Octonions

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1 Introduction

This is the second talk in a projected series of five. I shall try to make them as independent as possible, so that it will not be necessary to attend talk n in order to understand talk $n+1$. However, there are connections between the talks, so it will be helpful to have seen the previous talks.

Reflection groups. We saw last week that reflection groups in 2 dimensions can be described by complex numbers, and in 3 and 4 dimensions can be described in terms of quaternions.

We also saw that the exceptional series of reflections groups stop in dimension 2 (the dihedral groups of order at least 12) or 4 (the groups F_4 and H_4) or 8 (the group E_8).

Composition algebras. The doubling of dimensions from the real numbers to the complex numbers to the quaternions can be taken one stage further, to the *octonions* (or Cayley numbers).

From $\mathbb{R}$ to $\mathbb{C}$, we lose the ordering.

From $\mathbb{C}$ to $\mathbb{H}$, we lose commutativity.

From $\mathbb{H}$ to $\mathbb{O}$, we lose associativity, as we shall see.

If you try to double the dimension again, you lose the multiplicative property of the norms.

2 Basics of octonions

The projective plane of order 2 can be labelled so that the points are labelled by $0, 1, 2, 3, 4, 5, 6 \in \mathbb{Z}/7\mathbb{Z}$, and the lines are $\{t, t+1, t+3\}$.

Define $\mathbb{O} = \mathbb{R}[i_0, i_1, \ldots, i_6]$ such that the lines are quaternion algebras, in the sense that (i_t, i_{t+1}, i_{t+3}) multiply together like (i, j, k) in the quaternions. Thus $i_0 i_1 = -i_1 i_0 = i_3$ and images under the subscript permutations $t \mapsto t+1$ and $t \mapsto 2t$. (There is another symmetry $(1,3)(2,6)(0,\overline{0})(4,\overline{4})$, that is the subscript permutation $(1,3)(2,6)$ followed by negating i_0 and i_4, which is not quite so easy to see. Together these generate a non-split group $2^3L_3(2)$.)

Observe that $i_0(i_1i_2) = i_0i_4 = i_5$ but $(i_0i_1)i_2 = i_3i_2 = -i_5$, so the octonions are non-associative.

Octonion conjugation is the linear map which fixes 1 and negates all i_t. Thus if $x = a_\infty + \sum_{t=0}^{6} a_t i_t$ then $\overline{x} = a_\infty - \sum_{t=0}^{6} a_t i_t$. We have $\overline{xy} = \overline{y}.\overline{x}$ as in the quaternions, because i and j anticommute.

The *norm* is $N(x) = x\overline{x}$ and satisfies $N(xy) = N(x)N(y)$ (needs to be proved!). The associated inner product is

$$(x,y) = \frac{1}{2}(N(x+y) - N(x) - N(y)) = \frac{1}{2}(x\overline{y} + y\overline{x}) = \Re(x\overline{y}).$$

A trilinear form. The algebra product gives a trilinear form (or *scalar triple product* in applied mathematical language).

$$T(x,y,z) = (xy, \overline{z}) = \Re((xy)z)$$

But if x has norm 1 then left multiplication by $\overline{x}$ preserves norms, and therefore inner products, so this is also equal to

$$(y, \overline{x}.\overline{z}) = (y, \overline{zx}) = \Re(y(zx))$$

and (using right-multiplication by $\overline{y}$) to

$$(x, \overline{z}.\overline{y}) = (x, \overline{yz}) = \Re(x(yz))$$

so we can remove the brackets in the triple products, and get a cyclically symmetric trilinear form.

The units $\pm 1, \pm i_t$ form a 'non-associative group', or more precisely a *Moufang loop*. In place of associativity we have the three *alternative laws*

$$\begin{aligned} (xy)x &= x(yx) \\ x(xy) &= (xx)y \\ (yx)x &= y(xx) \end{aligned}$$

and the three *Moufang laws*

$$\begin{aligned} (xy)(zx) &= x(yz)x \\ x(y(xz)) &= (xyx)z \\ ((xy)z)y &= x(yzy) \end{aligned}$$

Indeed, all these laws hold in the whole of $\mathbb{O}$, though it takes a certain amount of effort to prove them.

In particular, the alternative laws imply that any 2-generator subalgebra is associative (in general it is a copy of the quaternions $\mathbb{H}$).

3 Reflections

Reflections in 8-space can be encoded by the octonions. Reflection in 1 is the map

$$x \mapsto -\overline{x},$$

so reflection in r is the map

$$x \mapsto -r\overline{\overline{r}x} = -r\overline{x}r$$

and the product of these two reflections in the rotation

$$x \mapsto rxr.$$

In particular, these bimultiplications generate $SO(8)$.

Left-multiplications by octonions of unit norm also preserve norms (and therefore inner products), and also generate $SO(8)$. But in a fundamentally different way, because -1 now acts as -1 instead of as $+1$.

Similarly for right-multiplications.

Triality is the name given to the connection between left-, right- and bi-multiplications by units. To see this connection more clearly, consider the set of triples (x, y, z) of octonions with $xyz = 1$. A triple of orthogonal linear maps (α, β, γ) is called an *isotopy* if

$$xyz = 1 \Rightarrow x^\alpha y^\beta z^\gamma = 1.$$

In particular if $u\overline{u} = 1$ then $(L_u, R_u, B_{\overline{u}})$ is an isotopy by the first Moufang law. [**Proof:** If $xyz = 1$ then

$$\begin{aligned} ((ux)(yu))(\overline{u}z\overline{u}) &= (u(xy)u)(\overline{u}z\overline{u}) \\ &= uz^{-1}u\overline{u}z\overline{u} \\ &= 1 \end{aligned}$$

since the algebra generated by u and z is associative.]

Now $xyz = 1$ is equivalent to $yzx = 1$, so if (α, β, γ) is an isotopy then so is (β, γ, α). In particular, $(R_u, B_{\overline{u}}, L_u)$ and $(B_{\overline{u}}, L_u, R_u)$ are isotopies.

If (α, β, γ) is an isotopy then so is $(-\alpha, -\beta, \gamma)$. But these are the only two isotopies of the form $(-, -, \gamma)$.

Proof We may assume $\gamma = 1$. Suppose that $1^\alpha = a$, necessarily of norm 1. Applying the definition of isotopy to the triple $(x, y, z) = (1, 1, 1)$ we have $1 = 1^\alpha 1^\beta 1 = a1^\beta$ so $1^\beta = \overline{a}$. Next, taking $(x, 1, x^{-1})$ we have

$$1 = x^\alpha 1^\beta x^{-1} = x^\alpha \overline{a} x^{-1}$$

so $x^\alpha = xa$ and so $\alpha = R_a$. Similarly, taking $(1, y, y^{-1})$ we have

$$1 = 1^\alpha y^\beta y^{-1} = a y^\beta y^{-1}$$

so $y^\beta = \overline{a} y$ and so $\beta = L_{\overline{a}}$.

Next we show that a is real: otherwise we can find x and w such that $(xa)w \neq x(aw)$. There exists y such that $\overline{a}y = w$, so

$$\begin{aligned} xy &= x((a\overline{a})y) \\ &= x(a(\overline{a}y)) \\ &\neq (xa)(\overline{a}y) \end{aligned} \tag{1}$$

In other words we have found x, y, z with $xyz = 1$ but $(xa)(\overline{a}y)z \neq 1$, which contradicts the assumption that $(R_a, L_{\overline{a}}, 1)$ is an isotopy.

Thus there are exactly two isotopies for every γ. The group of isotopies is a double cover of $SO(8)$, namely the spin group $Spin(8)$.

4 E_8 in the octonions

As we saw last week, we can make D_4 inside $\mathbb{H}$ by taking the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ and adjoining $\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$, to get a unit group $SL_2(3) \cong 2.A_4$.

If we do the same thing to $\mathbb{O}$ to make E_8, we take $\frac{1}{2}(\pm 1 \pm i_t \pm i_{t+1} \pm i_{t+3})$, and the 'complements' $\frac{1}{2}(\pm i_{t+2} \pm i_{t+4} \pm i_{t+5} \pm i_{t+6})$. As we saw last time, this is a copy of E_8, but it is not closed under multiplication: you can check by multiplying together the last element with $t = 0$ and $t = 1$:

$$\frac{1}{2}(1 + i_0 + i_1 + i_3).\frac{1}{2}(1 + i_1 + i_2 + i_4) = \frac{1}{2}(i_1 + i_3 + i_4 + i_6).$$

If this thing was an algebra I'd call it A. But it's not, so I'll call it B.

Reflecting in $\frac{1}{\sqrt{2}}(1 - i_0)$ has the effect of swapping 1 with i_0, and is exactly what we need to correct this closure problem. Equivalently, bimultiply by $\frac{1}{\sqrt{2}}(\pm 1 \pm i_0)$. (Indeed, we can bimultiply by $\frac{1}{\sqrt{2}}(\pm 1 \pm i_t)$ for any t: but we need to choose the t we are going to use, and stick with it.)

To prove the result is closed under multiplication, observe first that it is still invariant under the symmetries on subscripts: $t \mapsto 2t$ and $(1,3)(2,6)(0,\overline{0})(4,\overline{4})$. We only need to check multiplication by the $\frac{1}{2}^4$ type elements, and by symmetry we only need to check $\frac{1}{2}(1 + i_0 + i_1 + i_3)$ and $\frac{1}{2}(i_0 + i_1 + i_2 + i_4)$. Multiplying these by i_t is easy, so the only non-trivial case is to multiply these two together. The answer is $\frac{1}{2}(-1 + i_1 + i_4 + i_6)$.

So this gives us a (non-associative) ring. Let's call it $\mathbb{E} = \mathbb{E}_0$, because geometrically it is a copy of E_8.

Indeed, its units are the 240 roots of E_8 (which we saw last time), so this gives us a Moufang loop of order 240.

5 More copies of E_8

The set $(\pm 1 \pm i_0)B = \mathbb{E}(\pm 1 \pm i_0)$ has a nice description.

- $(\pm 1 \pm i_0)(\pm i_t) = \pm i_t \pm i_u$ where $\{0, t, u\}$ is a line of the projective plane.
- $(\pm 1 \pm i_0)(\frac{1}{2}(\pm 1 \pm i_0 \pm i_t \pm i_u))$ is also in a quaternion subalgebra, and we showed last week that this is of the form $x + y$ for $x, y \in Q_8$.
- $(1 + i_0)\frac{1}{2}(1 + i_1 + i_2 + i_4) = \frac{1}{2}(1 + i_0 + i_1 + i_2 + i_3 + i_4 + i_5 + i_6)$, and hence we get all even sign combinations. (Recall last week.)

Call this lattice R. It is again a copy of E_8, with roots now of norm 2.

Similarly $B(\pm 1 \pm i_0) = (\pm 1 \pm i_0)\mathbb{E}$ is just the same except that we get the odd sign combinations. Call this lattice L. Again it is a copy of E_8 with roots of norm 2.

6 Properties of these lattices

Using the Moufang laws it is quite easy to prove the following remarkable results:

$$\begin{aligned} LR &= 2B \\ BL &= L \\ RB &= R \end{aligned}$$

Proof

$$\begin{aligned} LR &= ((1 + i_0)\mathbb{E})(\mathbb{E}(1 + i_0)) \\ &= \{((1 + i_0)x)(y(1 + i_0) \mid x, y \in \mathbb{E}\} \\ &= \{(1 + i_0)(xy)(1 + i_0) \mid xy \in \mathbb{E}\} \\ &= 2B \end{aligned}$$

The other two are similar, using the other two Moufang laws.