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The weird and wonderful world of octonions

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Introduction. Unlike most of the seminars this term, my talk does not contain anything new, or even any of my own work. I chose the topic of octonions as I thought it would be a topic that few people knew about, but after I've been here a few months I find that there are several experts in the subject here. As far as I can tell, they tend to use real (or complex) octonions as a way of studying the exceptional Lie algebras, and hence a number of associated physical theories. But one can also study octonions over finite fields, and use them to study the *finite* exceptional groups of Lie type.

Complex numbers. I'll start by introducing the real octonions. Recall first the construction of the complex numbers from the real numbers. We introduce a new symbol i and define $\mathbb{C} = \{a+bi | a, b \in \mathbb{R}\}$, with *multiplication* defined by $(a+bi)(c+di) = (ac-bd)+(bc+ad)i$, and a complex *conjugation* defined by $\overline{a+bi} = a - bi$. We need to define conjugation like this in order for the *norm* to satisfy $N(a+bi) = a^2 + b^2 = (a+bi)(\overline{a+bi})$.

Quaternions. Now we can construct the quaternions from the complex numbers in a similar way. We introduce a new symbol j and define $\mathbb{H} = \{a + bj | a, b \in \mathbb{C}\}$, with *multiplication* defined by $(a + bj)(c + dj) = (ac - \overline{d}b) + (b\overline{c} + da)j$, a quaternion *conjugation* defined by $\overline{a + bj} = \overline{a} - bj$, and a norm $N(a + bj) = N(a) + N(b) = (a + bj)(\overline{a + bj})$. Notice that we had to introduce some complex conjugations into the multiplication formula we had before. This is because $N(a) + N(b) = a\overline{a} + b\overline{b}$ so to get the cross-terms to cancel out we need $bj\overline{a} = -a\overline{bj}$.

The usual notation for the quaternions has $k = ij$ and our multiplication formula includes the cases $ji = (1\bar{i})j = -ij$ and $jk = -\bar{i}1 = i$ and so on. In particular, the quaternions are non-commutative. They have gone out of fashion somewhat, but they are the origin of the familiar 3-dimensional vector calculus with unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and cross-products $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and so on.

Octonions. In fact we can go one step further and construct the (real) octonions from the quaternions by introducing a new symbol ℓ and defining a *multiplication* on $\mathbb{O} = \{a + b\ell | a, b \in \mathbb{H}\}$ by $(a + b\ell)(c + d\ell) = (ac - \bar{d}b) + (b\bar{c} + da)\ell$. We also have an octonion *conjugation* defined by $\overline{a + b\ell} = \bar{a} - b\ell$, and a norm $N(a + b\ell) = N(a) + N(b) = (a + b\ell)(\overline{a + b\ell})$. Notice that because the quaternions are non-commutative, the order of multiplication in the coefficients is important. Check that we have got it right by multiplying out $(a + b\ell)(\bar{a} - b\ell) = (a\bar{a} + \bar{b}b) + (ba + (-b)a)\ell = N(a) + N(b)$ as required.

In the octonions, however, we find some strange things going on. Let us try to work out the product $i(j\ell)$. The multiplication formula tells us that this comes to $(ji)\ell$, which is $(-ij)\ell$. So the octonions are not associative! In any case, we have an algebra of sorts which has dimension 8 over $\mathbb{R}$, and for convenience I am going to take the basis $\{1, i_0, \ldots, i_6\}$ defined by $i_0 = i$, $i_1 = j$, $i_3 = ij = k$, $i_2 = \ell$, $i_4 = j\ell$, $i_5 = -k\ell$ and $i_6 = -i\ell$. (We'll see the reason why in a moment.)

Notice that any two of these i_n anti-commute, and therefore generate a quaternion subalgebra. Moreover, we have chosen the basis so that $i_n i_{n+1} = i_{n+3}$ for each n, with subscripts read modulo 7: let's check one, say $i_5 i_6 = (-k\ell)(-i\ell) = -i(-k) = j = i_1$.

Laws in the octonions. Associativity is such a fundamental part of life that we tend to take it for granted, and use it all the time without noticing it. Without it we feel lost. We need something to replace it. For example, Lie algebras are non-associative, but the *Jacobi identity* $a(bc) + b(ca) + c(ab) = 0$ is an adequate replacement. We need to identify (no pun intended) some similar law or laws in the octonions which we can use instead of associativity.

One important law is that $N(ab) = N(a)N(b)$. You are familiar with this in the complex numbers, and possibly in the quaternions. (If not, check that $N((a + bj)(c + dj)) = N((ac - \bar{d}b) + (b\bar{c} + da)j) = N(ac - \bar{d}b) + N(b\bar{c} + da) = ac\overline{ac} + bd\overline{bd} - b\overline{acd} - acd\bar{b} + bc\overline{bc} + ad\overline{ad} + acd\bar{b} + b\overline{acd} = N(a + bj)N(c + dj)$.)

We can check this law in the octonions also (but not by this method, because it uses associativity).

It is easy to see that any two pure imaginary octonions anticommute iff they are orthogonal, and deduce that any two octonions whatsoever lie in a quaternion subalgebra, and therefore associate. In particular we have the so-called *alternative* laws, $x(xy) = x^2y$ and $x(yx) = (xy)x$ and $(yx)x = yx^2$.

The Moufang law. But we need a law involving three variables to replace the associative law. This is the so-called *Moufang* law, $(xy)(zx) = x(yz)x$ (note that the latter is well-defined, by the alternative law). To verify this, note that it is linear in y and z, so only needs to be verified for y and z being basis vectors. Moreover, it is true if y or z is real, and by symmetry we may assume $y = i$ and $z = j$ if we like. You can then verify the identity by direct calculation, although this is still rather tedious.

The Lie group of type G_2. With these tools at our disposal we are now able to discover lots of interesting things about the octonions. For example, if i', j' and ℓ' are three mutually orthogonal purely imaginary octonions of norm 1, and ℓ' is also orthogonal to $i'j'$, then the multiplication table in terms of i', j' and ℓ' is uniquely determined by the Moufang law, and is identical to that already given in terms of i, j, and ℓ. In particular there is a (unique) automorphism of the octonions mapping (i, j, ℓ) to (i', j', ℓ'). Thus the automorphism group is a Lie group of dimension $6 + 5 + 3 = 14$. This is G_2. (Physicists prefer to work with the Lie algebra of *derivations* of the octonions: these are maps D such that $D(xy) = x(Dy) + (Dx)y$. There is no essential difference between the two viewpoints.)

A Moufang loop. Perhaps it is time to look at a finite field version of this. First we abstract a structure analogous to the quaternion group $\{\pm1, \pm i, \pm j, \pm k\}$ which can be abstracted from the quaternions. This is the set $\{\pm1, \pm i_m \mid 0 \le m \le 6\}$ of 16 elements which has a non-associative multiplication defined on it. It is not a group, so we call it a *loop*. It satisfies the Moufang law, so we call it a *Moufang loop*.

Octonions over finite fields. Suppose that F is a finite field of odd characteristic. Define the octonion algebra over F as the algebra $\{a+\sum_{m=0}^{6} b_m i_m \mid a, b_m \in F\}$, with a bilinear multiplication defined by the multiplication table already defined on $\{\pm 1, \pm i_m\}$. This has much the same properties as the real octonion algebra. It is a non-associative algebra with a norm (which is a non-singular quadratic form), and its automorphism group is the exceptional group of Lie type called $G_2(F)$, or $G_2(|F|)$.

(I won't do the characteristic 2 case, as it is a lot more complicated.)

The finite simple groups $G_2(q)$. We can really study $G_2(q)$ by this method. For example we can see various interesting subgroups. The element negating ℓ (i.e. negating i_2, i_4, i_5 and i_6) is an automorphism, as are the coordinate permutations $(0,1,2,3,4,5,6)$ and $(1,2,4)(3,6,5)$ of the imaginary part (fixing the real part, of course). Together these generate a group of shape $2^3{:}7{:}3$. We can extend this to a maximal subgroup $2^3{\cdot}L_3(2)$ of $G_2(p)$ by adjoining the map $(i_0,\ldots,i_6) \mapsto (i_0,-i_1,i_4,-i_3,i_2,i_6,i_5)$. [These elements are defined by mapping (i,j,ℓ) to $(i,j,-\ell)$, (j,ℓ,ij), $(i,\ell,j\ell)$, and $(i,-j,j\ell)$ respectively.]

The centralizer of the involution negating ℓ acts irreducibly on both the fixed 3-space (spanned by i, j, k) and the negated 4-space. It has a homomorphic image $SO_3(F)$ acting on $\langle i,j,k\rangle$ and the kernel of this action still acts transitively on the vectors of norm 1 orthogonal to i,j,k (because these are the possible candidates for ℓ). In particular we get a quotient map $SO(4) \to SO(3)$.

Triality. (The rest is material I did not have time to cover in the talk.) We can do much more than study G_2, however. Moufang loops give rise to a bizarre concept of 'triality', somewhat reminiscent of 'duality' in the theory of vector spaces and matrices. In the latter situation, duality describes the way a matrix can be regarded either as acting on the left on column vectors, or as acting on the right on row vectors. There is a duality automorphism which swaps left and right multiplication (modulo technicalities). Triality introduces a third action: conjugation (or two-sided multiplication). It is only for orthogonal groups in dimension 8 that the groups have the right size for this to work. Or to put it another way, it is only in dimension 8 that there is a Moufang loop available to create a non-trivial triality.

Isotopies. If M is a Moufang loop, then an *isotopy* of M is a triple of maps $\alpha,\beta,\gamma : M \to M$ such that $x^\alpha y^\beta z^\gamma = 1$ whenever $xyz = 1$. For example, if u is an octonion of norm 1, so that $u^{-1} = \overline{u}$, let $L_u : x \mapsto ux$ be left-multiplication by u, and $R_u : x \mapsto xu$ be right-multiplication by u, and let $B_{\overline{u}} : x \mapsto \overline{u}x\overline{u}$ be bi-multiplication by $\overline{u}$. Then it follows from the Moufang law that $(L_u, R_u, B_{\overline{u}})$ is an isotopy.

Proof: $(ux)(yu)(\overline{u}z\overline{u}) = (u(xy)u)(\overline{u}z\overline{u}) = (uz^{-1}u)(\overline{u}z\overline{u}) = 1$ since the algebra generated by u and z is associative.

It is clear that if $xyz = 1$ then $z = (xy)^{-1} = y^{-1}x^{-1}$ so $yzx = y(y^{-1}x^{-1})x = 1$. Now an *orthogonal* isotopy is almost determined by one of its three components. On the one hand, $(-1, -1, 1)$ is an orthogonal isotopy, so there are at least two orthogonal isotopies (α, β, γ) for any given γ. On the other hand, if $(\alpha, \beta, 1)$ is an orthogonal isotopy, let $1^\alpha = a$, necessarily of norm 1. Applying the definition of isotopy to the triple $(x, y, z) = (1, 1, 1)$ we have $1 = 1^\alpha 1^\beta 1 = a1^\beta$ so $1^\beta = \overline{a}$. Next, taking $(x, 1, x^{-1})$ we have $1 = x^\alpha 1^\beta x^{-1} = x^\alpha \overline{a} x^{-1}$ so $x^\alpha = xa$ and so $\alpha = R_a$. Similarly, taking $(1, y, y^{-1})$ we have $1 = 1^\alpha y^\beta y^{-1} = a y^\beta y^{-1}$ so $y^\beta = \overline{a}y$ and so $\beta = L_{\overline{a}}$.

I claim that a is real, and therefore $a = \pm 1$. For $z^{-1} = xy = (xa)(\overline{a}y)$ and the latter equality now holds for all x and y. Some calculation shows that this cannot happen unless a is real.

The spin group. Now if u is purely imaginary of norm 1, then $B_1 B_u$ acts as (minus) the reflection in u on the purely imaginary octonions. These generate $SO(7)$. Also bimultiplications do not in general fix 1, so they generate at least $SO(8)$. Hence the group of isotopies is (more or less) the spin group, i.e. a double cover of the orthogonal group. And the map $(\alpha, \beta, \gamma) \mapsto (\beta, \gamma, \alpha)$ is a triality automorphism of this spin group.

The exceptional Jordan algebra and groups of type F_4 and E_6. By some fluke, 3×3 Hermitian matrices over the octonions are closed under *Jordan multiplication* $M_1 \circ M_2 = \frac{1}{2}(M_1 M_2 + M_2 M_1)$. The groups of type F_4 can be defined as automorphism groups of such *exceptional Jordan algebras*. There is a determinant map on such matrices also, and the group of linear maps preserving the determinant (but not the Jordan product) is a Lie group of type E_6.