A quaternionic approach to E_7

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Abstract

The classical Lie algebras over the complex numbers are all derived from associative algebras by defining a Lie bracket $[A, B] = AB - BA$, but the five exceptional Lie algebras are often defined directly as Lie algebras. In four cases, however, an alternative approach is available to construct both the algebra and the corresponding Lie group.

It is well-known that the Lie group G_2 is naturally defined as the automorphism group of the octonions (Cayley numbers), and F_4 as the automorphism group of the exceptional Jordan algebra, a 27-dimensional algebra of 3×3 Hermitian matrices over octonions with product $AB + BA$. Moreover, E_6 may be defined as the group of linear maps on the Jordan algebra which preserve a certain 'determinant' (but not the algebra product).

This leaves just E_7, which has a 56-dimensional representation whose structure is hard to describe. But recognising that this is really a 28-dimensional quaternionic representation simplifies things significantly, and reveals tantalising glimpses of an underlying 7-dimensional structure over the tensor product of two quaternion algebras.

1 Introduction

My motivation comes from finite simple groups, and in particular a desire to understand the finite groups of Lie type properly. More and more I am coming to realise that one must first understand the simple Lie groups (over $\mathbb{C}$ and $\mathbb{R}$).

Simple Lie groups over $\mathbb{C}$ were classified a long time ago: there are three types of classical groups (orthogonal, unitary and symplectic) and five types of exceptional groups (called G_2, F_4, E_6, E_7, E_8). It has long been understood that the orthogonal groups are essentially real, and the symplectic groups are essentially quaternionic, so that the classical groups over $\mathbb{C}$ are interpreted as series of one real, one complex, and one quaternionic group in each dimension.

Over $\mathbb{R}$ one obtains a number of 'real forms': for example, the orthogonal groups over $\mathbb{R}$ are parametrized by the signature of the quadratic form (up to sign), so there are $\lfloor \frac{n}{2} \rfloor + 1$ different real forms. Exactly one of these is compact, namely the group corresponding to a positive- (or negative-) definite form. At the opposite extreme is the split real form, with the numbers of positive and negative terms being either equal or differing by 1.

On the other hand, the exceptional groups are usually defined directly in terms of the Lie algebra. To put this in context, the Lie algebra for the orthogonal groups is the space of skew-symmetric matrices, acted on by conjugation. Thus it has dimension $n(n-1)/2$ rather than n for the natural representation. It should be clear that one should not use the Lie algebra unless one cannot avoid it.

The details for the exceptional groups are as follows:

	G_2	F_4	E_6	E_7	E_8
Lie algebra dimension	14	52	78	133	248
smallest representation	7	26	27	56	248
representing:			$3E_6$	$2E_7$	
underlying module		$\mathbb{R}^{26}$	$\mathbb{C}^{27}$	$\mathbb{H}^{28}$	

It is also known, but perhaps imperfectly understood, that the exceptional groups arise from octonions (Cayley numbers) in certain ways. The Cayley numbers are an 8-dimensional non-associative algebra, with an identity element, and G_2 is most naturally defined as the automorphism group of this algebra. Thus it has a 7-dimensional representation.

Now F_4 can be defined as the automorphism group of the exceptional Jordan algebra, which consists of 3×3 Hermitian matrices over the octonions (so its dimension is $3+3\times 8 = 27$), with product $AB+BA$. An alternative definition of the algebra is in terms of three invariant forms: the (linear) trace, the (cubic) determinant (which, surprisingly, is well-defined), and the quadratic form $\mathrm{Tr}(A\overline{A})$. Then E_6 is the group of linear maps which preserve the determinant.

Thus F_4 has a 26-dimensional representation, which (for the compact real form) is essentially real, and E_6 has a 27-dimensional representation, which (for the compact real form, but *not* for the split real form) is essentially complex. Now the smallest representation of E_7 is well-known to have dimension 56, which appears less closely related. But when you realise that, *for the compact real form*, this is really a 28-dimensional quaternionic representation, the pattern becomes intriguing.

2 A construction of E_7

The reflection group of type E_7. This is a (small!) finite group, usually called the Weyl group, which controls the structure of the much larger (infinite!) Lie group, so we need to understand it first. It is the group generated by the 63 reflections in the following vectors (called roots) in $\mathbb{R}^7$, and images under rotating the 7 coordinates:

- 7 pairs of vectors $\pm(2,0,0,0,0,0,0)$;
- $7 \times 8 = 56$ pairs of vectors $\pm(1,0,0,\pm1,0,\pm1,\pm1)$.

The first type of reflections just negate one of the seven coordinates. The second type fix three of the coordinates and act on the other four coordinates as a 4×4 matrix with entries $\pm\frac{1}{2}$.

The $\mathbb{Z}$-linear combinations of the roots form a lattice, whose dual lattice is by definition the set of vectors whose inner products with the roots are integers. If we scale this by a factor of 2 we find that the minimal vectors in the dual are the $7 \times 4 = 28$ pairs of vectors $\pm(0,1,\pm1,0,\pm1,0,0)$, together with all rotations of the 7 coordinates.

The quaternionic 28-space. Corresponding to each of these 28 pairs of vectors in $\mathbb{R}^7$, we take a copy of the quaternions. In order to emphasise the grouping of the 28 pairs of vectors into 7 sets of 4, I shall write elements of each of seven 4-dimensional (left-)quaternion-spaces as (H_t, I_t, J_t, K_t), for $t = 0,1,2,3,4,5,6$. The correspondence is given by

$$\begin{array}{rcl} H_0 & \mapsto & (0,1,1,0,1,0,0) \\ I_0 & \mapsto & (0,1,-1,0,-1,0,0) \\ J_0 & \mapsto & (0,-1,1,0,-1,0,0) \\ K_0 & \mapsto & (0,-1,-1,0,1,0,0) \end{array}$$

and increasing the suffix by 1 (mod 7) corresponds to rotating the coordinates *backwards*. Negation of a vector in $\mathbb{R}^7$ corresponds to multiplying the corresponding quaternion (on the right) by j.

Action of a fundamental $\mathrm{SU}(2)$**.** Corresponding to each of the 63 reflections in the Weyl group, there is a copy of $\mathrm{SU}(2)$ in the Lie group. Here we describe the action on the 28-dimensional quaternionic space of the $\mathrm{SU}(2)$ corresponding to the reflection in $\pm(2,0,0,0,0,0,0)$, which negates the first coordinate in $\mathbb{R}^7$. Now this reflection has exactly 6 orbits of size 2 on our 28 pairs of vectors, and these orbits correspond to quaternionic 2-spaces in the quaternionic 28-space. Moreover, there is a canonical choice of basis (only up to signs, unfortunately), whereby the first basis vector corresponds to the vector in $\mathbb{R}^7$ which has first coordinate 1, and the second basis vector corresponds to first coordinate -1.

The element $q = z + wj \in \mathrm{SU}(2) \subseteq \mathbb{H}$ (where $z, w \in \mathbb{C}$ and $q\overline{q} = 1$) acts on these 6 quaternionic 2-spaces in the following way (w.r.t. bases as described above, which we now specify precisely by giving the signs). The action of q on each of (H_1, I_1), (H_2, J_2) and (H_4, K_4) is then by right-multiplication by $\begin{pmatrix} z & wj \\ wj & z \end{pmatrix}$, in the sense that $H_1 \mapsto H_1 z + I_1 zj$, etc., and by $\begin{pmatrix} \overline{z} & \overline{w}j \\ \overline{w}j & \overline{z} \end{pmatrix}$ on (J_1, K_1), (K_2, I_2) and (I_4, J_4).

The other 56 root SU(2)s. In order to specify the other root groups, first look at the Weyl groups in the seven we have already constructed: that is the matrix $\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}$ corresponding to the quaternion $q = j \in \mathrm{SU}(2)$. These elements swap pairs of coordinates as described above, and can easily be seen to correspond in the reflection group to sign changes on three coordinates $(t, t+1, t+3)$. Therefore these are enough to fuse all the remaining 56 root groups into a single orbit.

Now we can specify one of them, for example the one given by $\begin{pmatrix} z & wj \\ wj & z \end{pmatrix}$ acting on the spaces (H_3, K_1), (H_6, I_2), (H_5, J_4), and $\begin{pmatrix} \overline{z} & \overline{w}j \\ \overline{w}j & \overline{z} \end{pmatrix}$ acting on (K_3, I_1), (I_6, J_2), and (J_5, K_4). We now have all 63 root groups, and therefore we have generators for (the double cover of) (the compact real form of) E_7.

3 Proofs

My proofs of the above results are mostly of the form: this construction is equivalent to some known construction in the literature. Ultimately I want more self-contained proofs, that enable us to get all the important properties of the groups of type E_7 directly.

The quartic form. All constructions of the 56-dimensional representation of $2E_7$ mention the quartic form (and/or a polarized version, i.e. a symmetric quadrilinear form). To describe this, we first split each quaternion into its complex and imaginary parts, say $q = q' + q''j$ for any quaternion q. Then the quartic form has 630 terms which are the images under the Weyl group of $H'_0 I'_0 J'_0 K'_0$, as well as 28 images of $-(H'_0)^2(H''_0)^2$ and 378 images of $\frac{1}{2}H'_0 H''_0 I'_0 I''_0$. I have proved explicitly that my generators preserve this quartic form. Notice that the sign problems which plague the construction of this form have almost completely disappeared in this treatment.

The stabilizer of a quaternionic coordinate. This stabilizer is a copy of $3E_6$. To prove this, one can look at the terms in the quartic form which involve H_0, and remove the H_0 factor. Then one obtains a cubic form (of 45 terms in 27 variables) which is easily shown to be equivalent to L.E. Dickson's cubic form from his 1901 construction of $3E_6$ (over finite fields).

One can deduce from this that the group which preserves the quartic form has dimension (as a variety) at most 133. Conversely, the group we have constructed has dimension at least 133. Thus the explicit group elements given above do indeed generate $2E_7$.

4 Finite groups

The split real form. There are various possible ways of getting an analogue over finite fields. Quaternions themselves do not really make sense over finite fields, so perhaps the best way is to convert to the split real form first, before tensoring with the appropriate finite field. This involves forgetting the quaternionic structure (in a suitable way—there is some choice here), so that the representation becomes 56-dimensional complex, and then restricting matrix entries to real numbers gives the split real form.

The result is as follows. To obtain generators for the first fundamental SL(2), each of the three quaternionic 2-vectors, (H_1, I_1), (H_2, J_2), $H_4, K_4)$ is replaced by a pair of real 2-vectors (H_1', I_1''), (I_1', H_1''), etc., on which we write down the usual generators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Similarly on the other three quaternionic 2-vectors, (J_1, K_1), (K_2, I_2), (I_4, J_4), we have the action as the transpose-inverse matrices on (J_1', K_1'') and (K_1', J_1'') etc. The other orbit of root SL(2) subgroups is obtained in the same way.

In particular, this gives explicit generators for the root subgroups, a maximal torus, the Weyl group and all the standard subgroups, such as the parabolic subgroups and the maximal rank subgroups.

The finite groups $(2)E_7(q)$. The given generators for the split real form can be interpreted over any field F whatever, and give generators for the split form of E_7 over F. In particular, if $F = \mathbb{F}_q$ is the field of order q, we obtain either $E_7(q)$ (if q is even) or a double cover $2E_7(q)$ (if q is odd).

We again obtain immediately generators for the root groups, maximal (split) torus, Weyl group, parabolic subgroups, and so on. For example, the stabilizer of one of the 56 coordinate vectors is a parabolic of shape $q^{27}(3)E_6(q)$, so by counting the images of this vector one can obtain the formula for the order of $E_7(q)$.

5 Speculations

The quartic form seems to pick out some 'quaternionic' structure to the 4-dimensional pieces like $\langle H_0, I_0, J_0, K_0\rangle$, so it is tempting to think of the 28-dimensional quaternionic space as a 7-dimensional 'space' over $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$. This interpretation leads to some nice formulae for actions of certain group elements.

It is also tempting to label the 7 'coordinates' by pure imaginary octonions, as there is a natural structure of the Fano plane on these 'coordinates'. This would give the whole representation of $2E_7$ the structure $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \otimes_{\mathbb{R}} \mathbb{O}^*$. Does this mean anything?

What about E_8? Is it 31-dimensional over $\mathbb{O}$ in any meaningful sense?