Octonions and the Leech lattice

Robert A. Wilson
School of Mathematical Sciences,
Queen Mary, University of London,
Mile End Road, London E1 4NS

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Abstract

We give a new, elementary, description of the Leech lattice in terms of octonions, thereby providing the first real explanation of the fact that the number of minimal vectors, 196560, can be expressed in the form $3 \times 240 \times (1 + 16 + 16 \times 16)$. We also give an easy proof that it is an even self-dual lattice.

1 Introduction

The Leech lattice occupies a special place in mathematics. It is the unique 24-dimensional even self-dual lattice with no vectors of norm 2, and defines the unique densest lattice packing of spheres in 24 dimensions. Its automorphism group is very large, and is the double cover of Conway's group Co_1 [2], one of the most important of the 26 sporadic simple groups. This group plays a crucial role in the construction of the Monster [13, 4], which is the largest of the sporadic simple groups, and has connections with modular forms (so-called 'Monstrous Moonshine') and many other areas, including theoretical physics. The book by Conway and Sloane [5] is a good introduction to this lattice and its many applications.

It is not surprising therefore that there is a huge literature on the Leech lattice, not just within mathematics but in the physics literature too. Many attempts have been made in particular to find simplified constructions (see for example the 23 constructions described in [3] and the four constructions described in [15]). In the latter are described a 12-dimensional complex Leech lattice, whose symmetry group is a sextuple cover of the sporadic Suzuki group [16]; a 6-dimensional quaternionic Leech lattice [17], whose symmetry group is a double cover of an exceptional group of Lie type, namely $G_2(4)$; and a 3-dimensional quaternionic version, known as the icosian Leech lattice [14, 1], which exhibits the double of cover of the Hall–Janko group as a group generated by quaternionic reflections. This last is based on the ring of icosians discovered by Hamilton, which is an algebraic version of the H_4 reflection group.

In this paper I show how to construct a 3-dimensional *octonionic* Leech lattice, based on the Coxeter–Dickson non-associative ring of integral octonions [8], which is an algebraic version of the E_8 lattice.

2 Octonions and E_8

The book by Conway and Smith [6] gives much useful background on octonions. The (real) octonion algebra is an 8-dimensional (non-associative) algebra with an orthonormal basis $\{1 = i_\infty, i_0, \ldots, i_6\}$ labeled by the projective line $PL(7) = \{\infty\} \cup \mathbb{F}_7$, with product given by $i_0 i_1 = -i_1 i_0 = i_3$ and images under the subscript permutations $t \mapsto t+1$ and $t \mapsto 2t$. The norm is $N(x) = x\overline{x}$, where $\overline{x}$ denotes the octonion conjugate of x, and satisfies $N(xy) = N(x)N(y)$.

The E_8 root system embeds in this algebra in various interesting ways. For example, we may take the 240 roots to be the 112 octonions $\pm i_t \pm i_u$ for any distinct $t, u \in PL(7)$, and the 128 octonions $\frac{1}{2}(\pm 1 \pm i_0 \pm \cdots \pm i_6)$ which have an odd number of minus signs. Denote by L the lattice spanned by these 240 octonions, and write R for $\overline{L}$. Let $s = \frac{1}{2}(-1 + i_0 + \cdots + i_6)$, so that $s \in L$ and $\overline{s} \in R$.

It is well-known that $\frac{1}{2}(1+i_0)L = \frac{1}{2}R(1+i_0)$ is closed under multiplication, and forms a copy of the Coxeter–Dickson integral octonions. Denote this non-associative ring by A, so that $L = (1+i_0)A$ and $R = A(1+i_0)$. It follows immediately from the Moufang law $(xy)(zx) = x(yz)x$ that $LR = (1+i_0)A(1+i_0)$. Writing $B = \frac{1}{2}(1+i_0)A(1+i_0)$, we have $LR = 2B$, and the other two Moufang laws imply that $BL = L$ and $RB = R$.

Since $\overline{s} \in R$ we have $L\overline{s} \subseteq LR = 2B$, but $2B$ is spanned by its 240 roots, all of which lie in $L\overline{s}$, so $L\overline{s} = 2B$. Indeed, the same argument shows that if ρ is any root in R then $L\rho = 2B$. More explicitly, it is easy to show (and presumably well-known) that the roots of B are $\pm i_t$ for $t \in PL(7)$ together with $\frac{1}{2}(\pm 1 \pm i_t \pm i_{t+1} \pm i_{t+3})$ and $\frac{1}{2}(\pm i_{t+2} \pm i_{t+4} \pm i_{t+5} \pm i_{t+6})$ for $t \in \mathbb{F}_7$. Hence $2L \subset 2B \subset L$, that is $2L \subset L\overline{s} \subset L$, from which we deduce also $2L \subset Ls \subset L$. Moreover, $L\overline{s} + Ls = L$, so by self-duality of L we have $L\overline{s} \cap Ls = 2L$.

3 The octonion Leech lattice

Using L as our basic copy of E_8 in the octonions, we define the *octonionic Leech lattice* $\Lambda = \Lambda_{\mathbb{O}}$ as the set of triples (x, y, z) of octonions, with norm $N(x, y, z) = \frac{1}{2}(x\overline{x} + y\overline{y} + z\overline{z})$, such that

1. $x, y, z \in L$;
2. $x + y, x + z, y + z \in L\overline{s}$;
3. $x + y + z \in Ls$.

It is clear that the definition of Λ is invariant under permutations of the three coordinates. We show now that it is invariant under the map $r_t : (x, y, z) \mapsto (x, yi_t, zi_t)$ which right-multiplies two coordinates by i_t. Certainly $Li_t = L$, so the first condition of the definition is preserved. Then $y(1 - i_t) \in LR = 2B = L\overline{s}$, so the second condition is preserved. Finally $(y+z)(1-i_t) \in 2BL = 2L \subset Ls$, so the third condition is preserved also. It follows that the definition is invariant under sign-changes of any of the three coordinates.

Suppose that λ is any root in L. Then the vector $(\lambda s, \lambda, -\lambda)$ lies in Λ, since $Ls \subseteq L$ and $\lambda s + \lambda = \lambda(s+1) = -\lambda\overline{s}$. Therefore Λ contains the vectors $(\lambda s, \lambda, \lambda) + (\lambda, \lambda s, -\lambda) = -(\lambda\overline{s}, \lambda\overline{s}, 0)$, that is, all vectors $(2\beta, 2\beta, 0)$ with β a root in B. Hence Λ also contains

$$(\lambda(1 + i_0), \lambda(1 + i_0), 0) + (\lambda(1 - i_0), -\lambda(1 + i_0), 0) = (2\lambda, 0, 0).$$

Applying the above symmetries it follows at once that Λ contains the following 196560 vectors of norm 4, where λ is a root of L and $j, k \in J = \{\pm i_t \mid t \in PL(7)\}$:

$(2\lambda, 0, 0)$	Number:	$3 \times 240 =$	720
$(\lambda\overline{s}, \pm(\lambda\overline{s})j, 0)$	Number:	$3 \times 240 \times 16 =$	11520
$((\lambda s)j, \pm\lambda k, \pm(\lambda j)k)$	Number:	$3 \times 240 \times 16 \times 16 =$	184320

4 Identification with the real Leech lattice

We show first that the 196560 vectors listed above span Λ. For if $(x, y, z) \in \Lambda$, then by adding suitable vectors of the third type, we may reduce z to 0. Then we know that $y \in L\overline{s}$, so by adding suitable vectors of the second type we may reduce y to 0 also. Finally we have that $x \in L\overline{s} \cap Ls = 2L$ so we can reduce x to 0 also.

At this stage it is easy to identify Λ with the Leech lattice in a number of different ways. First, let us label the coordinates of each brick of the MOG (see [9] or [7]) as follows:

-1	i_0
i_4	i_5
i_2	i_6
i_1	i_3

Then it is well-known (see for example [7]) that the map $i_t \mapsto i_{t+1}$ is a symmetry of the standard Leech lattice. Now L is spanned by $1 \pm i_t$ and s, and it is trivial to verify that the vectors $(1 \pm i_0)(s, 1, 1)$ and $s(s, 1, 1)$ are in this Leech lattice. These together with coordinate permutations, sign-changes and addition are enough to give all the minimal vectors, which span the lattice.

An alternative approach is to show directly from our definition that Λ is an even self-dual lattice with no vectors of norm 2, whence it is the Leech lattice by Conway's characterisation [5, Chapter 12]. For if (x, y, z) is in the dual of Λ then in particular its (real) inner product with $(2\lambda, 0, 0)$ is integral, and since L is self-dual this implies $x \in L$. Similarly its inner product with $(\lambda\overline{s}, \lambda\overline{s}, 0)$ is integral, and since the dual of B is $2B$ this implies $x + y \in L\overline{s}$. Also $(\lambda s, -\lambda, -\lambda) + (0, -\lambda\overline{s}, -\lambda\overline{s}) = (\lambda s, \lambda s, \lambda s) \in \Lambda$ and the dual of Ls is $2Ls$, so $x + y + z \in Ls$. Thus Λ contains its dual.

Conversely, if $(x, y, z) \in \Lambda$ then

$$2N(x, y, z) = N(x + y) + N(x + z) + N(y + z) - N(x + y + z)$$

and all the terms on the right hand side are divisible by 4, so Λ is an even lattice, and in particular is contained in its own dual. That Λ has no vectors of norm 2 is easy to see: if $(x, y, z) \in \Lambda$ has norm 2 then at least one coordinate is 0, so the other coordinates lie in $L\overline{s}$; therefore there is only one non-zero coordinate, which lies in $2L$, so the vector has norm at least 4.

5 Applications

The above construction is deceptively simple. In fact, however, finding the correct definition was not at all easy. Over the years, many people have noticed the suggestive fact that

$$196560 = 3 \times 240 \times (1 + 16 + 16 \times 16),$$

and tried to build the Leech lattice from triples of integral octonions (see for example [10, 11, 12]), but until now no-one has provided a convincing explanation for this numerology.

An alternative definition of an octonion Leech lattice using the 'natural' norm $N(x, y, z) = x\overline{x} + y\overline{y} + z\overline{z}$ may be obtained by a change of basis:

1. $x, y, z \in B$;
2. $x + y, y + z \in Bs = L$;
3. $x + y + z \in B\overline{s}$.

In a further paper [19] I shall show how to generate the automorphism group of the lattice in terms of 3×3 matrices with octonion entries, and give nice descriptions of many of its maximal subgroups. This will include elementary constructions of all the Suzuki-chain subgroups, which up till now have not been easy to describe directly in terms of the lattice.

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