248 and all that

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21/09/07, R.T. Curtis 60th birthday conference

Abstract

The Lie algebras of types G_2, F_4 and E_8 have dimensions 14, 52 and 248 respectively. The classical explanation of these numbers is as the rank plus the number of roots. Half the roots are positive and half are negative: the number of positive roots is 6, 24 and 120 in the three cases. That is 3!, 4!, and 5!. Why?

Perhaps there is another 'explanation': divide by 2 and we get 7, 26 and 124. Add 1 and we get 2^3, 3^3 and 5^3. Why?

1 Introduction

I have known Rob Curtis for over 20 years, during which time he has meant a lot to me as mentor, colleague and, above all, friend. In fact our association goes back further than that, to the 'Atlas of Finite Groups', which we both worked on, though not together. (This remains, I am slightly embarassed to say, our only joint mathematical publication.)

We are here to celebrate (somewhat belatedly) Rob's 60th birthday, and in particular his contributions to mathematics. One of his most important contributions is surely the invention of the MOG (Miracle Octad Generator) for facilitating calculation in the Mathieu group M_{24}. Before then, if you wanted to generate octads, you had to look up the list of all 759 of them in a paper of Todd. Afterwards, all the octads were drawn on a postcard (not quite a postage-stamp!) and calculation became immeasurably easier. The MOG was an essential ingredient in the constructions of J_4 and the Monster, and remains an indispensable tool for working in many of the sporadic groups. The real measure of Rob's achievement here is that nobody calls it Curtis's MOG any more—it is just *the* MOG.

The same principle of trying to find better definitions and better calculating tools for studying groups (and other mathematical objects) can be found in much of Rob's work, and also underlies a great deal of the work of our mathematical father, John Conway. Some of this has rubbed off on me, and I would like to tell you today about some recent progress in this direction.

For some years I have been trying to understand the exceptional groups of Lie type, and to find effective ways of constructing them and calculating with them. Many of you have heard me talk about some new ways of constructing the Ree groups. You will be relieved to hear I am not going to talk about that now. Instead I want to turn my attention to the exceptional Lie algebras.

2 Lie algebras of type A_1

The Cartan–Killing–Weyl–Chevalley theory of Lie algebras has been so successful that there is a tendency to think that there is no other way to describe finite-dimensional Lie algebras, Lie groups, algebraic groups or finite groups of Lie type. But in fact there are many other aspects to these theories which can be illuminating or fruitful.

To take a simple case, consider the Lie algebra of type A_1. This is a 3-dimensional algebra over the complex numbers, which may be described in various ways. The standard way is to define it as the algebra of 2×2 matrices of trace 0, with the Lie bracket $[AB] := AB - BA$ (it would really be better to take $\frac{1}{2}(AB - BA)$), and, following Chevalley, to take the basis

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that the multiplication table is

	e	f	h
e	0	h	$-2e$
f	$-h$	0	$2f$
h	$2e$	$-2f$	0

But this is not the only useful basis. Let us pick instead

$$\mathbf{i} := \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \mathbf{j} := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{k} := \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

so that the multiplication table is

	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	0	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$-\mathbf{k}$	0	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	0

which I am sure you recognise! What is particularly nice about this basis is that there is a symmetry of order 3 cyclically permuting the three basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

In both cases you will observe that the multiplication table itself only involves integer coefficients, so can be interpreted over any field whatsoever. In particular, over the real numbers these are non-isomorphic algebras. They are conventionally known respectively as the *split real form* and the *compact real form* of A_1.

A handy way to distinguish them is via the *Killing form*. Let $\mathrm{ad}x$ denote the linear map $y \mapsto [xy]$, and define $\langle x, y\rangle := \mathrm{Tr}(\mathrm{ad}x.\mathrm{ad}y)$ (the trace of the composite linear map). This is obviously a symmetric bilinear form, known as the Killing form. With respect to the ordered basis (e, f, h) the Killing form has matrix

$$\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix},$$

which over the real numbers has two positive and one negative eigenvalues. On the other hand, with respect to the ordered basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ the matrix of the Killing form is

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

which is negative definite over the real numbers.

In general, the *compact real form* of a Lie algebra is the one whose Killing form is negative definite. It can be shown that it is unique (up to real equivalence).

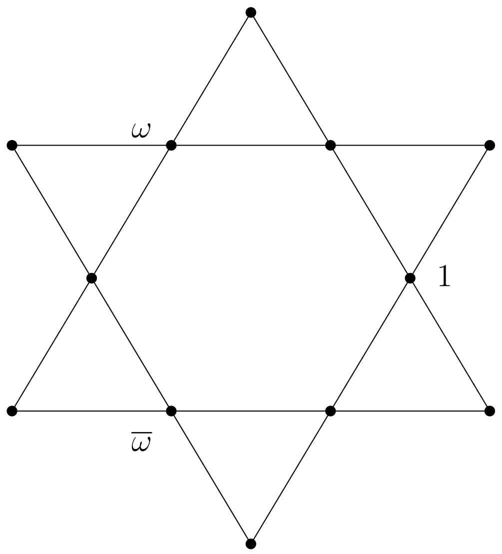
3 Composition algebras

You know, of course, how to construct the complex numbers from the real numbers, by adjoining a square root of -1, called i, and defining multiplication by $(a+bi)(c+di) = (ac-bd)+(bc+ad)i$. For each complex number $x = a+bi$ we define its *conjugate* $\overline{x} = a - bi$ and its *norm* $N(x) = x.\overline{x}$.

You probably also know how to construct the quaternions from the complex numbers, by adjoining another square root of -1 called j, and defining multiplication by $(a+bj)(c+dj) = (ac - b\overline{d}) + (b\overline{c} + ad)j$. For each quaternion $x = a + bj$ we define its *conjugate* $\overline{x} = \overline{a} - bj$ and its *norm* $N(x) = x.\overline{x}$. And of course you know that this product is no longer commutative because $j.i = (0+1j)(i+0j) = 0+(1\overline{i}+0)j = -i.j$. Writing $k = ij$ we see a remarkable similarity with the compact real form of the Lie algebra of type A_1 described above. (Multiply the matrices by 2 and use the ordinary matrix product instead of the Lie bracket.)

You may even know how to construct the octonions from the quaternions, by adjoining yet another square root of -1 called l, and defining multiplication by a $(a+bl)(c+dl) = (ac - \overline{d}b) + (b\overline{c} + da)l$, and conjugates and norms as before. This product is no longer associative, as $i(jl) = -(ij)l$.

I am interested in *integral* forms of these algebras. Let us start with the complex numbers. My favourite integral form is not $\mathbb{Z}[i]$, but $\mathbb{Z}[\omega]$, where ω is a primitive cube root of 1, that is a root of the polynomial x^2+x+1. The units in this algebra are ± 1, $\pm\omega$, $\pm\overline{\omega}$, which, geometrically, form (a scaled version of) a root system of type A_2. Adjoining also the sums of adjacent roots (that is, pairs of roots with inner product $1/2$) we obtain the root system of type G_2, in which the long roots are $\pm\theta$, $\pm\omega\theta$, $\pm\overline{\omega}\theta$, where $\theta := \omega - \overline{\omega} = \sqrt{-3}$. The *Weyl group* (generated by reflections in the roots) is a dihedral group of order 12 generated by multiplication by $-\omega$ together with complex conjugation.



What do you think is my favourite integral form of the quaternions? Surely I want it to contain a copy of $\mathbb{Z}[\omega]$! But now I can choose a particularly nice ω, that is $\omega = \frac{1}{2}(-1+i+j+k)$, so that $\mathbb{Z}[i,\omega]$ has a finite group of units

$$\{\pm 1, \pm i, \pm j, \pm k\}.\{1, \omega, \overline{\omega}\}.$$

I will leave it as an exercise for you if you are bored, to show that this is a group, of order 24, isomorphic to $SL_2(3)$. The crucial point is to observe that $i^\omega = j$.

These 24 units form a (scaled) copy of the root system of type D_4. There are also 24 elements of norm 2 in this algebra, which extend the root system to one of type F_4. The long roots are $i-j$ times the short roots, where $i-j$ is an (arbitrary) square root of -2 in the algebra. The Weyl group is generated by left- and right-multiplications by the group of units, together with quaternion conjugation, and the automorphism of the quaternions which negates i and swaps j with k. (I think this is right—I may have left something out.)

It is unlikely to surprise you at this point that there is an integral form of the octonions in which the units are precisely the 240 roots of the E_8 root system. I shall omit the details owing to shortage of time (and space). But note that the units do not form a group, due to the failure of the associative law in the octonions. However, the maps $\mathrm{ad}x$, where x is a root, generate the derived subgroup of the Weyl group (of index 2). This extends to the whole Weyl group by adjoining octonion conjugation.

4 Exceptional Lie algebras

The five exceptional Lie algebras are conventionally called G_2, F_4, E_6, E_7 and E_8, and their dimensions are respectively 14, 52, 78, 133 and 248. The subscript denotes the *rank*, which we may as well define as the dimension of the largest subalgebra on which the product is identically zero (a *Cartan subalgebra*). Considering the action of such a subalgebra (by Lie multiplication) on the algebra, we find that its eigenspaces are 1-dimensional. For each eigenspace, there is a corresponding hyperplane in the Cartan subalgebra which acts trivially. The geometry of these hyperplanes gives rise to much interesting combinatorics from which the classification of simple complex Lie algebras can be derived.

You may notice that the rank divides the dimension. (This is true for the classical Lie alegbras as well. Is there an obvious reason?) We have:

$$\begin{array}{rcl} 14 & = & 2 \times 7 \\ 52 & = & 4 \times 13 \\ 78 & = & 6 \times 13 \\ 133 & = & 7 \times 19 \\ 248 & = & 8 \times 31 \end{array}$$

You may feel that there should be some way of writing the Lie algebra as a direct sum of mutually orthogonal (with respect to the Killing form) Cartan subalgebras. Indeed, it is known that this can be done. What I want to do is to find the nicest possible way of doing this (over the complex numbers) so that the multiplication table is as nice as possible. In particular, I want the entries in the multiplication table to be integers so that they can be interpreted in any field.

I also want to use the *algebraic* structure that I have imposed on G_2, F_4 and E_8, if possible, to encode the Lie algebra. These three cases also exhibit some more fascinating numerology, as the numbers of Cartan subalgebras in the decomposition of the algebra are respectively

$$\begin{array}{rcl} 7 & = & 1+2+2^2 \\ 13 & = & 1+3+3^2 \\ 31 & = & 1+5+5^2 \end{array}$$

If this is not a complete coincidence, then surely these Cartan subalgebras must be indexed by the points of a projective plane, of order 2, 3 and 5 respectively?! The experts in the audience will also be thinking already of the non-toroidal local subgroups

$$\begin{array}{rcl} 2^3.L_3(2) & < & G_2(\mathbb{C}) \\ 3^3.L_3(3) & < & F_4(\mathbb{C}) \\ 5^3.L_3(5) & < & E_8(\mathbb{C}) \end{array}$$

These groups act irreducibly on the Lie algebra, and in each case the representation may be obtained by inducing up a suitable 2-dimensional representation of the inertial subgroup $p^3.p^2.SL_2(p)$. This gives a slightly different interpretation of the dimensions, which is also very suggestive:

$$\begin{array}{rcl} 14 & = & 2 \times (2^3 - 1) \\ 52 & = & 2 \times (3^3 - 1) \\ 248 & = & 2 \times (5^3 - 1) \end{array}$$

5 Projective planes

The projective plane of order 2 is well-studied. The best way of labelling it is with the field of order 7: the lines are then the subsets $\{t, t+1, t+3\}$. Then the symmetries $t \mapsto t+1$ and $t \mapsto 2t$ are clear, and generate a (maximal) subgroup 7:3 of $PSL_3(2)$.

The projective plane of order 3 may similarly be labelled by the field of order 13, with the lines $\{t, t+1, t+3, t+9\}$. Then the symmetries $t \mapsto t+1$ and $t \mapsto 3t$ are clear and generate a (maximal) subgroup 13:3 of $PSL_3(3)$. For reference, here is the affine part of plane, where the line at infinity is $\{0, 1, 3, 9\}$.

7	10	8
4	6	12
5	11	2

We also want to label the projective plane of order 5 by the field of order 31, so that $t \mapsto t+1$ and $t \mapsto 5t$ are obvious symmetries. There are ten different ways of doing this: the one I chose has the lines $\{t+1, t+5, t+25, t+11, t+24, t+27\}$. For reference, here is the affine part of plane, where the line at infinity is $\{1, 5, 25, 11, 24, 27\}$.

14	22	26	17	15
16	30	6	29	10
21	23	28	20	7
13	9	2	4	19
0	3	12	18	8

You can easily work out for yourselves which directions point to which of the points at infinity.

6 The compact real form of G_2

I won't bore you with the details of the calculations I did (with some help from MAGMA), but just present you with the nicest definition I have so far for the compact real form of G_2.

Take seven 2-spaces L_t, labelled with the elements $t \in \mathbb{F}_7$. In each L_t take three vectors $u = u_t$, $v = v_t$, $w = w_t$ summing to zero. The Lie bracket is defined by the products

	v_1	w_1
u_0	w_3	u_3
v_0	u_3	w_3

and their images under the group of order 21 generated by

$$\begin{array}{rcl} \alpha = (0,1,2,3,4,5,6) & : & u_t \mapsto u_{t+1}, v_t \mapsto v_{t+1}, w_t \mapsto w_{t+1} \\ \beta = (1,2,4)(3,6,5)(u,v,w) & : & u_t \mapsto v_{2t}, v_t \mapsto w_{2t}, w_t \mapsto u_{2t} \end{array}$$

together with the anti-symmetry.

It is obvious that this definition is invariant under the map negating L_2, L_4, L_5 and L_6, which extends the symmetry group to $2^3{:}7{:}3$. It is possible to write down another symmetry which extends it to $2^3L_3(2)$, such as the following:

$$\begin{array}{rcl} u_0, v_0, w_0 & \mapsto & -u_0, -v_0, -w_0 \\ u_4, v_4, w_4 & \mapsto & -v_4, -u_4, -w_4 \\ u_5, v_5, w_5 & \leftrightarrow & w_5, v_5, u_5 \\ u_1, v_1, w_1 & \leftrightarrow & v_3, u_3, w_3 \\ u_2, v_2, w_2 & \leftrightarrow & v_6, w_6, u_6 \end{array}$$

To show that this is a form of G_2, observe that L_0 is a Cartan subalgebra. Therefore its non-trivial eigenspaces are 1-dimensional, and can be explicitly calculated. Thus we may obtain an explicit base change between my basis and a Chevalley basis. It is then not hard, merely tedious, to verify that over the complex numbers they are the same algebra.

In fact, I have deliberately misled you into thinking there is more symmetry than there really is. If I extend my multiplication table to include the three distinguished vectors in each 2-space, you see something a little different:

	u_1	v_1	w_1
u_0	v_3	w_3	u_3
v_0	v_3	u_3	w_3
w_0	$-2v_3$	v_3	v_3

You also see that something remarkable happens in characteristic 3.

7 The compact real form of F_4

Our construction here should make the Lie algebra as a direct sum of thirteen 4-spaces, each of which is a Cartan subalgebra so has an F_4 root system naturally embedded in it. However, there is actually more structure than this (and so less symmetry), as we are really inducing up from a 2-dimensional complex representation of $3^3.3^2.SL_2(3) \cong 3^4.(3 \times 2A_4)$ contained in $3^4.W(F_4)$. Thus we centralize a fixed-point-free element of order 3 in $W(F_4)$, giving F_4 the structure of a 2-dimensional lattice over $\mathbb{Z}[\omega]$. There are then four 1-spaces containing six short roots each, namely the (left) unit multiples of 1, i, j and k respectively. Similarly there are four 1-spaces containing six long roots each, namely the (left) unit multiples of $i-j$, $j+k$, $k+i$, and $i+j$.

We have thirteen 4-spaces, labelled by the integers mod 13, and each coordinatised by $\mathbb{Z}[i,\omega]$. We have a symmetry of order 13 which maps $1_t \mapsto 1_{t+1}$ and $i_t \mapsto i_{t+1}$, etc. We also have an element of order 3 acting as *left* multiplications by $(\overline{\omega}, \overline{\omega}, \overline{\omega}, 1, \omega, 1, \omega, \overline{\omega}, 1, 1, \overline{\omega}, \overline{\omega}, \omega)$. These together generate 3^3:13. The rest of the symmetries are not quite so easy to describe... but once we have them, we only need to specify the Lie product on a few basis vectors, thus:

	1_1	i_1
1_0	$-\omega i_3 - k_9$	$\overline{\omega} k_3 - \omega i_9$
i_0	$\omega 1_3 + i_9$	$\overline{\omega} i_3 + \omega j_9$

(my current best shot, but I am sure this can be improved: I keep changing the basis to make it look simpler) and then take images under the action of $3^3{:}SL_3(3)$.

(The details still remain to be sorted out.)

8 The compact real form of E_8

We should be able to do the same sort of thing with E_8, but I haven't done it yet.