The O'Nan–Scott Theorem

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1 Introduction

The O'Nan–Scott theorem gives us a classification of the maximal subgroups of the alternating and symmetric groups. It does not tell us exactly what the maximal subgroups are—that is too much to ask, rather like asking what are all the prime numbers. It does however provide a first step towards writing down the list of maximal subgroups of A_n or S_n for any particular reasonable value of n.

Theorem 1 *If H is any proper subgroup of S_n other than A_n, then H is a subgroup of one or more of the following subgroups:*

1. *An intransitive group $S_k \times S_m$, where $n = k + m$;*
2. *An imprimitive group $S_k \wr S_m$, where $n = km$;*
3. *A primitive wreath product, $S_k \wr S_m$, where $n = k^m$;*
4. *An affine group $AGL_d(p) \cong p^d{:}GL_d(p)$, where $n = p^d$;*
5. *A group of shape $T^m.(\mathrm{Out}\,(T) \times S_m)$, where T is a non-abelian simple group, acting on the cosets of the 'diagonal' subgroup $\mathrm{Aut}\,(T) \times S_m$, where $n = |T|^{m-1}$;*
6. *An almost simple group acting on the cosets of a maximal subgroup.*

Note that the theorem does not assert that all these subgroups are maximal in S_n, or in A_n. This is a rather subtle question, particularly as far as subgroups of A_n are concerned. The last category of subgroups also requires us to know all the finite simple groups, or at least those with a maximal subgroup of index n. In practice, this means that we can only ever hope to get a *recursive* description of the maximal subgroups of A_n and S_n.

2 Preliminaries

In this section we collect a number of general facts about groups which will be useful in the proof of the O'Nan–Scott theorem. Some you may already know, others may be unfamiliar. They are all important!

Lemma 1 *Every normal subgroup N of a primitive group H is transitive,*

Proof. Otherwise the orbits of N form a block system for H. $\square$

Lemma 2 *Any two distinct minimal normal subgroups N_1 and N_2 of any group H commute.*

Proof. We have $[N_1, N_2] \leq N_1 \cap N_2 \trianglelefteq H$ so by minimality $[N_1, N_2] = N_1 \cap N_2 = 1$. $\square$

A subgroup K of a group N is called *characteristic* if it is fixed by all automorphisms of N. The following is obvious:

Lemma 3 *If K is characteristic in N and N is normal in H then K is normal in H.*

A group $K \neq 1$ is called *characteristically simple* if K has no proper non-trivial characteristic subgroups.

Lemma 4 *If K is characteristically simple then it is a direct product of isomorphic simple groups.*

Proof. If T is any minimal normal subgroup of K, then so is T^α for any $\alpha \in \operatorname{Aut} K$. So by Lemma 2 either $T^\alpha = T$ or $T \cap T^\alpha = 1$. In the latter case $TT^\alpha = T \times T^\alpha$ is a direct product. Since K is characteristically simple, it is generated by all the T^α. By induction we obtain that K is a direct product of a finite number of such T^α. But then any normal subgroup of T is normal in K, so by minimality of T, T is simple. $\square$

Corollary 1 *Every minimal normal subgroup N of a finite group H is a direct product of isomorphic simple groups (not necessarily non-abelian).*

Proof. By minimality, N is characteristically simple. $\square$

A group N is called *regular* on Ω if for each pair of points a and b in Ω, there is exactly one element of N mapping a to b. In particular N is transitive and $|N| = |\Omega|$, and every non-identity element of N is fixed-point-free.

Lemma 5 *If H is primitive, and N is a non-trivial normal subgroup of H, then either $C_H(N)$ is trivial, or $C_H(N)$ is regular and $|C_H(N)| = |\Omega|$.*

Proof. By Lemma 1, if $C_H(N) \neq 1$ then both N and $C_H(N)$ are transitive. Moreover, if $1 \neq x \in C_H(N)$ has any fixed points, then the set of fixed points of x is preserved by N. This contradiction implies that every element of $C_H(N)$ is fixed-point-free. This means that $C_H(N)$ is regular. $\square$

This has a number of important consequences.

Corollary 2 *If H is primitive, and N_1 and N_2 are normal subgroups of H, and $[N_1, N_2] = 1$, then $N_2 = C_H(N_1)$ and vice versa. In particular, H contains at most two minimal normal subgroups, and if it has an abelian normal subgroup then it has only one minimal normal subgroup.*

Proof. By Lemma 1, N_1 is transitive, and by Lemma 5, $C_H(N_2)$ is regular. But $N_1 \subseteq C_H(N_2)$, and therefore N_1 and $C_H(N_2)$ have the same order and are equal. □

Corollary 3 *With the same notation,* $N_1 \cong N_2$.

Proof. Fix a point $x \in \Omega$, and let K be the stabilizer of x in the group N_1N_2. Then $K \cap N_1 = K \cap N_2 = 1$ as N_1 and N_2 are regular. Therefore $KN_1 = KN_2 = N_1N_2$, and by the isomorphism theorem

$$K \cong K/K \cap N_1 \cong KN_1/N_1 = N_2N_1/N_1 \cong N_2$$

and similarly $K \cong N_1$. □

Lemma 6 *Suppose that H is primitive and N is a minimal normal subgroup of H. Let K be the stabilizer in H of a point. Then $KN = H$.*

Proof. K does not contain N, for if it did then N would act trivially on Ω, whence $N = 1$. Moreover, K is maximal in H, since H is primitive, and therefore $KN = H$ for otherwise $K < KN < H$ contradicting maximality. □

A subgroup X of H is called K-invariant if $K \leq N_H(X)$. In fact,

Lemma 7 *Suppose that H is primitive and N is a minimal normal subgroup of H. Let K be the stabilizer in H of a point. Then $K \cap N$ is maximal among K-invariant subgroups of N.*

Proof. If $K \cap N < X < N$ and $K \leq N_H(X)$ then KX is a subgroup of H and $K < KX < H$, contradicting maximality of K. □

3 The proof of the O'Nan–Scott theorem

Reduction to the case N unique and non-abelian. Let H be a subgroup of S_n not containing A_n. Then H is either intransitive (giving case 1 of the theorem), or transitive imprimitive (giving case 2 of the theorem), or primitive. If it is primitive, let N be a minimal normal subgroup of H.

If N is abelian, it is an elementary abelian p-group, acting regularly, and $N = C_H(N)$. Therefore H is affine (case 4 of the theorem).

Otherwise, all minimal normal subgroups of H are non-abelian. If there is more than one minimal normal subgroup, say N_1 and N_2, then $N_1 \cong N_2$ and both N_1 and N_2 act regularly on Ω.

Thus N_1 and N_2 act in the same way on the n points, so there is an element x of S_n conjugating N_1 to N_2. Moreover, by Corollary 2, $N_2 = C_{S_n}(N_1)$. Therefore x conjugates N_2 to N_1, and $\langle H, x\rangle$ has a unique minimal normal subgroup $N = N_1 \times N_2$. So this case reduces to the case when there is a unique minimal normal subgroup.

The case N unique and non-abelian. From now on, we can assume that H has a unique minimal normal subgroup, N, which is non-abelian. If N is simple, we are in case 6 of the theorem. Otherwise, N is non-abelian, non-simple, say $N = T_1 \times \cdots \times T_m$ with $T_m \cong T$ simple and $m > 1$, and H permutes the T_i transitively by conjugation.

For each i, let K_i be the image of $K \cap N$ under the natural projection from N to T_i. We divide into two cases: either $K_i \neq T_i$ for some i (and therefore for all i) or $K_i = T_i$ for all i.

Case 1. In the former case, K normalizes $K_1 \times \cdots \times K_m$, so that, by maximality, we have $K \cap N = K_1 \times \cdots \times K_m$, and K permutes the K_i transitively. Let k be the index of K_i in T_i. Then H is evidently contained in the group $S_k \wr S_m$ acting in the product action (case 3 of the theorem).

Case 2. There remains the case where $K_i = T_i$ for all i. Let Ω_1 be a minimal subset of $\{1, \ldots, m\}$ such that $K \cap N$ contains an element whose support is $\{T_i : i \in \Omega_1\}$. Then the elements with this support still map onto T_i, $i \in \Omega_1$, since they map onto a whole conjugacy class and T_i is simple. Now if Ω_2 is another such set, intersecting Ω_1 non-trivially, we have elements x and y in $K \cap N$ such that $[x, y]$ has support $\{T_i : i \in \Omega_1 \cap \Omega_2\}$. Minimality of Ω_1 implies that it is a block in a block system invariant under K, and therefore under $H = KN$.

The blocks cannot have size 1, for then K contains N, a contradiction. If the block system is non-trivial, with l blocks of size k, say, and $l > 1$, then $n = |T|^{(k-1)l}$ and H lies inside $S_r \wr S_l$, in its product action, where $r = |T|^{k-1}$. This is case 3 of the theorem again.

Otherwise, the block system is trivial, so $K \cap N$ is a diagonal copy of T inside $T_1 \times \cdots \times T_m$, and we can choose our notation such that it consists of the elements $(g, g, \ldots, g)$ for all $g \in T$. Also $n = |T|^{m-1}$, and the n points can be identified with the n conjugates of $K \cap N$ by elements of N. The largest subgroup of S_n preserving this setup is as in case 5 of the theorem. This concludes the proof of the O'Nan–Scott theorem.