On the compact real form of the Lie algebra $\mathfrak{g}_2$

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Abstract

We give an elementary construction of the compact real form of the Lie algebra $\mathfrak{g}_2$.

It is known that every simple complex Lie algebra has a (unique) compact real form, defined by the property that the Killing form is negative definite. For example, the compact real form of $\mathfrak{a}_1$ is just the well-known 3-dimensional vector cross product given by $i \times j = k = -j \times i$ and images under the symmetry rotating $i$, $j$ and $k$.

However, I have been unable to find in the literature any nice constructions of the compact real forms of the exceptional Lie algebras. In this note I give an elementary construction of the compact real form of $\mathfrak{g}_2$, exhibiting an irreducible subgroup $2^3 \cdot L_3(2)$ of the automorphism group.

This group acts on the Lie algebra by permuting seven mutually orthogonal Cartan subalgebras, and the stabilizer of one of these is a group $2^3 \cdot S_4 \cong 4^2.(2 \times S_3)$. The latter description shows that the induced action on the Cartan subalgebra is the full Weyl group, $W(G_2) \cong 2 \times S_3$, and that the representation of $2^3 \cdot L_3(2)$ on the Lie algebra is induced from the natural 2-dimensional representation of the Weyl group. (In fact, there are two such representations, but they are interchanged by the outer automorphism of $2^3 \cdot L_3(2)$, so it does not matter which one we pick.)

To construct this representation, we first take the 7-dimensional representation of $2^3 \cdot L_3(2)$ generated with respect to a basis $\{i_t \mid t \in \mathbb{F}_2\}$ of a (real) vector space $V$ by the maps $\alpha : t \mapsto t + 1$, $\beta : t \mapsto 2t$ and the involution $\gamma = (i_2, -i_2)(i_4, -i_4)(i_3, i_5)(i_6, i_0)$. It is clear that these maps preserve the set of lines $\{t, t + 1, t + 3\}$ of the projective plane of
order 2, and it is easy to check that the kernel of the action is precisely the group $2^3$ of sign-changes on the complements of the lines. Thus they generate a group of shape $2^3 \cdot L_3(2)$.

Now construct the exterior square of this representation, on the basis

$$\{ u_t = i_{t+2} \wedge i_{t+6}, v_t = i_{t+4} \wedge i_{t+5}, w_t = i_{t+1} \wedge i_{t+3} \}.$$  

The three generators given above act as follows (where $x$ stands for an arbitrary one of $u, v, w$):

$$\begin{align*}
\alpha : & \quad x_t \mapsto x_{t+1} \\
\beta : & \quad u_t \mapsto v_{2t} \mapsto w_{4t} \mapsto u_t \\
\gamma : & \quad u_1 \leftrightarrow v_1, x_2 \leftrightarrow -x_2, u_4 \leftrightarrow -w_1, \\
& \quad u_3 \leftrightarrow v_5, v_3 \leftrightarrow w_5, w_3 \leftrightarrow u_5 \\
& \quad u_6 \leftrightarrow w_0, v_6 \leftrightarrow v_0, w_6 \leftrightarrow u_0
\end{align*}$$

There is a submodule spanned by the $u_t + v_t + w_t$, isomorphic to the original 7-dimensional module spanned by the $i_t$. Factoring this out leaves the 14-dimensional module we require. We use the same notation, now with the understanding that $u_t + v_t + w_t = 0$. Write $L_t$ for the 2-space spanned by $u_t, v_t, w_t$, and let $L = \bigoplus_{t \in F_7} L_t$.

Next consider what products on this 14-space are invariant under the group $2^3 \cdot L_3(2)$. Invariance under the normal $2^3$ implies that the product of a vector in $L_r$ with one in $L_s$ lies in $L_t$, where $\{r, s, t\}$ is a line in the projective plane. Invariance under $\alpha$ means we only need to consider the products on one line, say the line $\{1, 2, 4\}$. Modulo the sign-changes on the $L_t$, the stabilizer of this line is a group $S_4$, generated by $\beta, \gamma, \text{ and } \gamma^{\alpha^{-1}}$.

Invariance under $\gamma$ implies that $[u_1 + v_1, x_2]$ is a scalar multiple of $u_4 + w_4$, and that $[u_1 - v_1, x_2]$ is a scalar multiple of $u_4 - w_4$. So we may assume that $[u_1, v_2] = \lambda w_4 + \mu u_4$ and $[v_1, v_2] = \mu w_4 + \lambda u_4$, and scale so that $\lambda + \mu = 1$. Applying the symmetry $\gamma^\beta$ gives the values of $[u_1, w_2]$ and $[v_1, w_2]$, and the multiplication table can be filled in using the relations $u_t + v_t + w_t = 0$:

<table>
<thead>
<tr>
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<th>$u_2$</th>
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<tbody>
<tr>
<td>$u_1$</td>
<td>$u_4$</td>
<td>$\lambda w_4 + \mu u_4$</td>
<td>$\lambda u_4 + \mu w_4$</td>
</tr>
<tr>
<td>$v_1$</td>
<td>$v_4$</td>
<td>$\lambda u_4 + \mu w_4$</td>
<td>$\lambda w_4 + \mu u_4$</td>
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<tr>
<td>$w_1$</td>
<td>$-2v_4$</td>
<td>$v_4$</td>
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</table>

This gives us a 1-parameter family of (non-associative) algebras invariant under $2^3 \cdot L_3(2)$. The symmetry $\gamma^{\beta^{-1}}$ shows that the multiplication is anti-commutative, and the symmetry $\gamma$ shows that the multiplication is zero on $L_2$, and therefore on each $L_t$.

**Lemma 1.** The multiplication given above satisfies the Jacobi identity if and only if $\mu = 0$ (equivalently, $\lambda = 1$).
Compact real form of $G_2$

**Proof.** The Jacobi identity $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ holds trivially if the three vectors $a, b, c$ are respectively in $L_r$, $L_s$ and $L_t$ where $\{r, s, t\}$ is a line. Now the symmetries show that it is necessary and sufficient for the Jacobi identity to hold for the triples $(u_1, v_1, v_2)$, $(u_3, u_6, v_5)$, and $(u_3, v_6, u_5)$. In the first case we have

$$[[u_1, v_1], v_2] + [[v_1, v_2], u_1] + [[v_2, u_1], v_1] = 0 + [\lambda u_4 + \mu w_4, u_1] - [\lambda w_4 + \mu u_4, v_1]$$

$$= 0$$

since $[u_4, u_1] = [w_4, v_1]$ and $[w_4, u_1] = [u_4, v_1]$. So this case of the identity holds whatever the values of $\lambda$ and $\mu$. In the second case we have

$$[[u_3, u_6], v_5] + [[u_6, v_5], u_3] + [[v_5, u_3], u_6] = -[\lambda w_4 + \mu v_4, v_5] - [v_1, u_3] + 2[w_2, u_6]$$

$$= -\lambda v_0 - \lambda \mu u_0 - \mu^2 w_0 - v_0 + 2v_0$$

$$= \mu (v_0 - \lambda u_0 - \mu w_0)$$

But the expression in brackets is never zero, so $\mu = 0, \lambda = 1$, and the multiplication table is determined. We can now verify the third case of the Jacobi identity as follows:

$$[[u_3, v_6], u_5] + [[v_6, u_5], u_3] + [[u_5, u_3], v_6] = -[u_4, u_5] - [w_1, u_3] - [w_2, v_6]$$

$$= -v_0 - u_0 - w_0 = 0$$

□

**Corollary 2.** Up to scalar multiplication, there is a unique 14-dimensional Lie algebra invariant under the given action of $2^3 L_3(2)$.

As usual, we denote by $ad x$ the linear map $y \mapsto [x, y]$ on $L$, and define the Killing form by $(x, y) := \text{Tr} (ad x \cdot y)$. It is obvious that the Killing form is a symmetric bilinear form.

**Lemma 3.** The Killing form on $L$ is negative definite.

**Proof.** First observe that if $x \in L_0$ then $ad x$ maps $L_t$ into $L_{\pi(t)}$ where $\pi$ is the permutation $(1,3)(2,6)(4,5)$. Similarly if $y \in L_1$ then $ad y$ effects the permutation $(2,4)(3,0)(5,6)$. Therefore $ad x \cdot ad y$ maps every $L_t$ into a different $L_t$, so has trace 0. Hence the $L_t$ are mutually orthogonal with respect to the Killing form.

Now we can calculate the Killing form on $L_0$ using the following two rows of the multiplication table of the algebra:

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<tr>
<th>$u_0$</th>
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<th>$w_1$</th>
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<td>$u_0$</td>
<td>$w_3$</td>
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<td>$w_6$</td>
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<td>$-v_1$</td>
<td>$u_5$</td>
<td>$u_5$</td>
<td>$-u_4$</td>
<td>$-w_4$</td>
<td>$-w_2$</td>
<td>$-u_2$</td>
</tr>
</tbody>
</table>

We find that $\text{Tr} (ad u_0 \cdot ad u_0) = -16$ and $\text{Tr} (ad u_0 \cdot ad v_0) = 8$, so that the Killing form is negative definite on each $L_t$. □
Theorem 4. The Lie algebra $L$ is the compact real form of $\mathfrak{g}_2$.

Proof. We have shown that the Killing form is non-singular, which implies that the Lie algebra $L$ is semisimple. It is easy to see that each $L_t$ is a Cartan subalgebra, so that $L$ has rank 2. The classification of complex semisimple Lie algebras shows immediately that $L$ is of type $G_2$. Since the Killing form is negative definite, $L$ is the compact real form.

Alternatively, a proof from first principles can be obtained by explicitly diagonalising $\text{ad} u_0$ and $\text{ad} v_0$ simultaneously (over $\mathbb{C}$). Their simultaneous eigenspaces are the root spaces, and one recovers the standard construction of the split real form. □

Since each $L_t$ is a Cartan subalgebra, it contains a natural copy of the $G_2$ root system. Our spanning vectors have been chosen so that the short roots in $L_t$ are (up to a suitable scaling factor) $\pm u_t$, $\pm v_t$ and $\pm w_t$. The long roots, similarly, are $\pm (u_t - v_t)$, $\pm (v_t - w_t)$ and $\pm (w_t - u_t)$.

Remark. The multiplication table

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<td>$u_4$</td>
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</tbody>
</table>

can be used to define a (non-associative) algebra on a 21-space over any field of characteristic not 2, invariant under the given action of $2^3 L_3(2)$. The 7-dimensional subspace spanned by the $u_t + v_t + w_t$ is a subalgebra (indeed, an ideal) isomorphic to the algebra of pure imaginary octonions. This subalgebra satisfies the Jacobi identity if and only if the field has characteristic 3. Moreover, its orthogonal complement is closed under multiplication if and only if the field has characteristic 3. In this case, the multiplication on the orthogonal complement also satisfies the Jacobi identity, and we recover the fact that the pure imaginary octonion algebra is a subalgebra (and an ideal) of the Lie algebra of type $\mathfrak{g}_2$ in characteristic 3.