Umbral groups

Robert A. Wilson

LMS–EPSRC Durham Symposium, 4th August 2015

1 Introduction

Background When I was first asked to give a talk about umbral groups, I declined, on the grounds that I did not know what umbral groups were. The organisers then sent me two papers about umbral groups, totalling 299 pages, and persuaded me to reconsider. These two papers gave me two different answers to the question of what umbral groups were, despite the fact that the two papers were written by the same three people. The first told me there were six umbral groups, and the second told me there were 23. It was at this point that I realised that nobody knows what an umbral group is. Neither of these papers contains a definition of umbral group: in both cases they are defined by the list of examples.

Six umbral groups In the beginning, there was one umbral group, namely the Mathieu group M_{24}. But wherever M_{24} goes, M_{12} or its double cover is sure to follow. And before long there were 6 umbral groups, one for each divisor of 12.

Group	d	$l = d + 1$	$n = 24/d$
$1.M_{24}$	1	2	24
$2.M_{12}$	2	3	12
$2.2^3L_3(2)$	3	4	8
$2.S_5$	4	5	6
$2.A_4$	6	7	4
2.2	12	13	2

Except for M_{24} itself, all these groups have a central involution, and the quotient group has a natural permutation representation on $24/d$ points.

Notice that $2^3L_3(2) = AGL_3(2)$ and $S_5 = PGL_2(5)$, acting on 8 and 6 points respectively. You may well ask, why divisors of 12, not 24? And you'd be right to ask that question. Indeed, it seems that not only can you extend to divisors of 24, but that it is possible to define an umbral group for each Niemeier lattice.

Niemeier lattices The Niemeier lattices are defined as the even ($x.x$ is even) unimodular (one point per unit volume) integral lattices ($x.y$ is an integer) in 24 dimensions (of which there are 24, including the Leech lattice). As the Leech lattice is special in so many ways, it is usually treated separately, and the other 23 are collectively known as the Niemeier lattices. They all have roots (that is vectors of norm 2 with the property that reflections in these roots are automorphisms of the lattice), but, except in one case, are not spanned by the roots. Irreducible root systems are classified by the A/D/E Dynkin diagrams, and the property that characterises the root systems of Niemeier lattices is that the Coxeter numbers of all the components are equal. If I understand correctly, the Coxeter number is known here as the *lambency*, although I did not find a definition of lambency, rather a list of examples.

23 umbral groups For ease of reference, I will divide the Niemeier lattices into pure A, pure D/E, and mixed types. The umbral group here is the automorphism group of the lattice, modulo the normal subgroup generated by reflections in the roots.

Roots	Group	ℓ	Roots	Group	ℓ	Roots	Group	ℓ
A_1^{24}	$1.M_{24}$	2	$A_5^4D_4$	$2.S_4$	6	D_4^6	$3.S_6$	6
A_2^{12}	$2.M_{12}$	3	$A_7^2D_5^2$	2.2^2	8	D_6^4	$1.S_4$	10
A_3^8	$2.2^3L_3(2)$	4	$A_9^2D_6$	2.2	10	D_8^3	$1.S_3$	14
A_4^6	$2.S_5$	5	$A_{11}D_7E_6$	2.1	12	D_{12}^2	1.2	22
A_6^4	$2.A_4$	7	$A_{15}D_9$	2.1	16	D_{24}	1.1	46
A_8^3	$2.S_3$	9	$A_{17}E_7$	2.1	18	$D_{10}E_7^2$	1.2	18
A_{12}^2	2.2	13				$D_{16}E_8$	1.1	30
A_{24}	1.2	25				E_6^4	$2.S_4$	12
						E_8^3	$1.S_3$	30

2 The Leech lattice

Deep holes Now the Niemeier lattices themselves look like a fairly motley crew, but the best way to make sense of them is via the Leech lattice. What distinguishes the Leech lattice from the rest of the Niemeier lattices is the fact that it has no roots: the smallest norm of a lattice vector is therefore 4. (NB I use norm here in the sense of *squared* length.) It is therefore possible to make a right-angled triangle with hypotenuse of length 2, and endpoints two neighbouring lattice vectors, and the other two sides of length $\sqrt{2}$. It turns out, in fact, that you can do this in such a way that the right angle of the triangle is at least $\sqrt{2}$ away from *every* lattice vector, in exactly 23 different ways, one corresponding to each of the Niemeier lattices. It is therefore possible to study all the Niemeier lattices at once inside the Leech lattice. These points are at maximum distance from the lattice vectors, so are called *deep holes*. The deep holes were classified by Conway, Parker and Sloane.

Twenty-three constructions Conway and Sloane went on to give 23 constructions of the Leech lattice, one from each Niemeier lattice. The original construction of Leech is equivalent to the Conway–Sloane construction from the A_1^{24} Niemeier lattice. The construction from A_2^{12} is more or less the same as the construction of the complex Leech lattice, whose automorphism group is $6.Suz$. The construction from D_4^6 corresponds to a quaternionic version of the Leech lattice which I also studied in my thesis. The construction from E_8^3 was described in a great deal more detail by Lepowsky and Meurman (1982), but the corresponding octonionic construction had to wait a good deal longer (W., 2010).

Conway's group Moreover, the umbral groups appear not only as quotients of the automorphism groups of the Niemeier lattices, but also as quotients of subgroups of the automorphism group of the Leech lattice. It seems likely, therefore, that a unified theory of umbral groups is more likely to come from studying the Leech lattice itself, and perhaps from studying subgroups of its automorphism group, that is Conway's group $2.Co_1$. And now it becomes clear why I have been asked to give this lecture, since my PhD thesis was mainly about classifying the maximal subgroups of Conway's group.

Some subgroups At least seven of the deep holes in the Leech lattice are associated to maximal subgroups.

Deep hole	Subgroup	Code over:	Length	Dimension
A_1^{24}	$2^{12}M_{24}$	$\mathbb{F}_2$	24	12
A_2^{12}	$2 \times 3^6.2M_{12}$	$\mathbb{F}_3$	12	6
A_4^6	$2.5^3.(4 \times A_5).2$	$\mathbb{F}_5$	6	3
A_6^4	$2 \times 7^2.(3 \times 2A_4)$	$\mathbb{F}_7$	4	2
D_4^6	$2^{5+12}(S_3 \times 3S_6)$	2_+^{1+4}	6	3
E_6^4	$2.3^{3+4}.2(S_4 \times S_4)$	3_+^{1+2}	4	2
E_8^3	$2^{3+12}(A_8 \times S_3)$	2_+^{1+6}	3	2

The first two are associated to the well-known Golay codes, over the fields of order 2 and 3 respectively, the next two to some less interesting codes over the fields of orders 5 and 7. The last three I described in terms of 'codes' over certain extraspecial groups.

Umbral series It is now understood that the best way to construct M_{24} and the binary Golay code is from the hexacode, which corresponds to the D_4^6 lattice and the umbral group $3S_6$. Moreover, the hexacode itself can be constructed from the umbral group S_3 associated to the E_8^3 lattice.

In a similar way, the best way to construct the ternary Golay code and the umbral group $2M_{12}$ is from the tetracode and the umbral group $2S_4$, corresponding to the E_6^4 lattice. Since this case is less well-known, I will concentrate on this one.

3 3-local structure

The tetracode The basic version is a length 4, dimension 2, linear code over the integers modulo 3. We will write $0, +, -$ for $0, 1, 2$ respectively, and often omit the 0s. The codewords are $(0,0,0,0)$, $\pm(0,+,+,+)$, $\pm(+,0,+,-)$ and those obtained by rotating the last three coordinates. The umbral group acts as automorphisms of this code: the central involution negates all codewords, there is a 3-cycle as already described, and a tranposition which swaps the first two coordinates and negates the fourth coordinate. Thus the group is a double cover of S_4, which acts on the code as $GL_2(3)$.

The ternary Golay code First draw an array with three rows and four columns: the columns correspond to the coordinate positions of the tetracode, and the rows to the coordinate values $0, +, -$. Each of the nine tetracode words now corresponds to a set of four positions in the array. Now put entries $0, +, -$ in the array, subject to the following rules:

- the sum of the four entries in any tetracode word is the same;
- this equals the negative of the sum of the three entries in any column.

The symmetry group and the codewords There is a group 3^2 acting on the columns in the same way as the tetracode, together with the automorphism group of the code. This gives a group $3^2.GL_2(3) = AGL_2(3)$ permuting the 12 entries.

One can now write down all the 3^6 codewords. The weight distribution is $0^1 6^{264} 9^{440} 12^{24}$, and the weight 12 words are: 6 which have $+$ on two columns and $-$ on the other two; and 9 which have $+$ on a tetracode word and $-$ elsewhere; together with their negatives. These are enough to span the code, so if you are bored you can work out the rest.

It can be shown that the full automorphism group of the code (i.e. monomial permutations of the 12 entries) is $2.M_{12}$.

Up to the next level We can again construct an extension of the Golay code, of order 729, by the automorphism group, to obtain a group of shape $3^6{:}2M_{12}$. Rather than construct this as a permutation group on 729 points, however, we can construct it as a group of 12×12 complex matrices, by writing the code multiplicatively, using the group $\{1, \omega, \bar{\omega}\}$ of complex cube roots of unity instead of the additive group $\mathbb{Z}_3$.

Writing $\theta = \omega - \bar{\omega} = \sqrt{-3}$, we take coordinates in $\mathbb{Z}[\omega]$ which satisfy

- $z_i \equiv m \bmod \theta$,
- $\sum_i z_i \equiv -3m \bmod 3\theta$,
- $(z_i - m)/\theta \bmod \theta$ is a word in the ternary Golay code.

With a suitable interpretation, this is the Leech lattice.

Leech lattice vectors The minimal vectors have shape:

- $(3, -3, 0^{10})$, of which there are $12.11.3^2 = 1188$;
- $(\theta^6, 0^6)$, of which there are $264.3^5 = 64152$;
- $(-2^2, 1^{10})$, of which $12.11.3^6 = 96228$;
- $(-2 - 3\omega, 1^{11})$ and complex conjugates, of which $2^2.12.3^6 = 35992$.

Return to E_6 The minimal vectors which lie in one column of the 3×4 array are 54 of shape $(3, -3, 0)$, and correspond to the minimal vectors of the dual lattice of E_6. The roots can be obtained from the next norm up, that is $(3\theta, 0, 0)$ and $(3, 3, 3)$, of which there are $18 + 54 = 72$. Modulo the scalars of order 3 on each column, we see both permutations and diagonal elements $(1, \omega, \bar{\omega})$, made up of tetracode words across the four columns. In this sense we have blown up the tetracode so that the individual coordinates lie in the group 3_+^{1+2}. There is also a group $2S_4$ acting as outer automorphisms of 3^{1+2}, in the same way on all four columns.

I do not know if this part of the group might also be of use in the study of umbral groups. Notice that $3^{1+2}2S_4$ is a maximal subgroup of the Weyl group of E_6.

4 2-local structure

The tricode group A similar procedure to the above can be used to construct the binary Golay code from the hexacode, and the hexacode from an even easier code. The construction of the complex Leech lattice using the A_2^{12} hole relies on the fact that the roots of A_2 are a scaled copy of the units of the Eisenstein integers $\mathbb{Z}[\omega]$. In a similar way, the roots of D_4 are a scaled copy of the units of the Hurwitz ring $\mathbb{Z}[i, j, k\omega = \frac{1}{2}(-1+i+j+k)]$ of integral quaternions.

I will spare you the details, but perhaps just describe what happens when we return to E_8^3 at the end of the construction. Fixing the three blocks of 8 coordinates, we restrict to the group $2^{3+12}A_8$, which is made out of an almost-maximal subgroup $2^{1+6}A_8$ of the Weyl group of type E_8 in much the same way that the E_6^4 case was made out of $3^{1+2}2S_4$. There are three copies of 2^{1+6}, acting on the three blocks, and the condition on the three elements is simply that their product is ± 1.

An octonionic version It is also possible to use the Coxeter–Dickson integral octonions, whose units are a scaled copy of the roots of E_8, to give a construction of the Leech lattice with triples of octonions. This is tricky, however. I tried it in 1980, and wrote down 196560 triples of octonions which I thought corresponded to the minimal vectors of the Leech lattice. They were wrong, however. Geoffrey Dixon in the mid 1990s wrote down the same wrong list of triples, and claimed they were the Leech lattice. (The easiest way to see it is not the Leech lattice is to note that Dixon's set of vectors is invariant under a group of order 7^3, but there is no such subgroup of Conway's group.) Others also tried and failed.

The difficulty can perhaps be traced to the fact that such a construction must contain a hidden construction of the binary Golay code. In Dixon's version, and my earlier versions, the code that emerged was instead the sum of three copies of the Hamming code of length 8. The lattice spanned by our vectors was presumably the Niemeier lattice of type E_8^3. It took me months of hard work, and many hundreds of pieces of paper, to get a correct version, which I was then able to simplify as follows.

E_8 in octonions We construct multiple copies of E_8 in octonions. First take the octonions spanned by $1 = i_\infty$, $i_0, \ldots, i_6$, with subscripts modulo 7 and multiplication rules defined by i_t, i_{t+1}, i_{t+3} behaving like quaternions i, j, k. Now take L to be the lattice spanned by $1 \pm i_t$ and $s = \frac{1}{2}(-1 + i_0 + \cdots + i_6)$, and R to be $\bar{L}$. Define $B = LR/2$. Then B is spanned by units, but is not closed under multiplication. However, it turns out that $BL = L$ and $RB = R$, which is sufficient for our purposes.

An octonionic Leech lattice Define Λ to be the set of triples $(x, y, z) \in L^3$ satisfying

- $x + y, y + z \in L\bar{s}$;
- $x + y + z \in Ls$.

Then it can be shown that Λ is the Leech lattice.

What is much more interesting, however, is that many of the maximal subgroups of Conway's group can be described in terms of the action of certain 3×3 octonionic matrices. But that is a story for another day.