radius $R = 10^{15}$ m. What is the mass of the cloud above which it will just undergo gravitational collapse ? Express your answer in units of the Solar mass. [2]

(e) If the cloud collapses to form a protostar at the centre, containing most of the mass of the cloud, estimate the size of the disc which will form around the protostar due to the rotation of the cloud. Express you answer in Astronomical Units. [2]

Question 2 This question concerns the vertical structure of an accretion disc.

(a) Beginning with the equation of motion in the direction z perpendicular to the plane of an accretion disc around a star of mass M, derive the equation of vertical hydrostatic equilibrium

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z} = -\frac{GM}{(R^2+z^2)^{3/2}}\, z \, ,$$

on the assumption that the mass of the disc is negligible compared with that of the star, where ρ is the density at a point in a cylindrical coordinate system (R, ϕ, z) centred on the star, P is the gas pressure, and G is the constant of gravitation. [2]

(b) Solve the equation of hydrostatic equilibrium for a thin ($z \ll R$), isothermal Keplerian accretion disc to show that the density ρ at a distance z from the central plane of the disc is

$$\rho(R, \phi, z) = \rho_0(R, \phi) \exp(-z^2/2H^2) \, ,$$

where $H(R, \phi) = \sqrt{RT/\mu\Omega^2}$, $\Omega(R)$ is the angular velocity at radius R, T is the temperature and μ is the mean molecular weight. [6]

(c) Show that the surface density $\Sigma(R)$ at a distance R from the central star in an axisymmetric accretion disc is given by the relation $\Sigma(R) = \sqrt{2\pi} H \rho_0(R)$. The standard result

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\, \mathrm{d}x = \sqrt{\pi}$$

may be useful. [3]

(d) If the sound speed is $c_s = \sqrt{\mathcal{R}T/\mu}$, what is the relationship between c_s, H and the angular velocity Ω for the same isothermal disc? [1]

Question 3 The drag force on a spherical particle moving through a gas when the radius a of the particle is smaller than the mean free path of gas molecules can be represented by $F_{drag} = \pi a^2 \rho\, c_s u$, where ρ is the density of the gas, c_s is the speed of sound through the gas, and u is the velocity of the particle relative to the gas.

(a) Derive from this the equation of motion for a spherical dust grain settling under gravity to the central plane of a non-turbulent Keplerian protoplanetary disc,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{3\rho\, c_s}{4\rho_{gr}\, a} v - \Omega^2 z \,,$$

where z is the distance from the plane, t is time, $v = \mathrm{d}z/\mathrm{d}t$ is the velocity of the grain relative to the gas, ρ is the density of the gas, ρ_{gr} is the density of the material of the grain, and $\Omega(R)$ is the angular velocity of the disc at a distance R from the the central star. Assume that $z \ll R$. [5]

(b) Assuming that the dust particles in question (a) fall from rest at a height above the disc midplane equal to the half-thickness H of the disc, and that they quickly reach their terminal velocity, show that the terminal velocity v_t is

$$v_t \simeq \frac{4a\rho_{gr}\Omega^2 H}{3\rho c_s} \sim \frac{8a\rho_{gr}\Omega H}{3\Sigma} \,,$$

where Σ is the local surface density. [6]

(c) Hence obtain an approximate expression for the settling time of the dust grains. [2]

Question 4 The final stages of terrestrial planet formation are thought to involve the collision and agglomeration of Mars-mass protoplanets ($M_{Mars} \simeq 0.1 M_{Earth}$). The minimum mass solar nebula model contains approximately 2.5 Earth masses of solid material interior to 1 AU, which at the beginning of the final accretion phase is expected to have agglomerated into embryos whose masses are approximately equal to that of Mars.

(a) Estimate the time required for terrestrial planet formation to be completed, assuming that the formation time scale is equal to the collision time between embryos, and the terrestrial planet zone is centred around 1 AU. You may assume that Mars-mass embryos have a radius of 3000 km. [8]

(b) The time of formation of the Earth after the formation of the Sun has been estimated using cosmochemical data. Describe briefly the method used to estimate this time scale, and explain the basic principles on which the technique is based. Comment on how well your answer in part (a) of this question agrees with the measured time for formation of the Earth. [5]

Section B: Each question carries 25 marks. You may attempt all questions but only marks for the best two questions will be counted, except for the award of a bare pass.

Question 5 Consider an axisymmetric accretion disc with surface density Σ, kinematic viscosity ν, and in which forces due to pressure and self-gravity may be neglected.

(a) Derive the result

$$R\,\Sigma\,v_R \;=\; \left(\frac{\partial(R^2\Omega)}{\partial R}\right)^{-1}\frac{\partial}{\partial R}\left(R^3\nu\Sigma\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\ ,$$

where R is the radial coordinate in the disc, Ω is the angular velocity, and v_R is the radial component of the gas velocity. In doing this, you may assume that the torque acting in the direction of speeding up the disc, due to material interior to R, is given by

$$\mathcal{T} \;=\; -\,2\pi\,R^3\,\nu\,\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\ ,$$

and the rate of change of the specific angular momentum j with time t is

$$\frac{\mathrm{d}j}{\mathrm{d}t} \;=\; v_R\,\frac{\partial j}{\partial R}$$

for an axisymmetric disc. [8]

(b) The continuity equation is $\dfrac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0$, where ρ is the density and $\mathbf{v}$ the velocity of the fluid. The divergence of a vector $\mathbf{A}$ in a cylindrical coordinate system (R,ϕ,z) is

$$\boldsymbol{\nabla}\cdot\mathbf{A} \;=\; \frac{1}{R}\frac{\partial(RA_R)}{\partial R} \;+\; \frac{1}{R}\frac{\partial A_\phi}{\partial\phi} \;+\; \frac{\partial A_z}{\partial z}$$

where A_R, A_ϕ and A_z are the components of $\mathbf{A}$ in the R, ϕ and z directions. Hence derive the disc surface density evolution equation in the form

$$\frac{\partial\Sigma}{\partial t} \;+\; \frac{1}{R}\frac{\partial}{\partial R}\left[\left(\frac{\partial(R^2\Omega)}{\partial R}\right)^{-1}\frac{\partial}{\partial R}\left(R^3\nu\Sigma\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\right] \;=\; 0\ .$$

[8]

(c) Show that if the disc is in a state of Keplerian rotation, the surface density evolution equation is

$$\frac{\partial\Sigma}{\partial t} \;=\; \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left(R^{1/2}\nu\Sigma\right)\right]\ .$$

[5]

(d) Using a dimensional analysis, show that the evolutionary timescale τ_{ev} of an accretion disc depends on the kinematic viscosity ν and the radius R of the disc as

$$\tau_{ev} \;=\; \frac{R^2}{3\nu}.$$

You may assume that the dimensions of kinematic viscosity are $[\text{length}]^2[\text{time}]^{-1}$. [2]

(e) In the alpha model of viscosity, the kinematic viscosity is represented as $\nu = \alpha c_s H$, where α is a dimensionless parameter, c_s is the sound speed, and H is the half-thickness of the accretion disc. Show that kinematic viscosity in a Keplerian disc at a distance R from a central star of mass M is given by

$$\nu \simeq \alpha \sqrt{GM} \left(\frac{H}{R}\right)^2 R^{1/2} .$$ [2]

Question 6 A rotating star has a strong, dipolar magnetic field that corotates with the star. The magnetic field threads through a surrounding circumstellar disc out to some radius. The interaction between the rotating magnetic field and the more slowly rotating parts of the disc causes these regions to be repelled from the star. This process may lead to the truncation of the disc inner edge at some distance, R_{in}, from the star.

(a) Working in cylindrical polar coordinates (R, ϕ, z), assuming that the system is axisymmetric, and that $B_R \ll B_z$, that $B_R \ll B_\phi$ and that $\partial(RB_\phi)/\partial R \ll R\partial B_\phi/\partial z$, show that the magnetic torque per unit volume acting on the disc may be written as

$$\mathcal{T}_V = \frac{RB_z}{\mu_0} \frac{\partial B_\phi}{\partial z} ,$$

where B_R, B_ϕ, and B_z are the R, ϕ, and z components of the magnetic flux density respectively. [8]

(b) Hence show that the total torque can be written

$$\dot{J} = \int_{z=-H}^{H} \int_{R=R_{in}}^{\infty} \frac{2\pi R^2 B_z}{\mu_0} \frac{\partial B_\phi}{\partial z} \, \mathrm{d}R \, \mathrm{d}z ,$$

where H is the disc semi-thickness. [4]

(c) Assuming that the magnitude of the flux density of the stellar magnetic field at a distance R from the star is given by

$$B(R) = B_* \left(\frac{R_*}{R}\right)^3 ,$$

where R_* is the stellar radius and B_* is a constant, and using simple arguments, show that an estimate of the torque is given by

$$\dot{J} = \frac{4\pi}{3\mu_0} \frac{B_*^2 R_*^6}{R_{in}^3} .$$

In doing this, the disc can be represented as having an upper surface where $B_\phi = B_\phi^+$ and a lower surface where $B_\phi = -B_\phi^+$. B_z can be assumed to be constant over z, and $B_z B_\phi^+ \simeq B(R)^2$. [6]

(d) Show further that the truncation radius of the disc, R_{in}, is given by

$$\frac{R_{in}}{R_*} = \left(\frac{4\pi B_*^2 R_*^{5/2}}{3\sqrt{GM_*}\, \dot{m}_d \, \mu_0}\right)^{2/7} ,$$

where M_* is the mass of the star and $\dot{m}_d$ is the steady-state mass flow rate through the disc. In deriving this result you may assume that the viscous torque exerted at the inner edge of the disc is

$$\dot{J}_{visc} \quad = \quad -3\pi\nu\,\Sigma\,R_{in}^2\,\Omega(R_{in}) \ ,$$

and $\dot{m}_d = 3\pi\nu\Sigma$. Here, ν is the kinematic viscosity and Σ is the disc surface density. [5]

(e) An accretion disc surrounds a rapidly rotating neutron star. The neutron star has a mass $M_* = 3 \times 10^{30}$ kg, a radius $R_* = 10^4$ m and has a strong magnetic field with a flux density $B_* = 7 \times 10^7$ Tesla at its surface. The magnetic field interacts with the more slowly rotating disc, and it truncates the disc at a radius R_{in}. If the mass transfer rate through the disc is $\dot{m}_d = 10^{15}\ \mathrm{kg\,s^{-1}}$, use the formula for R_{in}/R_* in part (a) above to estimate R_{in}. [2]

Question 7 This question concerns the observation of extrasolar planets and extraction of information from those observations.

(a) Describe three observational techniques which have been used successfully to detect extrasolar planets. For each method describe the basic principles behind the technique, and describe which properties of the planets and their orbits may be determined. [9]

(b) Describe the relative merits and drawbacks of each of the three extrasolar planet detection techniques that you have described when answering the previous subquestion. [3]

(c) Show that the relation between the observed radial velocity amplitude, v_{obs}, of a star and the mass of a planet on a circular orbit around the star may be written as

$$m_p \sin{(i)} = v_{obs} M_* \sqrt{\frac{D}{GM_*}}.$$

where m_p is the planet mass, M_* is the mass of the star, D is the orbital distance between the star and the planet. Explain why the mass of the planet is modulated by the $\sin{(i)}$ factor. [6]

(d) Absorption lines in the spectrum of a 0.3 Solar mass star are observed to red-shift and blue-shift repeatedly, with an amplitude which varies sinusoidally and which corresponds to a peak radial velocity of 2 ms^{-1}. The period of variation is 20 days. Estimate the mass of the orbiting planet, expressing your answer in units of Earth masses. [3]

(e) Photometric monitoring of the star described in the previous subquestion shows that there is a periodic reduction of the observed stellar luminosity with a period of 20 days. The degree of dimming corresponds to 0.15 percent of the stellar luminosity. What can be inferred about the physical nature of the planet given that the stellar radius is estimated to be 0.38 Solar radii. Speculate on the composition of the planet. [4]

End of Paper - A one page appendix follows

Useful information

In this paper π represents the conventional mathematical constant.
G represents the gravitational constant, with $G = 6.7 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}s^{-2}}$.
c is the velocity of light, with $c = 3 \times 10^8\ \mathrm{m\,s^{-1}}$.
The solar mass $\mathrm{M}_\odot = 2 \times 10^{30}$ kg.
The solar luminosity $\mathrm{L}_\odot = 4 \times 10^{26}$ W.
The solar radius $\mathrm{R}_\odot = 7 \times 10^8$ m.
The mass of Earth $\mathrm{M}_\oplus = 6 \times 10^{24}$ kg.
The radius of Earth $\mathrm{R}_\oplus = 6.3 \times 10^6$ m.
The Stefan-Boltzmann constant $\sigma = 6 \times 10^{-8}\ \mathrm{J\ s^{-1}\ m^{-2}\ K^{-4}}$.
The permeability of free space is $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$.

The isothermal sound speed c_s in a gas is related to the temperature T by

$$c_s \;=\; \sqrt{\frac{\mathcal{R}T}{\mu}}\ ,$$

where $\mathcal{R}$ is the gas constant and μ is the mean molecular weight.
$\mathcal{R} = 8.3\ \mathrm{J\ mol^{-1}\ K^{-1}}$ and you may assume that $\mu = 2 \times 10^{-3}\ \mathrm{kg\ mol^{-1}}$.

The ideal gas law relates the pressure P, temperature T, and density ρ at a point in an ideal (perfect) gas by

$$P \;=\; \frac{\mathcal{R}}{\mu}\,\rho\,T.$$

The internal gravitational potential energy E_g of a uniform sphere of mass M and radius R is

$$E_g \;=\; -\,\frac{3}{5}\,\frac{GM^2}{R}.$$

The moment of inertia of a uniform sphere of mass M and radius R is

$$I \;=\; \frac{2}{5}MR^2.$$

You may find useful the vector identities

$$\begin{aligned}
(\mathbf{A}\,.\,\boldsymbol{\nabla})f \;&\equiv\; \boldsymbol{\nabla}.(f\mathbf{A}) \;-\; f\,\boldsymbol{\nabla}.\mathbf{A} \\
\int_V (\boldsymbol{\nabla}.\mathbf{A})\,\mathrm{d}V \;&\equiv\; \int_S \mathbf{A}.\mathrm{d}\mathbf{S} \qquad \text{(Divergence theorem)} \\
\boldsymbol{\nabla}.\mathbf{r} \;&\equiv\; 3 \\
\boldsymbol{\nabla} \times \mathbf{A} \;&=\; \hat{\mathbf{e}}_{\mathbf{R}} \left(\frac{1}{R}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \;+\; \hat{\mathbf{e}}_{\boldsymbol{\phi}} \left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R} \right) \\
&\quad +\; \hat{\mathbf{e}}_{\mathbf{z}}\,\frac{1}{R} \left(\frac{\partial}{\partial R}(RA_\phi) - \frac{\partial A_R}{\partial \phi} \right) \ .
\end{aligned}$$

for any vector $\mathbf{A}$, scalar f and position vector $\mathbf{r}$.