A simple construction of the Ree groups of type $^2F_4$

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Abstract

We give an elementary construction of Ree's family of finite simple groups $^2F_4(q)$, avoiding the need for the machinery of Lie algebras, algebraic groups, or buildings. We calculate the group orders and prove simplicity from first principles. Moreover, this is a practical construction in the sense that it gives an explicit description of the generalized octagon, and from it generators for many of the maximal subgroups may be easily obtained.

Introduction

The last infinite family of finite simple groups to be discovered was the family of large Ree groups $^2F_4(2^{2n+1})$, which were constructed by Rimhak Ree [9] in 1961, by twisting the groups $F_4(2^{2n+1})$ of Lie type with an outer automorphism (see also [1]). A construction in terms of buildings was given by Tits [10], and another in terms of Moufang polygons by Tits and Weiss in Chapter 32 of [11]. However, none of these constructions is easy, and the groups remain somewhat inaccessible and little studied as a result. It is the aim of the present paper to provide a new construction and existence proof which is considerably easier and shorter than any of the previous constructions.

My construction was inspired by [12], in which I gave a new construction of the Ree groups of type $^2G_2$. However, the extra difficulties encountered in 26 dimensions as opposed to 7 forced me to make further simplifications, which in turn led to improvements in my treatment of the small Ree groups [13] and of the Suzuki groups [14]. Instead of using Lie theory, I describe a new algebraic structure whose automorphism group is the Ree group. The generalized octagon has a completely natural definition within this new structure. I then calculate the point stabilizer, count the points and prove transitivity, in order to compute the group order and prove simplicity (for $n > 0$).

On the way, I construct many of the maximal subgroups, and for the benefit of Lie theorists point out the relationships to the long and short root elements, parabolic subgroups, maximal torus, Weyl group, and so on.

After this work was done, I became aware of the work of Kris Coolsaet [2, 3, 4] who has independently, and somewhat earlier, come up with a description of the Ree–Tits octagon which is almost identical in substance to my description below. His motivation is more geometrical than group-theoretical, in particular to provide a practical framework for working with the geometry, and to solve van Maldeghem's Problem 10 for generalized polygons [8]. He does not consider the main problem which I solve here, namely to give an elementary existence proof for the Ree groups. However, his work already includes explicit formulae for the long and short root elements, and explicit generators for the maximal parabolic subgroups as well as $\mathrm{SL}_2(q) \wr 2$ and $\mathrm{Sz}(q) \wr 2$.

Even where my work overlaps with Coolsaet's, I feel that the treatment is sufficiently different to justify including it here in full.

1 Definitions

Let $F = \mathbb{F}_q$ be the field of order $q = 2^{2n+1}$, and let V be the vector space of dimension 27 over F spanned by vectors w_t, w'_t and w''_t for $-4 \le t \le 4$. We write the typical vector of V as

$$v = \sum_{t=-4}^{4} (\lambda_t w_t + \lambda'_t w'_t + \lambda''_t w''_t).$$

Let W be a vector space of dimension 26, which will be identified either with a subspace of V by imposing the condition $\lambda_0 + \lambda'_0 + \lambda''_0 = 0$, or with a quotient of V by imposing the condition $w_0 + w'_0 + w''_0 = 0$.

Define a quadratic form Q_V on V by

$$Q_V(v) = \lambda_0\lambda'_0 + \lambda_0\lambda''_0 + \lambda'_0\lambda''_0 + \sum_{t=1}^{4}(\lambda_t\lambda_{-t} + \lambda'_t\lambda'_{-t} + \lambda''_t\lambda''_{-t}).$$

Let $Q = Q_W$ be the restriction of Q_V to W regarded as a subspace of V. The associated symmetric bilinear form is $B(v, w) = Q(v + w) + Q(v) + Q(w)$ in the usual way. It is easy to see that this quadratic form is non-degenerate, of minus type. (The radical of B_V defined by $B_V(v, w) = Q_V(v + w) + Q_V(v) + Q_V(w)$ is the 1-space spanned by $w_0 + w'_0 + w''_0$.)

Define a cubic form C_V on V by

$$C_V(v) = \sum_{(i,j,k)} \lambda_i\lambda'_j\lambda''_k + \sum_{t=-1}^{1} \lambda_t\lambda'_t\lambda''_t + \sum_{t=1}^{4}(\lambda_t\lambda_0\lambda_{-t} + \lambda'_t\lambda'_0\lambda'_{-t} + \lambda''_t\lambda''_0\lambda''_{-t})$$

where the first sum is over triples (i, j, k) which are cyclic permutations of

$$\pm(1, -2, 2), \pm(1, -3, 3), \pm(1, 4, -4), \pm(-4, 2, 3), \pm(-4, 3, 2).$$

Denote by $C = C_W$ the restriction of C_V to W, regarded as a subspace of V. The associated symmetric trilinear form T is defined by

$$T(u, v, w) = C(u)+C(v)+C(w)+C(u+v)+C(u+w)+C(v+w)+C(u+v+w).$$

There is now a natural bilinear, commutative, product $u \circ v$ defined on W by the identity

$$T(u, v, w) = B(u \circ v, w)$$

for all $u, v, w \in W$. Notice that $T(v, v, w) = 0$, and therefore $v \circ v = 0$ for all $v \in W$. The 'interesting' part of the multiplication table is given by

$\circ$	w''_{-4}	w''_{-3}	w''_{-2}	w''_{-1}	w''_1	w''_2	w''_3	w''_4
w'_{-4}					w_{-4}	w_{-3}	w_{-2}	w_1
w'_{-3}			w_{-4}	w_{-3}			w_{-1}	w_2
w'_{-2}		w_{-4}		w_{-2}		w_{-1}		w_3
w'_{-1}	w_{-4}			w_1		w_2	w_3	
w'_1		w_{-3}	w_{-2}		w_{-1}			w_4
w'_2	w_{-3}		w_1		w_2		w_4	
w'_3	w_{-2}	w_1			w_3	w_4		
w'_4	w_{-1}	w_2	w_3	w_4				

and images under the 'triality map' $w_t \mapsto w'_t \mapsto w''_t \mapsto w_t$. The rest of the multiplication is given by $w_0 \circ w'_0 = w''_0$, $w_0 \circ w'_t = w'_t$ and $w_t \circ w_{-t} = w_0$ for $t \neq 0$, and images under triality. All other products of basis vectors are 0. (Here our notation for W is as a quotient space of V, so a little care is needed when calculating Q and C. In particular, the equation $w_0 \circ w'_t = w'_t$ looks wrong, but is in fact correct, for if we want W as a subspace of V we must replace w_0 by $w'_0 + w''_0$, and the equation becomes $(w'_0 + w''_0) \circ w'_t = w'_t$, which is what one would expect.)

Before we define the rest of the algebraic structure, consider the following coordinate permutations, which act both on V, and on W.

$$\begin{aligned} \rho \;=\; & (w_{-1}w_1)(w_{-2}w_2) \\ & (w'_{-4}w'_{-3})(w'_{-2}w'_{-1})(w'_1w'_2)(w'_3w'_4) \\ & (w''_{-4}w''_{-3})(w''_{-2}w''_1)(w''_{-1}w''_2)(w''_3w''_4) \end{aligned}$$

$$\begin{array}{rcl}\sigma & = & (w_0w''_0)(w_{-4}w''_{-4})(w_{-1}w''_1)(w_1w''_{-1})(w_4w''_4)\\ & & (w_{-3}w''_{-2})(w_{-2}w''_{-3})(w_3w''_2)(w_2w''_3)\\ & & (w'_{-3}w'_{-2})(w'_{-1}w'_1)(w'_2w'_3).\end{array}$$

It is easy to verify that the product of these two involutions has order 8, and therefore they generate a dihedral group D of order 16. This will shortly be revealed as the Weyl group. It is also quite easy to check that ρ and σ preserve the quadratic form Q and the cubic form C, in the sense that $Q(w^\rho) = Q(w^\sigma) = Q(w)$ and $C(w^\rho) = C(w^\sigma) = C(w)$ for all $w \in W$. They both act by permuting the terms in the definitions. Indeed, there are just seven orbits, of lengths 1, 4, 4, 4, 8, 8, 16, of D on the 45 terms in the definition of C.

Finally we shall define a new symmetric (partial) product $\bullet$ on W (regarded as a quotient of V), such that $u \bullet v$ is defined whenever $u \circ v = 0$, and satisfies

$$\begin{array}{rcl}u \bullet v & = & v \bullet u\\ u \bullet (v+w) & = & u \bullet v + u \bullet w\\ u \bullet (\lambda v) & = & \lambda^{2^n}(u \bullet v)\\ v \bullet v & = & 0.\end{array}$$

In order to make it clear that the product is well-defined it is useful to define it for all values of u and v, but it must be borne in mind that only the values $u \bullet v$ for which $u \circ v = 0$ will necessarily be invariant under the group. By the twisted linearity laws above it is sufficient to define $\bullet$ on a basis. The 'interesting' part of the multiplication is given by

$$\begin{array}{c}w_3 \bullet w_4 = w'_3 \bullet w'_4 = w''_3 \bullet w''_4 = w_4\\ w'_1 \bullet w'_2 = w''_{-1} \bullet w''_2 = w_{-3} \bullet w_4 = w_3\\ w_1 \bullet w_4 = w'_2 \bullet w'_3 = w'_4\end{array}$$

and images under the action of D. The only other non-zero terms may be taken to be

$$\begin{array}{rcl}w_{-2} \bullet w_2 = w'_{-2} \bullet w'_2 = w''_{-2} \bullet w''_2 & = & w_0\\ w_{-3} \bullet w_3 = w'_{-3} \bullet w'_3 = w''_{-3} \bullet w''_3 & = & w''_0\\ w_{-4} \bullet w_4 = w'_{-4} \bullet w'_4 = w''_{-4} \bullet w''_4 & = & w'_0\end{array}$$

which are *not* invariant under the action of D. This product is written out in more detail in the Appendix, to facilitate calculations. Note in particular that all 'mixed' products are zero, that is $w_i \bullet w'_j = w_i \bullet w''_k = w'_j \bullet w''_k = 0$.

We see that the terms involving a zero subscript are of the form $u \bullet v$ where $u \circ v \neq 0$. From them, however, we obtain values such as

$$\begin{array}{rcl}(w_{-1}+w_{-3}) \bullet (w_1+w_3) & = & w_{-3} \bullet w_1 + (w_{-1} \bullet w_1 + w_{-3} \bullet w_3) + w_{-1} \bullet w_3\\ & = & w''_{-1} + w''_0 + w''_1\end{array}$$

which is properly defined, since $(w_{-1}+w_{-3})\circ(w_1+w_3)=0$.

Let $\mathbb{W}$ denote the vector space W endowed with the forms Q and C, and the partial product $\bullet$. Let $R=R(2^{2n+1})$ be the automorphism group of $\mathbb{W}$, i.e. the group of linear maps g on W which preserve Q, C and $\bullet$, in the sense that $Q(w^g)=Q(w)$ and $C(w^g)=C(w)$ for all w, and $v^g\bullet w^g=(v\bullet w)^g$ whenever $v\circ w=0$. I claim, and shall prove, that R is the Ree group ${}^2F_4(2^{2n+1})$.

A crucial ingredient of the proof is a grading of the coordinates. In fact, we define three gradings on the coordinates which are compatible with the various products, in a sense to be explained shortly. The grade of w_{-t} is minus the grade of w_t, and the positive grades are as follows:

Vector	w'_1	w_1	w''_{-1}	w_2	w''_2	w_3	w'_2	w''_3	w'_3	w'_4	w''_4	w_4
A grade	1	2	3	4	5	6	7	8	9	10	11	12
B grade	1	1	2	3	4	5	5	6	7	9	10	11
C grade	1	1	2	2	3	3	4	4	5	6	7	8

(Mnemonic: A stands for absolute, B for bullet (product) and C for cubic (form).) Then for all three gradings, all the terms in the quadratic form have total grade 0. The terms in the cubic form have total C-grade 0. And the B-grade of $v\bullet w$ is the sum of the C-grades of v and w.

Now define the (A-, B-, C-)grade of an arbitrary vector to be the largest (A-, B-, C-)grade of its non-zero coefficients, and define the (A-, B-, C-)leading term of w to be the term with the highest (A-, B-, C-)grade. In particular the 'leading term' may depend on which grade is being used, and may not be uniquely defined. For most purposes, we regard the coordinates as being in the order of their A grade, as follows (with some ambiguity in the coordinates with subscript 0):

$$\begin{array}{l} w_{-4}, w''_{-4}, w'_{-4}, w'_{-3}, w''_{-3}, w'_{-2}, w_{-3}, w''_{-2}, w_{-2}, w''_1, w_{-1}, w'_{-1}, \\ w_0, (w'_0,) w''_0, w'_1, w_1, w''_{-1}, w_2, w''_2, w_3, w'_2, w''_3, w'_3, w'_4, w''_4, w_4. \end{array}$$

2 Remarks on the definitions

The motivation for most of the above definitions comes from consideration of the exceptional Jordan algebra. This consists of 3×3 Hermitian matrices over octonions, which we write in the shorthand notation

$$(x,y,z\mid X,Y,Z)=\begin{pmatrix} x & Z & \overline{Y} \\ \overline{Z} & y & X \\ Y & \overline{X} & z \end{pmatrix}$$

where x,y,z are scalars and X,Y,Z are octonions. The cubic form C is a characteristic 2 version of the so-called determinant, which defines the exceptional group $E_6(q)$. Fixing a particular element of determinant 1, and calling it the identity element 1, defines the Jordan algebra itself. In characteristic 2, we may restrict

C to the trace 0 part of the exceptional Jordan algebra, and define a quadratic form $Q(v) = C(v) + C(1+v)$ on this space. This is the same as the form Q defined above, and $\circ$ is the Jordan product (modulo the identity element).

Our basis is chosen so that w_0, w_0' and w_0'' correspond to diagonal matrices $(1,0,0 \mid 0,0,0)$, $(0,1,0 \mid 0,0,0)$ and $(0,0,1 \mid 0,0,0)$ modulo the identity matrix (or to $(0,1,1 \mid 0,0,0)$, $(1,0,1 \mid 0,0,0)$ and $(1,1,0 \mid 0,0,0)$ in the subspace of trace 0 matrices); and $w_i = (0,0,0 \mid x_i,0,0)$, $w_i' = (0,0,0 \mid 0,x_i,0)$ and $w_i'' = (0,0,0 \mid 0,0,x_i)$, where $\{x_{\pm i} \mid 1 \leq i \leq 4\}$ forms a basis for the split form of the octonion algebra. The triality map then corresponds to a cyclic permutation of the rows and columns of the matrices: $(a,b,c \mid A,B,C) \mapsto (c,a,b \mid C,A,B)$. The correspondence between the Jordan multiplication and the octonion multiplication is given by $w_i w_j' = w_k''$ when $x_i x_j = \overline{x_k}$. Moreover, $\overline{x_1} = x_{-1}$ and $\overline{x_k} = x_k$ for all $k \neq \pm 1$.

Coolsaet's definitions [2, 3] are expressed in terms of a different definition of the determinant, but amount to the same thing. A similar construction of Q and C (but obviously not $\bullet$) works in arbitrary characteristic, but then one has to introduce some signs into the definitions. This has also been done by Rylands and Taylor [7].

The 24 coordinates with non-zero subscript can be identified with the short roots of the F_4 root system. Here is one possible labelling, where $+$ stands for $\frac{1}{2}$ and $-$ stands for $-\frac{1}{2}$:

Vector	Root	Vector	Root	Vector	Root
w_{-4}	1000	w'_{-4}	$+++-$	w''_{-4}	$++++$
w_{-3}	0100	w'_{-3}	$++-+$	w''_{-3}	$++--$
w_{-2}	0010	w'_{-2}	$+-++$	w''_{-2}	$+-+-$
w_{-1}	0001	w'_{-1}	$+---$	w''_{-1}	$-++-$
w_1	$-(0001)$	w_1'	$-+++$	w_1''	$+--+$
w_2	$-(0010)$	w_2'	$-+--$	w_2''	$-+-+$
w_3	$-(0100)$	w_3'	$--+-$	w_3''	$--++$
w_4	$-(1000)$	w_4'	$---+$	w_4''	$----$

The triples which occur in the definition of C and have no 0 subscript are precisely the triples of short roots whose sum is 0. In other words, the Jordan product of two short roots is the vector corresponding to the sum of these roots whenever this sum is itself a short root.

Now consider the map

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

which maps short roots to long roots. The $\bullet$ product is defined for any two perpendicular short roots, and its value corresponds to the preimage of their sum under this map.

The $\bullet$ product also takes values w_0, w_0' and w_0'' which are less easy to motivate in this context. We justify our definition by the fact that it works. A different justification is given in [15], where we show that the $\bullet$ product is uniquely determined by the irreducible subgroup $\mathrm{SL}_3(3)$.

A more formal definition of the $\bullet$ product may be obtained by first noting that the Jordan product $\circ$ is equivalent to a linear map $j : W \wedge W \to W$, where we may write $u \circ v = j(u \wedge v)$. Similarly, the new product is equivalent to a map $\pi : \ker(j) \to W$ which is $\mathbb{F}_2$-linear but not F-linear. Indeed, $\pi(\lambda w) = \lambda^{2^n} w$ for any $w \in W \wedge W$ and any $\lambda \in F$, and we write $u \bullet v = \pi(u \wedge v)$. To check that π or $\bullet$ is invariant under a particular linear map, it may be necessary (and sufficient) to check it on a basis for $\ker(j)$, which has dimension 299, so this may not be a trivial task!

The grading can be explained by one of the standard orderings on the roots. For example, the C-grade is the inner product of the short root with the vector $-(8, 3, 2, 1)$, and the B-grade is the inner product with $-(11, 5, 3, 1)$. Alternatively, the B-grade of a short root is the C-grade of the corresponding long root.

3 Some symmetries of the algebra

In this section I shall define some linear maps and prove that they preserve the algebra $\mathbb{W}$. Later on these maps will be identified with elements of the torus, Weyl group, and root subgroups of the Ree group. We have already proved that $\mathbb{W}$ is invariant under the Weyl group D of order 16 generated by the coordinate permutations ρ and σ.

Next define the following diagonal elements, where α and β are arbitrary non-zero elements of the field F. The coordinate vectors are eigenvectors, and the eigenvalues on w_t are the inverses of those on w_{-t} (and similarly for w_t' and w_t'').

Vector	Eigenvalue
w_{-4}	α
w''_{-4}	β
w'_{-4}	$\alpha^{2^{n+1}-1}\beta^{2^{n+1}-1}$
w'_{-3}	$\alpha\beta^{1-2^{n+1}}$
w''_{-3}	$\alpha^{2^{n+1}}\beta^{-1}$
w'_{-2}	$\alpha^{1-2^{n+1}}\beta$

Vector	Eigenvalue
w_{-3}	$\alpha^{2^{n+1}-1}$
w''_{-2}	$\beta^{2^{n+1}-1}$
w_{-2}	$\alpha^{-1}\beta^{2^{n+1}}$
w''_1	$\alpha^{2-2^{n+1}}\beta^{1-2^{n+1}}$
w_{-1}	$\alpha^{1-2^{n+1}}\beta^{2-2^{n+1}}$
w'_{-1}	$\alpha\beta^{-1}$

Again it is easy to check that these elements preserve Q, C and $\bullet$. Indeed, they fix each term of Q and C individually. They form what will shortly be revealed as the maximal torus, which is normalised by the Weyl group already given. Indeed, σ interchanges α and β, so it is easy to check by eye that it normalises the torus just defined. The other generator ρ fixes α and maps β to $\beta^{-1}\alpha^{2^{n+1}}$ so takes only a little more effort to check.

Next we wish to define a long root element, t say. We regard W as a quotient of V, so that $w_0 + w_0' + w_0'' = 0$. It is sufficient for the definition of t to say that it fixes w_4', w_{-4}' and w_{-1}' and maps

$$w_1' \mapsto w_1' + w_0' + w_{-1}'.$$

Then the fact that it fixes $\bullet$ implies that it fixes w_{-3}' and w_2', and maps

$$\begin{array}{rcl} w_3' & \mapsto & w_3' + w_2' \\ w_{-2}' & \mapsto & w_{-2}' + w_{-3}' \\ w_0 & \mapsto & w_0 + w_{-1}' \end{array}$$

It then follows that $w_0' = w_2' \bullet w_{-2}' + w_3' \bullet w_{-3}'$ is also fixed. Then the multiplication table in the Appendix implies that t fixes w_{-4}, w_1, w_2, w_{-3}'', w_1'', w_4'', and maps

$$\begin{array}{rcl} w_{-3} & \mapsto & w_{-3} + w_{-3}'' \\ w_{-2} & \mapsto & w_{-2} + w_{-2}'' + w_{-3} + w_{-3}'' \\ w_{-1} & \mapsto & w_{-1} + w_1'' \\ w_3 & \mapsto & w_3 + w_2 \\ w_4 & \mapsto & w_4 + w_4'' \\ w_{-4}'' & \mapsto & w_{-4}'' + w_{-4} \\ w_{-2}'' & \mapsto & w_{-2}'' + w_{-3}'' \\ w_{-1}'' & \mapsto & w_{-1}'' + w_1 \\ w_2'' & \mapsto & w_2'' + w_2 \\ w_3'' & \mapsto & w_3'' + w_3 + w_2'' + w_2 \end{array}$$

To check that this element t preserves Q, C and $\bullet$ requires a certain amount of work, but is not as difficult as it at first appears. To see that t preserves Q, observe that it does the same to the λs as it does to the ws, so by substitution we obtain the new value of Q as

$$\begin{array}{rcl} Q(v') & = & (\lambda_0 + \lambda_{-1})^2 + (\lambda_0')^2 + (\lambda_0 + \lambda_{-1})\lambda_0' \\ & & + (\lambda_4 + \lambda_4'')\lambda_{-4} + \lambda_4'\lambda_{-4}' + \lambda_4''(\lambda_{-4}'' + \lambda_{-4}) \\ & & + (\lambda_3 + \lambda_2)(\lambda_{-3} + \lambda_{-3}'') + (\lambda_3' + \lambda_2')\lambda_{-3}' + \lambda_{-3}''(\lambda_3'' + \lambda_3 + \lambda_2'' + \lambda_2) \\ & & + \lambda_2(\lambda_{-2} + \lambda_{-2}'' + \lambda_{-3} + \lambda_{-3}'') + \lambda_2'(\lambda_{-2}' + \lambda_{-3}') + (\lambda_2'' + \lambda_2)(\lambda_{-2}'' + \lambda_{-3}'') \\ & & + \lambda_1(\lambda_{-1} + \lambda_1'') + (\lambda_1' + \lambda_0' + \lambda_{-1}')\lambda_{-1}' + \lambda_1''(\lambda_{-1}'' + \lambda_1) \\ & = & (\lambda_0)^2 + (\lambda_0')^2 + \lambda_0\lambda_0' + \lambda_4\lambda_{-4} + \lambda_4'\lambda_{-4}' + \lambda_4''\lambda_{-4}'' + \lambda_3\lambda_{-3} + \lambda_3'\lambda_{-3}' + \lambda_3''\lambda_{-3}'' \\ & & +\lambda_2\lambda_{-2} + \lambda_2'\lambda_{-2}' + \lambda_2''\lambda_{-2}'' + \lambda_1\lambda_{-1} + \lambda_1'\lambda_{-1}' + \lambda_1''\lambda_{-1}'' \\ & = & Q(v) \end{array}$$

as required. Similarly to check that t fixes C it is sufficient to substitute in the formula, although the calculations are a little more substantial. Finally we sketch the proof that $\bullet$ is preserved. First look at the 10-space $W' = \langle w'_i, w_0 \rangle$. Most of the product on this space has been used to obtain the action of t on the remainder of the space. All that remains is to check the products which lie inside this 10-space. There are 16 such products, and they are all very easy to check, for example

$$(w'_3 + w'_2) \bullet (w'_{-2} + w'_{-3}) = w'_1 + w'_{-1} + w'_0.$$

Since t fixes W' and $W'^{\perp}$, it is immediate that the products $w'_i \bullet w_j$ and $w'_i \bullet w''_k$ remain zero. It is also easy to check the six required equations in which the product lies in $\langle w_0, w'_0 \rangle$, such as

$$w_1 \bullet (w_{-1} + w''_1) + w_2 \bullet (w_{-2} + w_{-3} + w''_{-2} + w''_{-3}) = w_0 + w'_{-1}.$$

This leaves the 16-space $W'^{\perp}$, which is a little more difficult. We illustrate the argument by showing that the products involving w_4 are preserved. The following deals with the products with w_i:

$$\begin{array}{rcl}
(w_4 + w''_4) \bullet (w_4 + w''_4) & = & 0 \\
(w_4 + w''_4) \bullet (w_3 + w_2) & = & w_4 + w''_4 \\
(w_4 + w''_4) \bullet w_2 & = & w''_4 \\
(w_4 + w''_4) \bullet w_1 & = & w'_4 \\
(w_4 + w''_4) \bullet (w_0 + w'_{-1}) & = & 0 \\
(w_4 + w''_4) \bullet (w_{-1} + w''_1) & = & w'_3 + w'_2 \\
(w_4 + w''_4) \bullet (w_{-2} + w_{-3} + w''_{-2} + w''_{-3}) & = & w_3 + w_2 + w''_3 + w''_2 \\
(w_4 + w''_4) \bullet (w_{-3} + w''_{-3}) & = & w_3 + w_2 \\
(w_4 + w''_4) \bullet w_{-4} & = & w'_0
\end{array}$$

Now the products with w''_i are trickier as the Jordan products are not always zero. Specifically, $w_4 \circ w''_1 = w'_4$, $w_4 \circ w''_{-2} = w'_3$, $w_4 \circ w''_{-3} = w'_2$ and $w_4 \circ w''_{-4} = w'_1$. Therefore we need to check the following sums, rather than individual terms:

$$\begin{array}{rcl}
w_4 \bullet w''_1 & + & w_3 \bullet w''_2 \\
w_4 \bullet w''_{-2} & + & w_3 \bullet w''_{-1} \\
w_4 \bullet w''_{-3} & + & w_2 \bullet w''_{-1} \\
w_4 \bullet w''_{-4} & + & w_{-1} \bullet w''_{-1}
\end{array}$$

Together with the other four products, this gives:

$$\begin{array}{rcl}
(w_4 + w''_4) \bullet w''_4 & = & 0 \\
(w_4 + w''_4) \bullet (w''_3 + w''_2 + w_3 + w_2) & = & w_4 + w''_4 + w_4 + w''_4 \\
(w_4 + w''_4) \bullet (w''_2 + w_2) & = & w''_4 + w''_4 \\
(w_4 + w''_4) \bullet w''_1 + (w_3 + w_2) \bullet (w''_2 + w_2) & = & w'_2 + w'_2 \\
(w_4 + w''_4) \bullet (w''_{-1} + w_1) & = & w'_4 + w'_4 \\
(w_4 + w''_4) \bullet (w''_{-2} + w''_{-3}) + (w_3 + w_2) \bullet (w''_{-1} + w_1) & = & w_2 + w''_2 + w_2 + w''_2 \\
(w_4 + w''_4) \bullet w''_{-3} + w_2 \bullet (w''_{-1} + w_1) & = & w_2 + w_2
\end{array}$$

$$(w_4 + w_4'') \bullet (w_{-4}'' + w_{-4}) + (w_{-1} + w_1'') \bullet (w_{-1}'' + w_1) \quad = \quad w_0' + w_0'$$

all of which evaluate to 0.

Similarly we may define the short root element x of order 4 fixing w_{-4}, w_3, w_{-2}, w_4, and mapping

$$\begin{aligned} w_{-1} &\mapsto w_{-1} + w_{-2}, \\ w_2 &\mapsto w_2 + w_1 + w_0 + w_{-2}. \end{aligned}$$

We then deduce that $w_{-3} = w_{-4} \bullet w_3$ is fixed, as is $w_0 = w_3 \bullet w_{-3} + w_4 \bullet w_{-4}$, and also

$$\begin{aligned} w_0' &\mapsto w_0' + w_{-1} \\ w_1 &\mapsto w_1 + w_0 + w_{-1} + w_{-2} \end{aligned}$$

Every other basis vector is a product of two of these, so we deduce that

$$\begin{aligned} w_{-4}' &\mapsto w_{-4}' + w_{-4}'' \\ w_{-3}' &\mapsto w_{-3}' + w_{-4}' + w_{-4}'' \\ w_{-3}'' &\mapsto w_{-3}'' + w_{-3}' + w_{-4}'' \\ w_{-2}'' &\mapsto w_{-2}'' + w_{-2}' \\ w_1'' &\mapsto w_1'' + w_{-2}'' + w_{-2}' \\ w_{-1}' &\mapsto w_{-1}' + w_1'' + w_{-2}' \\ w_{-1}'' &\mapsto w_{-1}'' + w_1' \\ w_2'' &\mapsto w_2'' + w_{-1}'' + w_1' \\ w_2' &\mapsto w_2' + w_2'' + w_1' \\ w_3' &\mapsto w_3' + w_3'' \\ w_4' &\mapsto w_4' + w_3' + w_3'' \\ w_4'' &\mapsto w_4'' + w_4' + w_3'' \end{aligned}$$

and w_{-2}', w_1', w_{-4}'' and w_3'' are fixed. Checking that x preserves $\bullet$ is a little more time-consuming than checking t, but no more difficult. Notice that in both cases the vectors which are added on to the listed basis vectors are of strictly smaller grade, with respect to all three gradings.

Coolsaet's definitions of the long and short root elements rely more on the Lie theory than our definitions do, but of course are equivalent. Explicit root elements have also been calculated by Howlett, Rylands and Taylor [5], in order to obtain explicit generators for the Ree groups as 26×26 matrices in the Magma computer algebra system.

4 The fundamental subgroups $\mathrm{SL}_2(q)$ and $\mathrm{Sz}(q)$

Consider the group generated by t, σ and the elements of the torus with $\beta = \alpha^{-1}$. These elements act on $\langle w_4'', w_4 \rangle$ as the standard generators

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

of $\mathrm{SL}_2(q)$, and on $\langle w_{-4}, w''_{-4}\rangle$, $\langle w''_{-1}, w_1\rangle$ and $\langle w_{-1}, w''_1\rangle$ in the same way. On the 2-spaces $\langle w'_2, w'_3\rangle$ and $\langle w'_{-3}, w'_{-2}\rangle$ they act in the natural action twisted by the field automorphism $x \mapsto x^{2^{n+1}}$. On the 4-spaces $\langle w''_{-3}, w_{-3}, w''_{-2}, w_{-2}\rangle$ and $\langle w_2, w''_2, w_3, w''_3\rangle$ the action is the tensor product of these two actions.

The remaining 6 coordinates are $w'_{\pm 4}$, $w'_{\pm 1}$, w_0 and w'_0. The first two are centralised by the given elements, and on the 4-space $\langle w'_{-1}, w_0, w''_0, w'_1\rangle$ we see the tensor product of the natural representation twisted by the field automorphism $x \mapsto x^{2^{2n}}$, with itself. In particular, this group is isomorphic to $\mathrm{SL}_2(q)$.

Notice that conjugating our $\mathrm{SL}_2(q)$ by σ^ρ gives us another copy of $\mathrm{SL}_2(q)$ which commutes with the first. Thus R contains a group $\mathrm{SL}_2(q) \wr 2$ generated by t, σ, σ^ρ and the torus. In fact, if we adjoin also ρ, then we obtain a maximal subgroup $\mathrm{Sp}_4(q).2$. (The complete list of maximal subgroups was determined by Malle [6].)

Similarly, we shall show that the subgroup generated by x, ρ, and the elements of the torus with $\alpha = 1$, is isomorphic to $\mathrm{Sz}(q)$. On the 4-space $\langle w''_3, w'_3, w'_4, w''_4\rangle$ we write down the generators as 4×4 matrices as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta^{2^{n+1}-1} & 0 & 0 \\ 0 & 0 & \beta^{1-2^{n+1}} & 0 \\ 0 & 0 & 0 & \beta^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and notice that these are the standard generators for the Suzuki group $\mathrm{Sz}(q)$. The given generators act in the same way on the 4-spaces $\langle w''_{-4}, w'_{-4}, w'_{-3}, w''_{-3}\rangle$, $\langle w'_1, w''_{-1}, w''_2, w'_2\rangle$ and $\langle w'_{-2}, w''_{-2}, w''_1, w'_{-1}\rangle$. Finally, they act in the exterior square of the natural representation on the 6-space $\langle w_{-2}, w_{-1}, w_0, w'_0, w_1, w_2\rangle$. In particular this group is $\mathrm{Sz}(q)$.

Notice that conjugating our $\mathrm{Sz}(q)$ by ρ^σ gives us another copy of $\mathrm{Sz}(q)$ which commutes with the first. Thus R contains a group $\mathrm{Sz}(q) \wr 2$ generated by x, ρ, ρ^σ and the torus.

5 The generalized octagon

In order to prove that R is a simple group of order $q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$ we shall construct the generalized octagon on which it acts.

Define a *point* to be a 1-dimensional subspace $\langle v\rangle$ of W such that $v = v \bullet w$ for some vector w. Define two points spanned by vectors u and v to be *adjacent* if $u \circ v = 0$ and $u \bullet v = 0$. A *line* is a space $\langle u, v\rangle$ spanned by two adjacent points $\langle u\rangle$, $\langle v\rangle$. In this section I shall show that the points are just the images under the automorphism group of $\mathbb{W}$ of $\langle w_{-4}\rangle$, and that the number of them is $1 + q + q^3 + q^4 + q^6 + q^7 + q^9 + q^{10} = (1+q)(1+q^3)(1+q^6)$.

First observe that the leading term of $v \bullet w$ is (up to a scalar multiplication) the product of the leading terms of v and w. Here the leading term is well-defined, because if two basis vectors have the same C-grade, then at most one of them has non-zero $\bullet$-product with any given basis vector. In particular, if $v = v \bullet w$ then the leading term of v is equal to the leading term of v times the leading term of w. Thus we read off from the multiplication table that the leading term of v is one of

$$w_{\pm 4}, w''_{\pm 4}, w''_{\pm 3}, w_{\pm 2}.$$

If the leading term is any of w_4, w''_3, w_{-2} or w''_{-4} we use the group $\mathrm{SL}_2(q)$ described above, generated by σ and t, and an element of the torus with $\alpha\beta = 1$, to map v to a vector with a lower-grade leading term. First observe that this group acts on the 2-spaces $\langle w_4, w''_4\rangle$ and $\langle w''_{-4}, w_{-4}\rangle$ in its natural representation, so acts transitively on the $q+1$ subspaces of dimension 1. This deals with the two cases w_4 and w''_{-4}.

In the other two cases, the action is on a 4-space, respectively $\langle w''_3, w_3, w''_2, w_2\rangle$ and $\langle w_{-2}, w''_{-2}, w_{-3}, w''_{-3}\rangle$, as the tensor product of two Frobenius automorphs of the natural representation. Without loss of generality, consider the first case. Thus the leading term of v is w''_3, and the leading term of w is w''_{-1}. Then we can use conjugates of t by elements of the torus to ensure that the coefficient of w''_2 in v is 0. Since $v \circ w = 0$, the coefficients of w_2 and w_3 in v are also 0. Hence applying σ reduces the grade of the leading term of v as claimed. [If the coefficient of w'_2 in v were non-zero, this would create a term in w'_3, which would then be the leading term of v. This is a contradiction.]

In the remaining cases we use instead the group generated by x and ρ, and the elements of the torus with $\alpha = 1$, which, as we have seen, is isomorphic to $\mathrm{Sz}(q)$. It acts on the following two 4-spaces:

$$\langle w''_4, w'_4, w'_3, w''_3\rangle$$
$$\langle w''_{-3}, w'_{-3}, w'_{-4}, w''_{-4}\rangle$$

We explain the first case: the other is similar. The leading term of v is w''_4 and the leading term of w is w''_2. First use a conjugate of x by an element of the torus to ensure the coefficient of w'_3 is 0. Then use a conjugate of x^2 to ensure the coefficient of w''_3 is 0. Then we have (modulo lower terms) $v = w''_4 + \alpha w''_3$ and $w = w''_2 + \lambda w''_1 + \mu w'_1$. Thus $v \bullet w = w''_4 + \alpha w'_3$ plus lower terms, whence $\alpha = 0$. Thus applying σ reduces the grade of the leading term of v.

In the final case, we have to consider the 6-space

$$\langle w_2, w_1, w_0, w'_0, w_{-1}, w_{-2}\rangle.$$

The leading term of v is w_2 and the leading term of w is w_1. Use a conjugate of x to ensure the coefficient of w_1 in v is 0, and then a conjugate of x^2 to ensure the coefficient of w_0 is 0. Now $w'_0 \circ w_1 = w_1$ so the coefficient of w'_0 in v is 0. At this point we can apply a conjugate of x^2 again to reduce the coefficient of w_{-2} to 0, and then ρ reduces the leading term of v, as required. [If the coefficient of

w''_{-1} or w'_1 in v were non-zero, this would create a term in w''_2 or w'_2, which would be the leading term of v. This is a contradiction.]

We have shown that all the points are images of $\langle w_{-4}\rangle$. Moreover, the argument in reverse shows us exactly how many images there are with each leading term. Indeed, there is just one point $\langle w_{-4}\rangle$ with leading term w_{-4}, and there are q points $\langle w''_{-4} + \lambda w_{-4}\rangle$ with leading term w''_{-4}. Each of the latter gives rise to q^2 points with leading term w''_{-3}, since in the 4-space $\langle w''_{-3}, w_{-3}, w'_{-4}, w''_{-4}\rangle$ there are exactly q^2+1 points of the Suzuki ovoid, one being $\langle w''_{-4}\rangle$ and the other q^2 having leading term w''_{-3}. Thus we have altogether q^3 points with leading term w''_{-3}. At the next stage, we have $\mathrm{SL}_2(q)$ acting on the 4-space $\langle w_{-2}, w''_{-2}, w_{-3}, w''_{-3}\rangle$, and the point $\langle w''_3\rangle$ has just $q+1$ images under this group, since it is fixed by the Borel subgroup of index $q+1$. Continuing in this way we obtain in total

$$1 + q + q^3 + q^4 + q^6 + q^7 + q^9 + q^{10} = (q^6+1)(q^3+1)(q+1)$$

points, since every time we use $\mathrm{SL}_2(q)$, we introduce a factor of q, and every time we use $\mathrm{Sz}(q)$, we introduce a factor of q^2.

6 The stabiliser of a point

In this section I show that the stabiliser of a point has order $q^{12}(q^2+1)(q-1)^2$. In fact this subgroup has shape $q.q^4.q.q^4.(\mathrm{Sz}(q) \times C_{q-1})$, and in Lie theoretic terms this is of course a maximal parabolic subgroup. First define two points $\langle u, v\rangle$ to be *opposite* if $B(u, v) \neq 0$. In the previous section we showed that for a fixed point $\langle u\rangle$ there are exactly q^{10} points $\langle v\rangle$ which are opposite to it, and that the stabiliser of $\langle u\rangle$ is transitive on them. Picking $u = w_{-4}$ and $v = w_4$, it now suffices to show that the subgroup fixing the vectors w_{-4} and w_4 is exactly the fundamental $\mathrm{Sz}(q)$ exhibited above. (The stabilizer of the corresponding points $\langle w_{-4}\rangle$ and $\langle w_4\rangle$ is $C_{q-1} \times \mathrm{Sz}(q)$.)

Now there are exactly q^2+1 points which are adjacent to $\langle w_{-4}\rangle$ and not opposite to $\langle w_4\rangle$, namely the q^2+1 points of the Suzuki ovoid in $\langle w''_{-4}, w'_{-4}, w'_{-3}, w''_{-3}\rangle$. Since the fundamental $\mathrm{Sz}(q)$ permutes these points faithfully and transitively, it suffices to show that the subgroup of R which fixes all these points, as well as the vectors w_{-4} and w_4, is trivial.

It is easy to see that any linear map on this 4-space which fixes all the points is a scalar, and the fact that $w''_{-3} \bullet w''_{-4} = w_{-4}$ implies that this scalar is the identity. Thus we may assume that $w_4, w_{-4}, w''_{-4}, w'_{-4}, w'_{-3}, w''_{-3}$ are all centralised. Now it easy to prove that all basis vectors are fixed. For example $w_0 = w_4 \circ w_{-4}$ is fixed, and $w'_2 = w_4 \circ w''_{-3}$ is fixed, and similarly $w''_2 = w_4 \circ w'_{-3}$ and $w''_{-1} = w_4 \circ w'_{-4}$ and $w'_1 = w_4 \circ w''_{-4}$ are fixed. Then $w''_1 = w'_{-4} \bullet w'_2$ and $w'_{-2} = w'_{-4} \bullet w'_1$ and $w'_{-1} = w'_{-3} \bullet w'_2$ and $w_3 = w'_2 \bullet w'_1$ and $w''_{-2} = w''_{-4} \bullet w''_2$ are all fixed. We leave the remaining few coordinates as an exercise for the reader.

7 Properties of R

Finally we are in a position to compute the order of $R = R(q)$ and prove that, provided $q > 2$, it is a simple group.

Theorem 1 *The order of* $R(q)$ *is* $q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$.

Proof. The number of points is $(q^6+1)(q^3+1)(q+1)$, and the order of the stabilizer of a point is $q^{12}(q^2+1)(q-1)^2$. $\square$

Theorem 2 *If* $q = 2^{2n+1} > 2$ *then* $R(q)$ *is simple.*

Proof. We saw that the Borel subgroup has orbits of sizes $1, q, q^3, q^4, q^6, q^7, q^9, q^{10}$ on the points, and these are fused by the point stabilizer into five orbits of sizes 1, $q+q^3$, q^4+q^6, q^7+q^9 and q^{10}. The element σ of the Weyl group interchanges w_{-4} with w''_{-4} and fuses each orbit with the next, so in any system of imprimitivity, the orbit of size $q+q^3$ cannot be in the same block as the fixed point. Similarly, the element ρ^σ interchanges w_{-4} with w_{-2}, and fuses the first, third and fifth orbits, so the orbit of size q^4+q^6 cannot lie in the same block as the fixed point. Finally, $\sigma^{\rho\sigma\rho}$ interchanges w_{-4} with w''_4 and fuses all the five orbits, so the orbit of size q^7+q^9 cannot lie in the same block as the fixed point. Therefore the action of $R(q)$ on the points of the octagon is primitive.

The point stabilizer has a normal abelian subgroup of order q consisting of conjugates of the involution x^2. These lie inside one of the factors $\mathrm{Sz}(q)$ of $\mathrm{Sz}(q) \times \mathrm{Sz}(q)$ inside $\mathrm{Sz}(q) \wr 2$, and since $\mathrm{Sz}(q)$ is simple (provided $q > 2$), the group $\mathrm{Sz}(q) \times \mathrm{Sz}(q)$ is generated by conjugates of x^2. But this group contains the full stabiliser of two opposite points, so by transitivity on pairs of opposite points it follows that the group generated by all conjugates of x^2 is the whole of $R(q)$. Clearly x^2 is a commutator since it lies in a simple subgroup $\mathrm{Sz}(q)$ (here we use once more the fact that $q > 2$), so $R(q)$ is perfect. Now Iwasawa's Lemma states that if G is a finite perfect group acting faithfully and primitively on a set, such that the point stabilizer has a normal abelian subgroup whose conjugates generate G, then G is simple. Since we have verified that $R(q)$ satisfies all these hypotheses (provided $q > 2$), it follows that $R(q)$ is simple in these cases. $\square$

Finally, I remark that almost everything in this paper, except this last section and anything else which explicitly depends on counting, goes through for infinite fields F of characteristic 2, provided that F is perfect (which means that the Frobenius endomorphism $x \mapsto x^2$ is an automorphism) and has an automorphism τ which squares to the Frobenius automorphism. It is possible to remove the requirement for F to be perfect, at the expense of replacing the multiplication by a co-multiplication $M : W \to W \wedge W/\mathrm{im}(J)$, where J is the Jordan co-multiplication, so that the twisted linearity law in the finite case becomes $M(\lambda v) = \lambda^{2^{n+1}} M(v)$, which generalises to $M(\lambda v) = \lambda^\tau M(v)$.

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Appendix

The partial product $\bullet$ takes the following values at basis vectors. All products of basis vectors which are defined but not listed here, are zero.

$\bullet$	w_{-4}	w_{-3}	w_{-2}	w_{-1}	w_1	w_2	w_3	w_4
w_{-4}		w_{-4}	w''_{-4}	w'_{-4}	w'_{-3}	w''_{-3}	w_{-3}	
w_{-3}	w_{-4}		w'_{-2}	w''_{-2}	w''_1	w'_{-1}		w_3
w_{-2}	w''_{-4}	w'_{-2}		w_{-2}	w_{-1}		w'_1	w''_3
w_{-1}	w'_{-4}	w''_{-2}	w_{-2}			w_1	w''_{-1}	w'_3
w_1	w'_{-3}	w''_1	w_{-1}			w_2	w''_2	w'_4
w_2	w''_{-3}	w'_{-1}		w_1	w_2		w'_2	w''_4
w_3	w_{-3}		w'_1	w''_{-1}	w''_2	w'_2		w_4
w_4		w_3	w''_3	w'_3	w'_4	w''_4	w_4	

$\bullet$	w'_{-4}	w'_{-3}	w'_{-2}	w'_{-1}	w'_1	w'_2	w'_3	w'_4
w'_{-4}		w_{-4}	w''_{-4}	w'_{-3}	w'_{-2}	w''_1	w_{-1}	
w'_{-3}	w_{-4}		w'_{-4}	w''_{-3}	w''_{-2}	w'_{-1}		w_1
w'_{-2}	w''_{-4}	w'_{-4}		w_{-3}	w_{-2}		w'_1	w''_{-1}
w'_{-1}	w'_{-3}	w''_{-3}	w_{-3}			w_2	w''_2	w'_2
w'_1	w'_{-2}	w''_{-2}	w_{-2}			w_3	w''_3	w'_3
w'_2	w''_1	w'_{-1}		w_2	w_3		w'_4	w''_4
w'_3	w_{-1}		w'_1	w''_2	w''_3	w'_4		w_4
w'_4		w_1	w''_{-1}	w'_2	w'_3	w''_4	w_4	

$\bullet$	w''_{-4}	w''_{-3}	w''_{-2}	w''_{-1}	w''_1	w''_2	w''_3	w''_4
w''_{-4}		w_{-4}	w''_{-4}	w'_{-2}	w'_{-4}	w''_{-2}	w_{-2}	
w''_{-3}	w_{-4}		w'_{-3}	w''_1	w''_{-3}	w'_{-1}		w_2
w''_{-2}	w''_{-4}	w'_{-3}		w_{-1}	w_{-3}		w'_1	w''_2
w''_{-1}	w'_{-2}	w''_1	w_{-1}			w_3	w''_3	w'_4
w''_1	w'_{-4}	w''_{-3}	w_{-3}			w_1	w''_{-1}	w'_2
w''_2	w''_{-2}	w'_{-1}		w_3	w_1		w'_3	w''_4
w''_3	w_{-2}		w'_1	w''_3	w''_{-1}	w'_3		w_4
w''_4		w_2	w''_2	w'_4	w'_2	w''_4	w_4	

$$
\begin{aligned}
w_1 \bullet w_{-1} + w_2 \bullet w_{-2} = w_3 \bullet w_{-3} + w_4 \bullet w_{-4} &= w_0 \\
w'_1 \bullet w'_{-1} + w'_2 \bullet w'_{-2} = w'_3 \bullet w'_{-3} + w'_4 \bullet w'_{-4} &= w_0 \\
w''_1 \bullet w''_{-1} + w''_2 \bullet w''_{-2} = w''_3 \bullet w''_{-3} + w''_4 \bullet w''_{-4} &= w_0 \\
w_1 \bullet w_{-1} + w_4 \bullet w_{-4} = w_2 \bullet w_{-2} + w_3 \bullet w_{-3} &= w'_0 \\
w'_1 \bullet w'_{-1} + w'_4 \bullet w'_{-4} = w'_2 \bullet w'_{-2} + w'_3 \bullet w'_{-3} &= w'_0 \\
w''_1 \bullet w''_{-1} + w''_4 \bullet w''_{-4} = w''_2 \bullet w''_{-2} + w''_3 \bullet w''_{-3} &= w'_0 \\
w_1 \bullet w_{-1} + w_3 \bullet w_{-3} = w_2 \bullet w_{-2} + w_4 \bullet w_{-4} &= w''_0 \\
w'_1 \bullet w'_{-1} + w'_3 \bullet w'_{-3} = w'_2 \bullet w'_{-2} + w'_4 \bullet w'_{-4} &= w''_0 \\
w''_1 \bullet w''_{-1} + w''_3 \bullet w''_{-3} = w''_2 \bullet w''_{-2} + w''_4 \bullet w''_{-4} &= w''_0
\end{aligned}
$$